CONTOUR PROCESSES OF RANDOM TREES

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Abstract. We study two stochastic processes describing the contour of simply generated random trees: the contour process as defined by Gutjahr and Pflug [?] and the traverse process constructed of the node heights during pre-order traversal of the tree. Using multivariate generating functions and singularity analysis we show the weak convergence of the contour process to Brownian excursion and obtain a new proof of the analogous result for the traverse process.

1. Introduction

Let $\mathcal{A}$ be a class of plane rooted trees and define for $T \in \mathcal{A}$ the size $|T|$ by the number of nodes of T. Furthermore there is assigned a weight $\omega(T)$ to each $T \in \mathcal{A}$. Let a_n denote the quantity

$$a_n = \sum_{|T|=n} \omega(T)$$

Besides, let us define the generating function (GF) corresponding to $\mathcal{A}$ by $a(z) = \sum_{n\geq 0} a_n z^n$. According to Meir and Moon [?] we call a family of trees *simply generated* if its GF satisfies a functional equation of the form

$$a(z) = z\varphi(a(z)), \tag{1.1}$$

where $\varphi(t) = \sum_{i\geq 0} \varphi_i t^i$ with $\varphi_i \geq 0, \varphi_0 > 0$.

Let $n_k(T)$ denote the number of nodes $v \in T$ with outdegree k (the outdegree of v is the number of edges incident with v that lead away from the root). Then we have for each simply generated tree T the relation

$$\omega(T) = \prod_{i\geq 0} \varphi_k^{n_k(T)}. \tag{1.2}$$

Consider a simply generated tree T of size n. The height $h_T(x)$ of a node $x \in T$ is defined to be the number of edges of the uniquely determined path that connects x with the root. Let $\hat{h}_T(m)$ denote the height of the m-th leaf of T supposing that the leaves are enumerated from left to right. In the following we will assume that for each n the set of all trees of size n is equipped with a probability distribution according to the weights (1.2). Then $\hat{h}_T(m)$ becomes a random variable which we denote by $\hat{H}_n(m)$. If we define the continuation of $\hat{H}_n(m)$ by linear interpolation, i.e.

$$\hat{H}_n(x) = (\lfloor x\rfloor + 1 - x)\, \hat{H}_n(\lfloor x\rfloor) + (x - \lfloor x\rfloor)\, \hat{H}_n(\lfloor x\rfloor + 1),$$

then we get a continuous stochastic process. The scaled process

$$\hat{X}_n(t) = \frac{1}{\sqrt{n}}\hat{H}_n(tn), \quad 0 \leq t \leq 1,$$

is called the contour process.

We show that for simply generated trees this process converges weakly to Brownian excursion (for the definition and basic properties see [?, pp.75]):

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Theorem 1.1. *Let $W^+(t)$ denote Brownian excursion of duration 1. Furthermore assume that $\varphi(t)$ has a positive or infinite radius of convergence R and $d = ggT\{i|\varphi_i > 0\} = 1$. Moreover suppose that the equation*

$$t\varphi'(t) = \varphi(t) \tag{1.3}$$

has a minimal positive solution $\tau < R$. Define

$$\sigma^2 = \frac{\tau^2\varphi''(\tau)}{\varphi(\tau)}.$$

Then the contour process $\hat{X}_n(t)$ converges weakly to Brownian excursion, i.e.

$$\hat{X}_n\left(\frac{\varphi_0}{\varphi(\tau)}t\right) \xrightarrow{w} \frac{2}{\sigma}W^+(t) \tag{1.4}$$

in $C[0,1]$.

For the class of binary trees (1.4) was established by Gutjahr and Pflug [**?**] but their method does not seem to be transferable to the general case, because it relies on exact enumeration formulae which are only available for binary trees.

Remark 1. The case $d > 1$ can be treated similarly, but is technically more involved. The only difference concerning the results is that the limit theorems hold only for $n \equiv 1 \mod d$ and that the limiting distribution in local limit theorems has to be multiplied by d. Thus we restrict ourselves to $d = 1$.

Remark 2. Simply generated trees may be considered as trees associated to Galton-Watson branching processes. In this context (1.3) means that the branching process is critical and σ^2 equals the variance of the offspring distribution. Thus the above theorem yields also a limiting distribution result for branching processes conditioned on the total progeny. For a more detailed discussion of the connection between trees and branching processes see Aldous [**?**].

In order to define the traverse process we have to use the tree T' defined to be the tree we obtain by attaching T to a single node which serves as the root of T'. Now consider the following traverse procedure:

1. If the current node is v, choose the left-most successor of v that has not been traversed yet (v' is called successor of v if it is adjacent to v and $h_T(v') > h_T(v)$). If no such successor does exist, go back to the previous node.
2. Start at the root and apply step (1) to its successor.

Since in (1) choosing the left-most successor v' is equivalent to choosing the edge (v, v'), each edge is traversed twice and thus the number of steps is $2n$. Let v_i denote the node we arrive at after i steps and define $h_n(i) = h_T(v_i),\ i = 0, \ldots, 2n$. Assuming again the probability model induced by the weights (1.2) $h_n(i)$ becomes a stochastic process $H_n(i)$ and as above we continue $H_n(i)$ by linear interpolation. The traverse process is defined by the scaled process

$$X_n(t) = \frac{1}{\sqrt{n}}H_n(2nt), \quad 0 \le t \le 1.$$

The GFs involved in the investigation of $\hat{X}_n(t)$ and $X_n(t)$ are closely related and thus we rather easily obtain from (1.4):

Theorem 1.2. *Under the assumptions of theorem 1.1 the traverse process $X_n(t)$ converges weakly to Brownian excursion, i.e.*

$$X_n(t) \xrightarrow{w} \frac{2}{\sigma}W^+(t)$$

in $C[0,1]$.

This limit theorem was established by Aldous [**?**] by means of probabilistic techniques (see [**?**, **?**]) and under the slightly weaker condition $\sigma^2 > 0$. Our approach yields a new proof of this result.

The paper is organized as follows: In section 2 we give a brief description of the basic methods used in the following sections, especially the combinatorial background. Section 3 is devoted to the proof of theorem 1.1. Therefore we have to show the weak convergence of the finite dimensional distributions and the tightness of the process (see Billingsley [**?**]). In order to settle the first part of the proof we first consider the three dimensional distributions where we prove an invariance property which enables us to simplify the rest of the proof essentially. The last section provides a brief discussion of the traverse process.

