WHAT IS AN OKA MANIFOLD?

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Dedication

I wish to dedicate this talk to the memory of Hans Grauert.

He was one of the founders and main contributors to modern complex analysis and geometry,

and the principal creator of the Oka-Grauert theory.

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The Oka Principle: There are only topological obstructions to solving certain complex-analytic problems on Stein manifolds and Stein spaces.

This is a complex-analytic analogue of the Hirsch-Smale-Gromov **h-principle** in smooth geometry.

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• S is **holomorphically convex**: For every compact set $K \subset S$, its $\mathcal{O}(S)$-convex hull $\widehat{K}$ is also compact:

$$\widehat{K} = \{x \in S : |f(x)| \leq \sup_K |f|,\ \forall f \in \mathcal{O}(S)\}$$

Equivalently, for every discrete sequence $a_j \in S$ there exists a holomorphic function f on S such that $|f(a_j)| \to +\infty$ as $j \to \infty$.

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A **Stein space** is a complex space with singularities satisfying these two axioms.

Examples of Stein manifolds

- Domains in $\mathbb{C}$, open Riemann surfaces (Behnke & Stein).
- $\mathbb{C}^n$, and domains of holomorphy in $\mathbb{C}^n$ (Cartan & Thullen).
- Closed complex submanifolds of $\mathbb{C}^N$.
- A closed complex submanifold of a Stein manifold is Stein.
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And a few examples of non-Stein manifolds:

- A Stein manifold does not contain any compact complex subvariety of positive dimension.
- Quotients of Stein manifolds need not be Stein.
- There exists a fiber bundle $E \to \mathbb{C}$ with fiber $\mathbb{C}^2$ and nonlinear structure group $\Gamma \subset \mathrm{Aut}\,\mathbb{C}^2$ such that E is non-Stein.

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Cartan-Serre Theorem B (1951-56): A complex manifold S is Stein iff for every coherent analytic sheaf $\mathscr{F}$ over S we have

$$H^k(S, \mathscr{F}) = 0 \quad \text{for all } k = 1, 2, \ldots .$$

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These results can be considered as a form of **Oka principle for linear complex-analytic problems** on Stein manifolds.

Theorems of Grauert and Lefschetz-Milnor

Grauert, 1958; Narasimhan, 1962: A complex manifold S is Stein iff it admits a *strongly plurisubharmonic exhaustion function* $\rho\colon S \to \mathbb{R}$,

$$\mathrm{dd}^c\rho = \mathrm{i}\partial\overline{\partial}\rho > 0 \quad \text{(a Kähler form)}.$$

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This implies the theorem of *Lefschetz and Milnor*:
A Stein manifold of complex dimension n is homotopy equivalent to a CW-complex of dimension at most n.

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Thus Stein manifolds are holomorphic analogues of affine algebraic manifolds. In fact, every relatively compact domain in a Stein manifold is biholomorphic to a domain in an affine algebraic manifold (Stout, 1984).

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The case $n = 1$, $N = 2$ remains a difficult open problem:

Is every open Riemann surface biholomorphic to some closed nonsingular complex curve in $\mathbb{C}^2$?

Recent advances in this direction: E. F. Wold & F.

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The weak homotopy equivalence principle: The inclusion $\mathcal{O}(S,X) \hookrightarrow \mathcal{C}(S,X)$ of the space of holomorphic maps into the space of continuouos maps induces isomorphisms of all homotopy groups (a *weak homotopy equivalence*):

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- *Schwarz lemma:* Let $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$, and let $f \colon \mathbb{D} \to \mathbb{D}$ be a holomorphic function. Then

$$\frac{|\mathrm{d}f(z)|}{1-|f(z)|^2} \le \frac{|\mathrm{d}z|}{1-|z|^2}, \quad z \in \mathbb{D}.$$

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None of these examples satisfies any Oka property.

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A compact complex manifold is Brody hyp. iff it is Kobayashi hyp. A majority of complex manifolds are close to hyperbolic. Every compact complex manifold of maximal Kodaira dimension is volume hyperbolic, and is conjecturally 'almost' hyperbolic.

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Runge Theorem. If $K \subset \mathbb{C}$ is a compact set with no holes, then every holomorphic function $K \to \mathbb{C}$ can be approximated uniformly on K by entire functions.

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A twist of philosophy: We can also view them as properties of the target manifold, the complex number field $\mathbb{C}$. We shall now formulate them as properties of an arbitrary target X.

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By Oka-Weil and Cartan, $\mathbb{C}$, and hence $\mathbb{C}^n$, satisfy BOP.