2. Basic Methods

In order to derive the above mentioned limit theorems we use the concept of combinatorial constructions introduced by Vitter and Flajolet [**?**]: Let $\circ$ denote a node and $\mathcal{A}$ a simply generated family of trees. Then every element in $\mathcal{A}$ has the form

$$\{\circ\} \times \mathcal{A} \times \cdots \times \mathcal{A}.$$

Taking into account that we are considering weighted trees we have to assign the weight φ_i to the above expression, if there are i factors $\mathcal{A}$. Thus we get the following symbolic recursion:

$$\mathcal{A} = \varphi_0 \cdot \{\circ\} \,\cup\, \varphi_1 \cdot \{\circ\} \times \mathcal{A} \,\cup\, \varphi_2 \cdot \{\circ\} \times \mathcal{A} \times \mathcal{A} \,\cup\, \cdots$$

Using the fact that the operations $\cup$ and $\times$ can be translated into sum and product of the corresponding GFs we obtain the functional equation (1.1).

Now let $\theta(T)$ be a characteristic of the tree T we are interested in. Then we mark the corresponding substructures of T which is equivalent to introducing a new variable in the GF. Thus we get a bivariate GF

$$a(z,u) = \sum_{m,n\geq 0} a_{mn} z^n u^m$$

The distribution of θ is given by

$$P\{\theta(T) = m : \ |T| = n\} = \frac{a_{mn}}{a_n}$$

where a_{mn} is the coefficient of $z^n u^m$ in $a(z,u)$, denoted by $[z^n u^m]a(z,u)$. We will calculate this distribution by deriving multivariate asymptotic expansions for a_{kn} with uniform error terms. Thus we get a local limit theorem and due to uniformity this implies the corresponding weak limit theorem.

Example . $\theta(T)$ equals the number of leaves of T. If a marked node is represented by $\bullet$ and the family of all trees with marked leaves is denoted by $\mathcal{Y}$, then we get the recursion

$$\mathcal{Y} = \{\bullet\} \,\cup\, \{\circ\} \times \Phi(\mathcal{Y}),$$

where

$$\Phi(\mathcal{Y}) = \varphi_1 \cdot \{\circ\} \times \mathcal{Y} \,\cup\, \varphi_2 \cdot \{\circ\} \times \mathcal{Y} \times \mathcal{Y} \,\cup\, \cdots$$

Due to the correspondence

$$\circ \leftrightarrow z$$
$$\bullet \leftrightarrow uz$$

translating into GFs gives

$$y(z,u) = \varphi_0 z(u-1) + z\varphi(y(z,u)). \tag{2.1}$$

For further demonstrations of these marking techniques we refer to [**?**].

In order to get asymptotic expansions we use Cauchy's integral formula combined with singularity analysis following the ideas of Flajolet and Odlyzko [?]. They used the fact that the coefficients of the power series of an analytic function are essentially determined by the behaviour of the function near its dominant singularities, i.e. those on the circle of convergence, and proved the following theorem:

Theorem 2.1 ([?]). *Let $f(z)$ be analytic in the domain*

$$\Delta = \{z \mid |z| \le z_0 + \eta, |arg(z - z_0)| \ge \vartheta\},$$

where $z_0, \eta > 0$ and $0 < \vartheta < \frac{\pi}{2}$. Furthermore let α be a real number satisfying $\alpha \notin \{0, 1, 2, \dots\}$ Then

$$f(z) \sim \left(1 - \frac{z}{z_0}\right)^{\alpha} \text{ for } z \to z_0 \text{ in } \Delta \implies [z^n]f(z) \sim \frac{1}{z_0^n n^{\alpha+1}\Gamma(-\alpha)}.$$

Analogous formulae hold for $\mathcal{O}$ and o instead of $\sim$.

Remark . Let $y(z, u)$ be the function defined by (2.1) and $y_{mn} = [u^m z^n]y(z, u)$. Then it can be shown that $\frac{y_{mn}}{a_n}$ satisfy a central limit theorem with mean

$$\frac{1}{a_n} \sum_{m \ge 0} m y_{mn} = \frac{\varphi_0}{\varphi(\tau)} n + \mathcal{O}(1) \tag{2.2}$$

and variance

$$\left(\frac{\varphi_0}{\varphi(\tau)} - \frac{\varphi_0^2}{\varphi(\tau)^2} - \frac{\varphi_0^2 \varphi'(\tau)}{\varphi(\tau)^3 \varphi''(\tau)}\right) n + \mathcal{O}(1), \tag{2.3}$$

where τ is the solution of (1.3) (see [?, ?]). If $L_n(T)$ denotes the number of leaves of a random tree T of size n, then (2.2) and (2.3) imply

$$\lim_{n \to \infty} \frac{L_n(T)}{n} = \frac{\varphi_0}{\varphi(\tau)}, \quad \text{a.s.}$$

Thus the restriction of $\hat{X}_n(t)$ to the interval $\left[0, \frac{\varphi_0}{\varphi(\tau)}\right]$ is justified.

3. The contour process

3.1. **Basic functions and their local expansions.** Let $\mathcal{A}$ be a family of simply generated trees with GF defined by (1.1) and $m_1 < m_2 < \cdots < m_p$. Consider the set $\mathcal{F}_{k_1 m_1 k_2 m_2 \dots k_p m_p n} \subseteq \mathcal{A}$ of all trees T with n nodes satisfying $\hat{h}_T(m_i) = k_i$ for $i = 1, \dots, p$. Set

$$a_{k_1 m_1 \dots k_p m_p n} = \sum_{T \in \mathcal{F}_{k_1 m_1 \dots k_p m_p n}} \omega(T),$$

where $\omega(T)$ denote the weight defined by (1.2). Then the finite dimensional distributions of $H_n(x)$ are given by

$$P\{H_n(m_1) = k_1, \dots, H_n(m_p) = k_p\} = \frac{a_{k_1 m_1 \dots k_p m_p n}}{a_n}.$$

Thus we need asymptotic expansions for $a_{k_1 m_1 \dots k_p m_p n}$ and a_n. When setting up the GFs it turns out that they are composed of three basic functions: Obviously the function $y(z, u)$ defined by (2.1) plays the most important role. The other two functions are composed of $y(z, u)$:

$$\begin{aligned} \phi_1(z, u, v) &= z \sum_{i \ge 1} \varphi_i \sum_{j_1 + j_2 = i-1} y(z, u)^{j_1} y(z, v)^{j_2} \\ &= z \frac{\varphi(y(z, u)) - \varphi(y(z, v))}{y(z, u) - y(z, v)} \end{aligned}$$

and

$$\phi_2(z, u, v, w) = z \sum_{i \ge 2} \varphi_i \sum_{j_1 + j_2 + j_3 = i-2} y(z, u)^{j_1} y(z, v)^{j_2} y(z, w)^{j_3}$$

Remark . These functions originate from the following setup: Consider a node to which we attach $i-1$ trees and a marked leaf b. Then leaves of the trees left from b contribute to the number of b while the others do not. Thus the trees left from b correspond to the GF $y(z,u)$ and the remaining trees to $y(z,1)$. Summing up over all node degrees and keeping in mind that nodes of degree i are weighted by φ_i we get the GF $u\phi_1(z,u,1)$. If we replace the marked node by more complicated structures we will get powers of ϕ_1 or ϕ_2. Of course there may occur functions $\phi_3, \phi_4, \ldots$ (it is obvious how to define them), if we mark more than two leaves, but they prove to be of no importance for the asymptotics in the following.

In order to proceed we need local expansions of these functions near their singularities. We have

Lemma 3.1 ([?])**.** *Let $\varphi(t)$ have a positive or infinite radius of convergence R. Furthermore assume that $d = ggT\{i|\varphi_i > 0\} = 1$ and that the equation $t\varphi'(t) = \varphi(t)$ has a minimal positive solution $\tau < R$. Then for an $\varepsilon > 0$ there exists a uniquely determined analytic function $z = f(u)$ on $|u-1| < \varepsilon$ such that $f(1) = \frac{1}{\varphi'(\tau)}$ and $y(f(u),u)$ satisfies*

$$\begin{aligned} y &= \varphi_0 z(u-1) + z\varphi(y), \\ 1 &= z\varphi'(y). \end{aligned}$$

$z = f(u)$ is the only singularity of $y(z,u)$ in the domain $|z| \le z_0 + \varepsilon$, $\arg\left(1 - \frac{z}{f(u)}\right) \ne \pi$, where $z_0 = f(1)$. Moreover, inside the domain $\left\{(z,u) : |z - f(u)| < \varepsilon,\ \arg\left(1 - \frac{z}{f(u)}\right) \ne \pi\right\}$ $y(z,u)$ admits the local representation

$$y(z,u) = g(z,u) - h(z,u)\sqrt{1 - \frac{z}{f(u)}},$$

where $g(z,u)$ and $h(z,u)$ are analytic functions satisfying

$$g(z_0,1) = \tau \quad \text{and} \quad h(z_0,1) = \sqrt{\frac{2\varphi(\tau)}{\varphi''(\tau)}}.$$

For $d > 1$ analogous representations in the vicinity of $f(u)\exp\left(j\frac{2\pi i}{d}\right)$, $0 \le j < d$ hold.

Corollary 1. *Assume $d = 1$. Then $a(z)$ has one and only one singularity $z = z_0$ on the circle of convergence. Furthermore the local representation*

$$a(z) = \tau - \frac{\tau\sqrt{2}}{\sigma}\sqrt{1 - \frac{z}{z_0}} + \mathcal{O}\left(\left|1 - \frac{z}{z_0}\right|^2\right).$$

holds near $z = z_0$

Corollary 2. *For $a_n = [z^n]a(z)$ we have*

$$a_n = \frac{\tau}{\sigma z_0^n \sqrt{2\pi n^3}}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right) \tag{3.1}$$

Remark . It is possible to exchange the roles of z and u in the theorem, that means we have also a local representation of the form

$$y(z,u) = \tilde{g}(z,u) - \tilde{h}(z,u)\sqrt{1 - \frac{u}{\tilde{f}(z)}}, \tag{3.2}$$

where $\tilde{g}(z,u)$, $\tilde{h}(z,u)$, and $\tilde{f}(z)$ are analytic functions.

Using this lemma local expansions for the above mentioned basic functions can easily be derived:

Lemma 3.2 ([?]). *Set $z = z_0(1+\frac{t}{n})$ and $u_i = 1 + \frac{s_i}{m_i}$, $i = 1,2,3$, where $\varepsilon < \frac{m_i}{n}\frac{\varphi(\tau)}{\varphi_0} < 1-\varepsilon$, for arbitrary $\varepsilon > 0$. Furthermore let $|t| \le \eta n$ and $|s_i| \le \eta m_i$ for sufficiently small $\eta > 0$. Then for $n \to \infty$ the following local expansions hold*

$$\begin{aligned}
y(z,u_1) - \tau &= -\sqrt{\frac{2\varphi(\tau)}{\varphi''(\tau)}}\sqrt{-\frac{t}{n} - \frac{\varphi_0}{\varphi(\tau)}\frac{s_1}{m_1}} + \mathcal{O}\left(\frac{|s_1|}{m_1} + \frac{|t|}{n}\right)\\
\phi_1(z,u_1,u_2) &= 1 - \frac{\sigma}{\sqrt{2}}\left(\sqrt{-\frac{t}{n} - \frac{\varphi_0}{\varphi(\tau)}\frac{s_1}{m_1}} + \sqrt{-\frac{t}{n} - \frac{\varphi_0}{\varphi(\tau)}\frac{s_2}{m_2}}\right)\\
&\quad + \mathcal{O}\left(\frac{|t|}{n} + \frac{|s_1|}{m_1} + \frac{|s_2|}{m_2}\right), \qquad (3.3)\\
\phi_2(z,u_1,u_2,u_3) &= \frac{z_0\varphi''(\tau)}{2} + \mathcal{O}\left(\sqrt{\frac{|t|}{n} + \frac{|s_1|}{m_1} + \frac{|s_2|}{m_2} + \frac{|s_3|}{m_3}}\right),
\end{aligned}$$

Since expressions like those in the previous lemma will frequently occur in the following sections we will from now on use the abbreviation

$$\begin{aligned}
c_{i_1 i_2 \dots i_k} &= \sqrt{-\frac{t}{n} - \frac{\varphi_0}{\varphi(\tau)}\left(\frac{s_{i_1}}{m_{i_1}} + \dots + \frac{s_{i_k}}{m_{i_k}}\right)}\\
c &= \sqrt{-\frac{t}{n}}
\end{aligned}$$

3.2. **An invariance property.** Drmota [?] used the above setup to determine the one and two dimensional distributions of the contour process. The method works in principle for higher dimensional distributions, too, but the expressions obtained in these cases get too complicated to cope with. If we combine this method with an idea of Gutjahr and Pflug [?] that works for binary trees, we will achieve an essential simplification. The idea is to introduce an additional quantity l_i which is defined as follows: Consider a simply generated tree T where the leaves with numbers $m_1 < m_2 < \cdots < m_p$ are marked. Then the paths connecting the root with the m_i-th and the m_{i+1}-st leaf, resp., have at least the root in common. Let V_i denote that of the common nodes which has maximal height and define $l_i := h_T(V_i)$.

Let us now consider the case $p = 3$. Define the GF

$$B_{k_1 l_1 k_2 l_2 k_3}(z,u_1,u_2,u_3) = \sum_{n,m_1,m_2,m_3 \ge 0} b_{k_1 m_1 l_1 k_2 m_2 l_2 k_3 m_3 n} z^n u_1^{m_1} u_2^{m_2} u_3^{m_3},$$

where $b_{k_1 m_1 l_1 k_2 m_2 l_2 k_3 m_3 n}$ denotes the sum of weights of all trees with n nodes and satisfying $\hat{h}_T(m_i) = k_i, i = 1,2,3$, and $h_T(V_j) = l_j$. For setting up this GF we have to distinguish three cases: $l_1 < l_2$, $l_1 > l_2$, and $l_1 = l_2$. The third one is asymptotically negligible since it corresponds to a hyperplane in $\mathbf{R}^5$ in the limit case and thus it has no influence on the density of the limiting distribution (for a detailed argumentation see [?]).