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Applying POP with parameter pairs $\emptyset \subset S^k$ and $S^k \subset B^{k+1}$ gives the *weak homotopy equivalence principle*:

$$\pi_k(\mathcal{O}(S,X)) \cong \pi_k(\mathcal{C}(S,X)), \quad \forall k \in \mathbb{N}.$$

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Example: Let G be a complex Lie group, acting holomorphically and transitively on a complex manifold X. Let $\mathfrak{g} = \mathrm{T}_1 G \cong \mathbb{C}^n$ be the Lie algebra of G. For every point $x \in X$ we have a holomorphic dominating map

$$s_x \colon \mathfrak{g} \cong \mathbb{C}^n \to X, \quad s_x(v) = \mathrm{e}^v \cdot x$$

such that $s_x(0) = x$ and $\mathrm{d}_0 s_x$ is surjective.

The Oka-Grauert Principle

Good candidates for having Oka properties are complex manifolds X with sufficiently many holomorphic maps $\mathbb{C}^n \to X$.

Example: Let G be a complex Lie group, acting holomorphically and transitively on a complex manifold X. Let $\mathfrak{g} = \mathrm{T}_1 G \cong \mathbb{C}^n$ be the Lie algebra of G. For every point $x \in X$ we have a holomorphic dominating map

$$s_x \colon \mathfrak{g} \cong \mathbb{C}^n \to X, \quad s_x(v) = \mathrm{e}^v \cdot x$$

such that $s_x(0) = x$ and $\mathrm{d}_0 s_x$ is surjective.

Grauert, 1957-58: **Every complex Lie group and, more generally, every homogeneous manifold, enjoys POP.**

The same holds for sections $S \to Z$ of holomorphic G-bundles $\pi \colon Z \to S$ (G a complex Lie group) over a Stein space S.

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For any complex Lie group G, the topological and the holomorphic isomorphism classes of principal G-bundles over any Stein space S are in one-to-one correspondence:

$$\mathcal{O}^G \hookrightarrow \mathcal{C}^G \text{ induces an isomorphism } H^1(S, \mathcal{O}^G) \cong H^1(S, \mathcal{C}^G).$$

The same is true for the associated fiber bundles with G-homogeneous fibers; in particular, for *complex vector bundles* (take the group $G = GL_k(\mathbb{C})$).

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Oka treated this as a *second Cousin problem*:
Given an open cover $\mathcal{U} = \{U_j\}$ of S and a collection of nonvanishing holomorphic functions $f_{ij} \colon U_{ij} \to \mathbb{C} \setminus \{0\}$ such that

$$f_{ii} = 1, \quad f_{ij} f_{ji} = 1, \quad f_{ij} f_{jk} f_{jk} = 1,$$

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Oka: *If the problem can be solved by continuous functions, then it can also be solved by holomorphic functions.*

Cohomological proof of Oka's theorem

Let $\sigma(f) = \mathrm{e}^{2\pi \mathrm{i} f}$. The *exponential sheaf sequence*:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{O} & \xrightarrow{\sigma} & \mathcal{O}^* & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{C} & \xrightarrow{\sigma} & \mathcal{C}^* & \longrightarrow & 1 \end{array}$$

Long exact sequence on cohomology:

$$\begin{array}{ccccccc} H^1(S,\mathcal{O}) & \longrightarrow & H^1(S,\mathcal{O}^*) & \xrightarrow{c_1} & H^2(S,\mathbb{Z}) & \longrightarrow & H^2(S,\mathcal{O}) \\ \downarrow & & \downarrow & & \| & & \downarrow \\ 0 & \longrightarrow & H^1(S,\mathcal{C}^*) & \xrightarrow{c_1} & H^2(S,\mathbb{Z}) & \longrightarrow & 0 \end{array}$$

If S is Stein then $H^1(S,\mathcal{O}) = 0 = H^2(S,\mathcal{O})$, and hence $H^1(S,\mathcal{O}^*) \cong H^1(S,\mathcal{C}^*) \cong H^2(S,\mathbb{Z})$.

Main analytic ingredients of Grauert's proof

This cohomological proof fails for nonabelian Lie groups. Grauert's proof is constructive and uses two main ingredients:

• *Homotopy version of Runge approximation theorem* for maps of Stein manifolds S to homogeneous manifolds X:

Given a pair $K \subset L$ of compact $\mathcal{O}(S)$-convex sets and a homotopy of holomorphic maps $f_t \colon K \to X$ $(t \in [0,1])$ such that f_0 is holomorphic on L, $\{f_t\}$ is approximable uniformly on K by holomorphic homotopy $\widetilde{f}_t \colon L \to X$ $(t \in [0,1])$ with $\widetilde{f}_0 = f_0$.