For convenience introduce a tree T_0 consisting of m_1, m_2, m_3, V_1, V_2, and the root of T. The edges of T_0 are the paths that connect its nodes in T. As described in the previous section each edge of T_0 corresponds to a power of ϕ_1 according to its length. The branching points V_1 and V_2 yield factors ϕ_2. Therefore we obtain for $l_1 < l_2$

$$\begin{aligned}
B_{k_1 l_1 k_2 l_2 k_3}(z,u_1,u_2,u_3) &= \varphi_0^3 z^3 u_1 u_2 u_3 \phi_2(z, u_1u_2u_3, u_2u_3, 1)\phi_2(z, u_2u_3, u_3, 1)\\
&\quad \times \phi_1(z, u_1u_2u_3, 1)^{l_1} \phi_1(z, u_1u_2u_3, u_2u_3)^{k_1 - l_1 - 1}\\
&\quad \times \phi_1(z, u_2u_3, 1)^{l_2 - l_1 - 1} \phi_1(z, u_2u_3, u_3)^{k_2 - l_2 - 1}\\
&\quad \times \phi_1(z, u_3, 1)^{k_3 - l_2 - 1}
\end{aligned}$$

Analogously we get the GF in the case $l_1 > l_2$:

$$\begin{aligned} B_{k_1l_1k_2l_2k_3}(z,u_1,u_2,u_3) = \varphi_0^3 z^3 u_1u_2u_3 &\phi_2(z,u_1u_2u_3,u_3,1)\phi_2(z,u_1u_2u_3,u_2u_3,u_3) \\ &\times \phi_1(z,u_1u_2u_3,1)^{l_2}\phi_1(z,u_1u_2u_3,u_2u_3)^{k_1-l_1-1} \\ &\times \phi_1(z,u_1u_2u_3,u_3)^{l_1-l_2-1}\phi_1(z,u_2u_3,u_3)^{k_2-l_1-1} \\ &\times \phi_1(z,u_3,1)^{k_3-l_2-1} \end{aligned}$$

If we consider random trees, then the heights of the leaves m_1, m_2, m_3 as well as the path lengths l_1, l_2 become random variables. Let us denote this multivariate random variable by $(K_1, L_1, K_2, L_2, K_3)$. Its distribution is determined by

$$[z^n u_1^{m_1} u_2^{m_2} u_3^{m_3}] B_{k_1l_1k_2l_2k_3}(z,u_1,u_2,u_3).$$

This coefficient can be calculated asymptotically by means of Cauchy's integral formula. The integration path is chosen in that way that one part lies close to the singularity (this part yields the main term) and the remaining part is asymptotically negligible (for details see next section). Thus the limiting distribution is completely determined by the local behaviour of the GF.

Let k_i, $i = 1,2,3$, and l_j, $j = 1,2$, be proportional to $\sqrt{n}$ and m_i, $i = 1,2,3$, satisfy the condition $\varepsilon < \frac{m_i}{n}\frac{\varphi(\tau)}{\varphi_0} < 1-\varepsilon$ for arbitrary $\varepsilon > 0$. Using lemma 3.2 and the fact that $k_i - 1 \sim k_i$ and $l_i - 1 \sim l_i$ it can be shown that $B_{k_1l_1k_2l_2k_3}(z,u_1,u_2,u_3)$ admits the local representation

$$\begin{aligned} &\frac{\varphi_0^3 z_0^5 \varphi''(\tau)^2}{4} \exp\Bigg(-\frac{\sigma}{\sqrt{2}}(l_1(c_{123}+c) + (k_1-l_1)(c_{123}+c_{23}) + (l_2-l_1)(c_{23}+c) \\ &\quad + (k_2-l_2)(c_{23}+c_3) + (k_3-l_2)(c_3+c))\Bigg) \\ &\quad \times \left(1 + \mathcal{O}\left(M\left(\frac{|t|}{n} + \frac{|s_1|}{m_1} + \frac{|s_2|}{m_2} + \frac{|s_3|}{m_3}\right)\right) + \mathcal{O}\left(\sqrt{\frac{|t|}{n} + \frac{|s_1|}{m_1} + \frac{|s_2|}{m_2} + \frac{|s_3|}{m_3}}\right)\right) \end{aligned} \tag{3.4}$$

for $l_1 < l_2$ and

$$\begin{aligned} &\frac{\varphi_0^3 z_0^5 \varphi''(\tau)^2}{4} \exp\Bigg(-\frac{\sigma}{\sqrt{2}}(l_2(c_{123}+c) + (k_1-l_1)(c_{123}+c_{23}) + (l_1-l_2)(c_{123}+c_3) \\ &\quad + (k_2-l_1)(c_{23}+c_3) + (k_3-l_2)(c_3+c))\Bigg) \\ &\quad \times \left(1 + \mathcal{O}\left(M\left(\frac{|t|}{n} + \frac{|s_1|}{m_1} + \frac{|s_2|}{m_2} + \frac{|s_3|}{m_3}\right)\right) + \mathcal{O}\left(\sqrt{\frac{|t|}{n} + \frac{|s_1|}{m_1} + \frac{|s_2|}{m_2} + \frac{|s_3|}{m_3}}\right)\right) \end{aligned}$$

for $l_1 > l_2$, respectively, where $M = \max(k_1,k_2,k_3)$. The difference of the exponents is

$$(l_1 - l_2)(c_{123} + c - c_{23} - c - c_{123} - c_3 + c_{23} + c_3) = 0$$

and thus (3.4) also holds for the case $l_1 > l_2$, that means that the local representation of $B_{k_1l_1k_2l_2k_3}(z,u_1,u_2,u_3)$ is invariant with respect to the shape of T_0. Obviously the above considerations also hold in the case of p leaves. We have

Lemma 3.3 (Invariance property)**.** *Let $B_{k_1l_1\dots k_{p-1}l_{p-1}k_p}(z,u_1\dots,u_p)$ be the GF of $(K_1, L_1,\dots,K_{p-1},L_{p-1},K_p)$ and $\mathcal{B}_p$ denote the set of all binary trees with p leaves. Assume that $T_0 \in \mathcal{B}_p$ and that the quantities $k_i^2, m_i, i = 1,\dots,p$, and $l_j^2, j = 1,\dots,p-1$, are asymptotically proportional to n. Then for $\|z - z_0, u_1 - 1,\dots,u_p - 1\|_{\max} = o(\sqrt{n})$ there exists a local asymptotic representation of $B_{k_1l_1\dots k_{p-1}l_{p-1}k_p}(z,u_1\dots,u_p)$ that holds for all $T_0 \in \mathcal{B}_p$.*

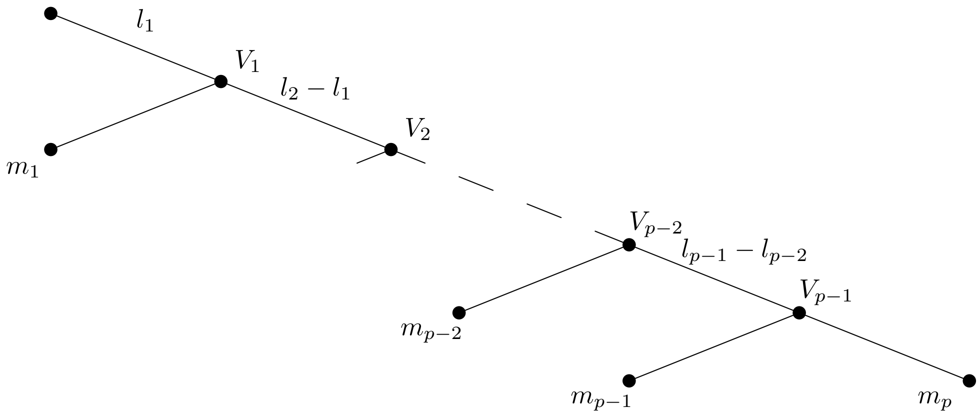

FIGURE 1. The shape of T_0

3.3. **The finite dimensional distributions.** Due to the above stated lemma it suffices to consider only the one special shape of T_0 which is most convenient to work with. Thus we choose the one that satisfies $l_1 < l_2 < \cdots < l_p$ (see figure 1) in order to get rid of the usually unpleasant terms $\min(l_i, l_j)$ and $\max(l_i, l_j)$ occurring in the GFs. This leads to the GF

$$\begin{aligned} B(z, u_1, \ldots, u_p) = \varphi_0^p z^p u_1 u_2^2 \cdots u_p^p \prod_{i=1}^{p-1} \big[&\phi_1(z, u_i \cdots u_p, u_{i+1} \cdots u_p)^{k_i - l_i - 1} \\ &\times \phi_1(z, u_i \cdots u_p, 1)^{l_i - l_{i-1} - 1} \phi_2(z, u_i \cdots u_p, u_{i+1} \cdots u_p) \big] \\ &\times \phi_1(z, u_p, 1)^{k_p - l_{p-1} - 1}, \end{aligned}$$

where we define $l_0 = -1$. Let $z = z_0 \left(1 + \frac{t}{n}\right)$ and $u_i = 1 + \frac{s_i}{m_i}$ be chosen in such a way that the assumptions of lemma 3.2 hold. Then we get for $k_i = \kappa_i \sqrt{n}$ and $l_i = \lambda_i \sqrt{n}$ the local representation

$$\begin{aligned} B(z, u_1, \ldots, u_p) &= \varphi_0^p z_0^p \left(\frac{z_0 \varphi''(\tau)}{2} \right)^{p-1} \exp\left(-\frac{\sigma}{\sqrt{2}} \left(\sum_{i=1}^{p-1} ((k_i - l_i)(c_{i \cdots p} + c_{i+1, \cdots, p}) \right.\right. \\ &\qquad \left.\left. + (l_i - l_{i-1})(c_{i \cdots p} + c)) + (k_p - l_{p-1})(c_p + c) \right) \right) \\ &\qquad \times \left(1 + \mathcal{O}\left(M_p \left(\frac{|t|}{n} + \sum_{i=1}^{p} \frac{|s_i|}{m_i} \right) \right) + \mathcal{O}\left(\sqrt{\frac{|t|}{n} + \sum_{i=1}^{p} \frac{|s_i|}{m_i}} \right) \right) \\ &= \varphi_0^p z_0^p \left(\frac{z_0 \varphi''(\tau)}{2} \right)^{p-1} \exp\left(-\frac{\sigma}{\sqrt{2}} \left(c_{1 \cdots p} k_1 + \sum_{i=1}^{p-1} c_{i+1, \ldots, p} w_i + c k_p \right) \right) \\ &\qquad \times \left(1 + \mathcal{O}\left(M_p \left(\frac{|t|}{n} + \sum_{i=1}^{p} \frac{|s_i|}{m_i} \right) \right) \right), \end{aligned} \tag{3.5}$$

where we have used the substitution $w_i = k_i + k_{i+1} - 2l_i$ and $M_p = \max_{1 \le i \le p} k_i$. By means of this formula we are able to prove

Theorem 3.1. *Let* $\varepsilon > 0$ *and* $w_i = k_i + k_{i+1} - 2l_i$. *Then we have uniformly for* $\frac{m_1}{n} \ge \varepsilon$, $\frac{m_{j+1} - m_j}{n} \ge \varepsilon$, $j = 1, \ldots, p-1$, $\frac{\varphi_0}{\varphi(\tau)} - \frac{m_p}{n} \le \varepsilon$ *and* $k_i = \mathcal{O}(\sqrt{n})$, $i = 1, \ldots, p$, $w_j = \mathcal{O}(\sqrt{n})$,

$j = 1, \ldots, p-1$,

$$[z^n u_1^{m_1} \cdots u_p^{m_p}]B = C_p' k_1 w_1 \cdots w_{p-1} k_p \times \left[m_1(m_2 - m_1) \cdots (m_p - m_{p-1}) \left(n - \frac{\varphi(\tau)}{\varphi_0} m_p \right) \right]^{-3/2} \times \exp\left(-\frac{\sigma^2}{8} \frac{\varphi_0}{\varphi(\tau)} \left(\frac{k_1^2}{m_1} + \sum_{i=2}^{p} \frac{w_{i-1}^2}{m_i - m_{i-1}} + \frac{k_p^2}{\frac{\varphi_0}{\varphi(\tau)} n - m_p} \right) \right) \times \left(1 + \mathcal{O}\left(\frac{1}{\sqrt{n}} \right) \right), \quad n \to \infty, \tag{3.6}$$

where

$$C_p' = \varphi_0^p z_0^{p-n} \left(\frac{\sigma}{\sqrt{2}} \right)^{p+1} \left(\frac{z_0 \varphi''(\tau)}{2} \right)^{p-1} \frac{1}{(2\sqrt{\pi})^{p+1}} \left(\frac{\varphi_0}{\varphi(\tau)} \right)^{p/2} = z_0^{-n} \frac{\tau}{2^{p+1}} \left(\frac{\sigma}{\sqrt{2}} \right)^{3p-1} \pi^{-(p+1)/2} \left(\frac{\varphi_0}{\varphi(\tau)} \right)^{3p/2}.$$