(Proof: Using the exponential map $\mathfrak{g} \times X \to X$, $(v, x) \mapsto \mathrm{e}^v x$, pull back $\{f_t\}_{t\in[0,1]}$ to a homotopy of sections of a vector bundle over S, apply the Oka-Weil theorem, then push back to X.)

• *Cartan's lemma*: Given a suitable pair of compact sets $K = K_0 \cup K_1$ in S, every holomorphic maps $f \colon K_0 \cap K_1 \to G$ splits as a product $f = f_0 \cdot f_1$, where $f_j \colon K_j \to G$ is holomorphic for $j = 0, 1$. (This is used for gluing holomorphic sections.)

Generalizations and applications of Grauert's theorem

- *Forster and Ramspott*, 1964–1970:
  - Generalization to certain stratified bundles with homogeneous fibers, and to Oka pairs of sheaves.
  - Optimal estimates for the number of holomorphic sections needed to generate a holomorphic vector bundle, or a coherent analytic sheaf over a Stein space.
  - The Oka principle for complete intersection subvarieties.
  - Embedding theorems for Stein manifolds.

- *Forster & Ohsawa*, 1984: Complete intersections in $\mathbb{C}^n$ for entire functions of finite order.

- *Heinzner and Kutzschebauch*, 1995: An equivariant version of Grauert's Oka principle.

- *Henkin and Leiterer*, 1986 and 1998: Different proof and an extension to 1-convex manifolds.

- *Leiterer and Vâjâitu, 2003:* Banach-valued bundles.

Sprays and elliptic manifolds

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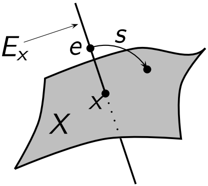

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- $\mathbb{P}^n \setminus A$ is subelliptic if A is a subvariety of codimension ≥ 2.

Oka principle for sections of elliptic submersions

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Gromov also obtained POP for sections of (non-locally trivial !) *elliptic holomorphic submersions* $Z \to S$ over a Stein S: Every point $x_0 \in S$ admits an open neighborhood $U \subset S$ and dominating sprays of the fibers Z_x, depending holomorphically on the point $x \in U$.

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Let $E \to S$ be a holomorphic vector bundle (or a projective bundle), and let $\Sigma \subset E$ be a complex subvariety with algebraic fibers $\Sigma_x \subset E_x \in \{\mathbb{C}^n, \mathbb{CP}^n\}$ of codimension ≥ 2. Then sections $S \to E \setminus \Sigma$ avoiding Σ satisfy POP.

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This result is crucial in the proof of the optimal embedding theorem for Stein manifolds:

$$S^n \hookrightarrow \mathbb{C}^{\left[\frac{3n}{2}\right]+1}.$$

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Gromov's questions:

- Can one characterize Oka properties by a Runge approximation property for maps $\mathbb{C}^n \to X$?
- Does BOP imply POP for every manifold X?

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It easily follows that all Oka properties are equivalent.
A complex manifold satisfying any of these properties is called an

OKA MANIFOLD

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  No Oka manifold is of Kodaira general type.

Examples of Oka manifolds

- $\mathbb{C}^n, \mathbb{P}^n$, complex Lie groups and their homogeneous spaces;
- $\mathbb{C}^n \setminus A$, where A is an algebraic subvariety of codim. ≥ 2;
- $\mathbb{P}^n \setminus A$, where A is a subvariety of codim. ≥ 2;
- rational surfaces (in particular, all Hirzebruch surfaces; these are $\mathbb{P}^1$ bundles over $\mathbb{P}^1$);
- Hopf manifolds (quotients of $\mathbb{C}^n \setminus \{0\}$ by cyclic groups);
- Algebraic manifolds that are covered Zariski locally affine ($\cong \mathbb{C}^n$);
- certain modifications of such (blowing up points, removing subvarieties of codim. ≥ 2);
- $\mathbb{C}^n$ blown up at all points of a tame discrete sequence;
- complex torus of dim> 1 with finitely many points removed, or blown up at finitely many points;
- toric varieties $X = (\mathbb{C}^m \setminus Z)/G$, where Z is a union of coordinate subspaces of $\mathbb{C}^m$, and G is a subgroup of $(\mathbb{C}^*)^m$ acting on $\mathbb{C}^m \setminus Z$ by diagonal matrices.