Dividing (3.6) by a_n yields the following local limit theorem:

Corollary . *Let $k_j = \kappa_j \sqrt{n} + o(\sqrt{n}) \in \mathbf{N}, j = 1, \ldots, p$ and $w_j = k_i + k_{i+1} - 2l_i = \omega_j \sqrt{n} + o(\sqrt{n}) \in \mathbf{N}, j = 1, \ldots, p-1$, satisfying $|\kappa_{j+1} - \kappa_j| \le \omega_j \le \kappa_{j+1} + \kappa_j$. Moreover assume $\frac{\varphi(\tau)}{\varphi_0} m_j = \mu_j n + o(n)$, where $0 < \mu_1 < \cdots < \mu_p < 1$ and let W_i denote the random variable $K_i + K_{i+1} - 2L_i$. Then the density*

$$P\{K_1 = k_1, W_1 = w_1, \ldots, K_{p-1} = k_{p-1}, W_{p-1} = w_{p-1}, K_p = k_p\}$$

of the random variable $(K_1, W_1, \ldots, K_{p-1}, W_{p-1}, K_p)$ admits the following asymptotic expansion:

$$n^{(2p-1)/2} \frac{b_{k_1 m_1 l_1 \cdots k_{p-1} m_{p-1} l_{p-1} k_p m_p n}}{a_n} = \frac{1}{(2\sqrt{\pi})^{p+1}} \left(\frac{\sigma}{\sqrt{2}} \right)^{3p} \kappa_1 \omega_1 \cdots \omega_{p-1} \kappa_p \times [\mu_1(\mu_2 - \mu_1) \cdots (\mu_p - \mu_{p-1})(1 - \mu_p)]^{-3/2} \times \exp\left(-\frac{\sigma^2}{8} \left(\frac{\kappa_1^2}{\mu_1} + \sum_{j=2}^{p} \frac{\omega_{j-1}^2}{\mu_j - \mu_{j-1}} + \frac{\kappa_p^2}{1 - \mu_p} \right) \right) + o(1) \tag{3.7}$$

for $n \to \infty$. The error term is uniform in ω_i, $i = 1, \ldots, p-1$, and for $\kappa_j \in [a_j, b_j]$, $b_j > a_j > 0$ and $\kappa_{j+1} - \kappa_j > \varepsilon > 0$, $j = 1, \ldots, p$.

Now the finite dimensional distribution of the contour process, i.e. the distribution of $(K_1, \ldots, K_p)$, can be calculated. Due to uniformity of the error term it suffices to determine the marginal density in $(\kappa_1, \ldots, \kappa_p)$ of (3.7). Doing this we obtain a multivariate Maxwell distribution which actually coincides with that of Brownian excursion. Thus the following theorem holds:

Theorem 3.2. *Let $\pi_{t_1, \cdots, t_k}$ be the projection defined by*

$$\pi_{t_1,\cdots,t_k} \colon C[0,1] \to \mathbf{R}^k, \qquad x(t) \mapsto (x(t_1), \ldots, x(t_k))$$

Then the following limit theorem holds:

$$\pi_{t_1,\ldots,t_k}\left(\hat{X}_n \left(\frac{\varphi_0}{\varphi(\tau)} t \right) \right) \xrightarrow{d} \pi_{t_1,\ldots,t_k}\left(\frac{2}{\sigma} W^+(t) \right)$$

Remark . Note that theorem 3.1 and its corollary only provide the distributions at the vertices of the polygon $\hat{X}_n(t)$. Thus they imply a slightly different form of the above limit theorem: We have to substitute $\hat{X}_n(t)$ by the corresponding step function process $\hat{X}_n(\lfloor t \rfloor / n)$. However, by means of the proof of tightness (see section **??**) we are able to prove the theorem as we stated it (see end of section 3).

3.4. **Proof of theorem 3.1: Determination of the main term.** In order to prove theorem 3.1 we use Cauchy's integral formula

$$[z^n u_1^{k_1} \cdots u_p^{k_p}] B(z, u_1, \dots, u_p) = \frac{1}{(2\pi i)^{p+1}} \int_{\Gamma_1} \cdots \int_{\Gamma_p} \int_{\Gamma_0} \frac{B(z, u_1, \dots, u_p)}{z^{n+1} u_1^{m_1+1} \cdots u_p^{m_p+1}} \, dz \, du_p \cdots du_1. \tag{3.8}$$

with the following integration contour: Let z run through the contour $\Gamma_0 = \Gamma_{01} \cup \Gamma_{02} \cup \Gamma_{03} \cup \Gamma_{04}$ defined by

$$\begin{aligned}
\Gamma_{01} &= \left\{ z = z_0 \left(1 + \frac{t}{n}\right) \middle| \Re t \le 0 \text{ und } |t| = 1 \right\} \\
\Gamma_{02} &= \left\{ z = z_0 \left(1 + \frac{t}{n}\right) \middle| \Im t = 1 \text{ und } 0 \le \Re t \le \log^2 n \right\} \\
\Gamma_{03} &= \overline{\Gamma}_{02} \\
\Gamma_{04} &= \left\{ z \middle| |z| = z_0 \left| 1 + \frac{\log^2 n + i}{n} \right| \text{ und } \arg\left(1 + \frac{\log^2 n + i}{n}\right) \le |\arg(z)| \le \pi \right\}.
\end{aligned}$$

Note that while z is running through Γ_0 the location of the singularity also changes. This fact has to be taken care of when choosing the integration contour for the other variables. The location of the singularity is determined asymptotically by the equations

$$\begin{aligned}
\frac{t}{n} &= 0 \\
\frac{\varphi_0}{\varphi(\tau)} \frac{s_p}{m_p} + \frac{t}{n} &= 0 \\
\frac{\varphi_0}{\varphi(\tau)} \left(\frac{s_{p-1}}{m_{p-1}} + \frac{s_p}{m_p} \right) + \frac{t}{n} &= 0 \\
\vdots \quad &\vdots \\
\frac{\varphi_0}{\varphi(\tau)} \left(\frac{s_1}{m_1} + \cdots + \frac{s_{p-1}}{m_{p-1}} + \frac{s_p}{m_p} \right) + \frac{t}{n} &= 0
\end{aligned}$$

as one can easily see by looking at (3.5). Thus as the integration contour of $u_1,\dots,u_p$ we may choose $\Gamma_j=\Gamma_{j1}\cup\Gamma_{j2}\cup\Gamma_{j3}\cup\Gamma_{j4}$ defined by

$$\Gamma_{j1}=\left\{u_j=\left(1+\frac{s_j}{m_j}\right)\middle|\,\Re s_j\le -R_j(s_{j+1},\cdots,s_p,t)\text{ and }\right.$$
$$\left.|s_j+R_j(s_{j+1},\cdots,s_p,t)+I_j(s_{j+1},\cdots,s_j,t)i|=1\right\}$$
$$\Gamma_{j2}=\left\{u_j=\left(1+\frac{s_j}{m_j}\right)\middle|\,\Im s_j=-I_j(s_{j+1},\cdots,s_p,t)+1,\right.$$
$$\left.-R_j(s_{j+1},\cdots,s_p,t)\le\Re s_j\text{ and }|u_j|\le\left|1+\frac{\log^2 m_j+i}{m_j}\right|\right\}$$
$$\Gamma_{j3}=\left\{u_j=\left(1+\frac{s_j}{m_j}\right)\middle|\,\Im s_j=-I_j(s_{j+1},\cdots,s_p,t)-1,\right.$$
$$\left.-R_j(s_{j+1},\cdots,s_p,t)\le\Re s_j\text{ and }|u_j|\le\left|1+\frac{\log^2 m_j+i}{m_j}\right|\right\}$$
$$\Gamma_{j4}=\left\{u_j\,\middle|\,|u_j|=\left|1+\frac{\log^2 m_j+i}{m_j}\right|\text{ and }\arg u_j\in[-\pi,\arg z_{j3}]\cup[\arg z_{j2},\pi]\right\},$$

where

$$R_j(s_{j+1},\cdots,s_p,t)=\begin{cases}\max\left(0,\dfrac{\varphi(\tau)}{\varphi_0}\dfrac{m_p}{n}\Re t\right) & \text{if } j=p\\ \max\left(0,\Re\left(\dfrac{\varphi(\tau)}{\varphi_0}\dfrac{tm_j}{n}+\dfrac{s_{j+1}m_j}{m_{j+1}}+\cdots+\dfrac{s_pm_j}{m_p}\right)\right) & \text{else,}\end{cases}$$

$$I_j(s_{j+1},\cdots,s_p,t)=\begin{cases}\max\left(n^{2/3},\dfrac{\varphi(\tau)}{\varphi_0}\dfrac{m_p}{n}\Im t\right) & \text{if } j=p\\ \max\left(n^{2/3},\Im\left(\dfrac{\varphi(\tau)}{\varphi_0}\dfrac{tm_j}{n}+\dfrac{s_{j+1}m_j}{m_{j+1}}+\cdots+\dfrac{s_pm_j}{m_p}\right)\right) & \text{else.}\end{cases}$$

and z_{jk} denotes the point of Γ_{jk} with maximal absolute value.

Remark . The functions R_j and I_j guarantee that the Hankel-like contours[1] $\Gamma_j'=\Gamma_{j1}\cup\Gamma_{j2}\cup\Gamma_{j3}$ follow the movement of the singularity while $z,u_{j+1},\dots,u_p$ are varying. It can be shown that for these variables moving away from the Hankel contour along $\Gamma_{.4}$ the singularity drifts out of the circle determined by Γ_{j4} and reaches a point x with $|x|=1+Cn^{-1/3}$ when one of the variables $z,u_{j+1},\dots,u_p$ arrives at distance $n^{-1/3}$ from the Hankel contour. Thus the term $n^{2/3}$ in the definition of I_j is justified.

Let us now consider the contribution of the Hankel integrals which yields the main term as we will show in the next section. If we apply the substitutions $z=z_0\left(1+\frac{t}{n}\right)$, $u_j=1+\frac{s_j}{m_j}$ to (3.8) and use the asymptotic expansion (3.5), then we get

$$\frac{C_p}{(2\pi i)^{p+1}}\int\limits_{\Gamma_0'}\int\limits_{\Gamma_1'}\cdots\int\limits_{\Gamma_p'}\exp\left(-\frac{\sigma}{\sqrt{2}}\left(c_{1\cdots p}k_1+\sum_{i=1}^{p-1}c_{i+1,\dots,p}w_i+ck_p\right)\right.$$
$$\left.-t-s_1-\cdots-s_p\right)\frac{ds_p}{m_p}\cdots\frac{ds_1}{m_1}\frac{dt}{n}\left(1+\mathcal{O}\left(M_p\left(\frac{1}{n}+\sum_{j=1}^{p}\frac{1}{m_j}\right)\right)\right),$$

[1] According to Hankel's representation of the Gamma function we will refer to the integration contour starting at $e^{2\pi i}\infty$, passing the origin clockwise and returning to $+\infty$ as Hankel contour. Similarly we will use the attribute Hankel for all related concepts like Hankel integral,. . .

where

$$C_p = \varphi_0^p z_0^{p-n} \left(\frac{z_0 \varphi''(\tau)}{2} \right)^{p-1}.$$

The shape of this integral suggests the substitution

$$\begin{pmatrix} \frac{\varphi_0}{\varphi(\tau) m_1} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & & \frac{\varphi_0}{\varphi(\tau) m_p} & 0 \\ 0 & & 0 & \frac{1}{n} \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_p \\ t \end{pmatrix} = \begin{pmatrix} \frac{\varphi_0}{\varphi(\tau) m_1} & \frac{\varphi_0}{\varphi(\tau) m_2} & \cdots & \frac{\varphi_0}{\varphi(\tau) m_p} & \frac{1}{n} \\ 0 & \frac{\varphi_0}{\varphi(\tau) m_2} & \cdots & \frac{\varphi_0}{\varphi(\tau) m_p} & \frac{1}{n} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \frac{\varphi_0}{\varphi(\tau) m_p} & \frac{1}{n} \\ 0 & \cdots & \cdots & 0 & \frac{1}{n} \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_p \\ t \end{pmatrix}$$

which finally leads to

$$\frac{C_p}{m_1 \cdots m_p n} \prod_{j=1}^{p} \int_{\gamma_j} e^{-\alpha_j \sqrt{-v_j} - \beta_j v_j} \, dv_j \int_{\gamma_0} e^{-\alpha_{p+1} \sqrt{-t} - \beta_{p+1} t} \, dt \left(1 + \mathcal{O}\left(\frac{1}{\sqrt{n}} \right) \right) \tag{3.9}$$

where

$$\begin{aligned} \alpha_1 &= \frac{\sigma k_1}{\sqrt{2 m_1}} \sqrt{\frac{\varphi_0}{\varphi(\tau)}}, & \beta_1 &= 1, \\ \alpha_j &= \frac{\sigma w_{j-1}}{\sqrt{2 m_j}} \sqrt{\frac{\varphi_0}{\varphi(\tau)}}, \quad j = 2, \dots, p, & \beta_j &= 1 - \frac{m_{j-1}}{m_j}, \quad j = 2, \dots, p, \\ \alpha_{p+1} &= \frac{\sigma k_p}{\sqrt{2n}}, & \beta_{p+1} &= 1 - \frac{m_p}{n} \frac{\varphi(\tau)}{\varphi_0}. \end{aligned}$$

and γ_j are Hankel contours meeting the constraint

$$\Re t \le \log^2 n, \quad \text{and} \quad \Re v_j \le \log^2 m_j, \ j = 1, \dots, p.$$