Methods to prove CAP $\Longrightarrow$ POP

A nonlinear Cousin-I problem:
Let (A, B) be a *Cartain pair* in a Stein manifold S (compacts such that $A \cup B$, $A \cap B$ have Stein neighborhood bases).

Given $f\colon A \to X$, $g\colon B \to X$ holomorphic, with $f \approx g$ on $A \cap B$, find a holomorphic map $\widetilde{f}\colon A \cup B \to X$ such that $\widetilde{f}|_A \approx f|_A$.

- Extend f, g to holomorphic maps

$$F\colon A \times \mathbb{B}^k \to X, \;\; G\colon B \times \mathbb{B}^k \to X,$$

submersive in $z \in \mathbb{B}^k \subset \mathbb{C}^k$; $f = F(\cdot\,, 0)$, $g = G(\cdot\,, 0)$.

- Find a holomorphic transition map $\gamma(x, z) = (x, c(x, z))$ over $(A \cap B) \times r\mathbb{B}^k$ $(r < 1)$, $\gamma \approx \mathrm{Id}$, such that $F = G \circ \gamma$.

- Split

$$\gamma = \beta \circ \alpha^{-1}, \quad \alpha, \beta \approx \mathrm{Id}.$$

Then $F \circ \alpha = G \circ \beta\colon (A \cup B) \times r\mathbb{B}^k \to X$ solves the problem.

Passing a critical point

Passing a critical point p_0 of a strongly plurisubharmonic exhaustion function $\rho\colon S \to \mathbb{R}$:

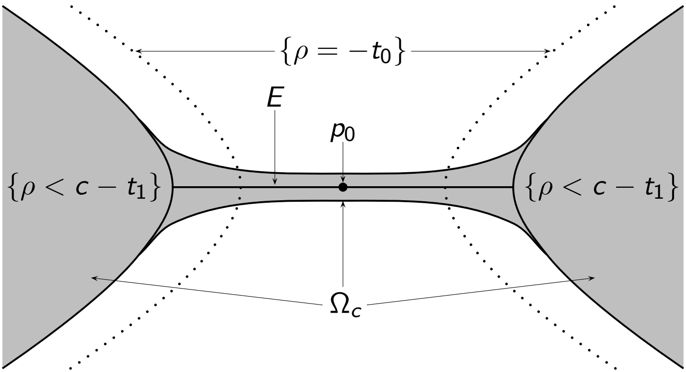

Figure: The set $\Omega_c = \{\tau < c\}$, $c > 0$.

From manifolds to maps

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Oka properties of a holomorphic map $\pi \colon E \to B$ pertain to liftings in the following diagram, with Stein source S:

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Let S be a Stein manifold and $f\colon S \to SL_m(\mathbb{C})$ a null-homotopic holomorphic mapping. There exist $k \in \mathbb{N}$ and holomorphic mappings $G_1, \ldots, G_k\colon S \to \mathbb{C}^{m(m-1)/2}$ such that

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Cohn (1966): The matrix

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Suslin (1977): For $m \geq 3$ (and any n) any matrix in $SL_m(\mathbb{C}^{[n]})$ decomposes as a finite product of unipotent matrices.

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In this model category, a complex manifold is:

- cofibrant iff it is Stein, and
- fibrant iff it is an Oka manifold.

Open problems

- Find a geometric characterization of Oka manifolds.
- In particular, is every Oka manifold also elliptic (or at least subelliptic)? Is every subelliptic manifold elliptic?
- Which complex surfaces of non-general type are Oka?
- In particular, which K3 surfaces are Oka? Is every Kummer surface Oka?
- Which modifications (such as blow-ups or blow-downs) preserve Oka property?
- Is $\mathbb{C}^n \setminus$ (closed ball) Oka?
- Topological restrictions on the class of Oka manifolds?

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Example: The complex surface $S = \mathbb{CP}^1 \times \mathbb{C}$ does not admit a non-exotic Stein structure, but it admits exotic Stein structures.

Additional reading...

– F. Lárusson: What is an Oka manifold?
Notices Amer. Math. Soc. 57 (2010), no. 1, 50–52.

– F. Forstnerič and F. Lárusson: Survey of Oka theory.
New York J. Math., 17a (2011), 1–28.

– T. Ohsawa: Topics in complex analysis from the viewpoint of Oka principle. Preprint (2011), 30 pp.

– F. Forstnerič: *Stein Manifolds and Holomorphic Mappings.*
(The Homotopy Principle in Complex Analysis.)
Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, vol. 56, Springer-Verlag, Berlin-Heidelberg (2011)