Lemma 3.4. *Let γ be a Hankel contour truncated at K. Then we have for $\alpha, \ \beta > 0$*

$$\frac{1}{2\pi i} \int_\gamma e^{-\alpha\sqrt{-t} - \beta t} \, dt = \frac{\alpha \beta^{-\frac{3}{2}}}{2\sqrt{\pi}} \exp\left(-\frac{\alpha^2}{4\beta} \right) + \mathcal{O}\left(\frac{1}{\beta} e^{-K\beta} \right). \tag{3.10}$$

Proof. Substitute $t = u^2$ and $\sqrt{\beta} u - \frac{i\alpha}{2\sqrt{\beta}} = v$. Then we get

$$\frac{\alpha \beta^{-\frac{3}{2}}}{2\pi} \exp\left(-\frac{\alpha^2}{4\beta} \right) \int_{-\infty + i\alpha/2\beta}^{\infty + i\alpha/2\beta} e^{-v^2} \, dv$$

and this immediately implies (**??**). ☐

Applying this lemma to (**??**) yields the main term of (3.6).

3.5. **The remainder integrals.** In this section we have to show that those parts of the Cauchy integral (3.8) where z or at least one of the u_j lie in Γ_{04} or Γ_{j4}, respectively, are asymptotically negligible. Therefore let I_p denote the integral (**??**) and R_p the remaining integral. Obviously we have

$$I_p = \mathcal{O}\left(z_0^{-n} n^{-p-1} \right), \ n \to \infty. \tag{3.11}$$

In order to estimate R_p observe that for $z \in \Gamma_{04}$ and $u_j \in \Gamma_{j4}$, respectively, the relations

$$|z^{-n-1}| = \mathcal{O}\left(z_0^{-n-1} e^{-\log^2 n} \right) \text{ and } |u_j^{-m_j - 1}| = \mathcal{O}\left(e^{-\log^2 m_j} \right),$$

hold. $B(z, u_1, \dots, u_p)$ is composed of $\phi_1(z, u, v)$ and $\phi_2(z, u, v, w)$. As both functions are analytic inside the integration domain (and thus bounded there) and moreover the latter one only appears to the first power, it suffices to study the behaviour of $\phi_1(z, u, v)$. Inside the domain

$\max(|z-z_0|,|u-1|,|v-1|)\le\varepsilon$, $\varepsilon>0$ sufficiently small, we may use the local representation (3.3) provided that ε is sufficiently small. Let $z=1+\frac{t}{n}$ and consider the expression

$$A=1-\frac{\sigma}{\sqrt{2}}\sqrt{-\frac{t}{n}}$$

for $t\in\Gamma_0$ and $z_0|\frac{t}{n}|\le\varepsilon$. If $t\in\Gamma_{01}$, then

$$-t=e^{i\psi},\quad \psi\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$

and immediately we get $|A|\le 1$. Let $t\in\Gamma_{02}\cup\Gamma_{03}$, that means $t=r\pm i$, where $0\le r\le\log^2 n$. Then

$$\sqrt{-\frac{t}{n}}=\frac{(1+r^2)^{1/4}}{\sqrt{n}}\exp\left(i\left(\frac{\pi}{2}-\frac{1}{2}\arctan\frac{1}{r}\right)\right)$$

and that implies

$$|A|^2=\begin{cases}1-\dfrac{\sigma}{\sqrt{n}}+\mathcal{O}\left(\dfrac{r}{\sqrt{n}}\right)+\mathcal{O}\left(\dfrac{1}{n}\right) & \text{for small } r\\ 1-\dfrac{\sigma}{\sqrt{2rn}}+\mathcal{O}\left(\dfrac{1}{\sqrt{r^5 n}}\right)+\mathcal{O}\left(\dfrac{\log^2 n}{n}\right) & \text{for large } r\end{cases}$$

It remains to investigate the case $z\in\Gamma_{04}$. In this case we have $\frac{z}{z_0}=ae^{i\psi/n}$, where

$$a=\left|1+\frac{\log^2 n+i}{n}\right|$$

and $\psi\le\varepsilon n$. An easy calculation shows

$$\sqrt{-\frac{t}{n}}\sim\sqrt{-\frac{\log^2 n}{n}-ia\frac{\psi}{n}}$$

and using this we immediately obtain $|A|\le 1$.

Obviously the above considerations are also valid, if we use

$$\sqrt{-\frac{\varphi_0}{\varphi(\tau)}\left(\frac{s_j}{m_j}+\cdots+\frac{s_p}{m_p}\right)-\frac{t}{n}}$$

or sums of terms of this form instead of $\sqrt{-\frac{t}{n}}$. Thus we have for $\max(|z-z_0|,|u-1|,|v-1|)\le\varepsilon$ the inequality

$$|\phi_1(z,u,v)|\le 1$$

which implies

$$R_p=\mathcal{O}\left(z_0^{-n}e^{-C\log^2 n}\right) \tag{3.12}$$

for a suitable constant C.

Now let (z,u,v) be outside the region where the local expansion of $\phi_1(z,u,v)$ is valid. Set $z=z_0\left(1+\frac{t}{n}\right)$, $u=1+\frac{s}{m}$ and $v=1+\frac{r}{l}$, l,m proportional to n and, for example, $\left|\frac{r}{l}\right|>\varepsilon$. Then $\phi_1(z,u,v)$ is analytic for $|u|\le\left|1+cm^{-1/3}\right|$ and $|z|\le z_0\left|1+c'n^{-1/3}\right|$. Thus it is bounded and as the exponents k_i and l_i are bounded by $\sqrt{n}$ we have

$$|B(z,u_1,\ldots,u_p)|=\mathcal{O}\left(e^{\sqrt{n}}\right).$$

On the other hand we may choose the circles $|u|=\left|1+cm^{-1/3}\right|$ and $|z|=z_0\left|1+c'n^{-1/3}\right|$ as integration contours for u and z. Thus we get finally

$$R_p=\mathcal{O}\left(z_0^{-n}\exp\left(\sqrt{n}-n^{2/3}\right)\right) \tag{3.13}$$

Finally, equations (**??**)–(**??**) imply that the remainder integrals are exponentially small and therefore negligible which completes the proof of theorem 3.1.

3.6. **Tightness.** In order to complete the proof of theorem 1.1 we have to prove that the process $\hat{X}_n(t)$ is tight. This can be done by employing theorem 12.3 of [**?**]: The first condition is trivial, as $P\{\hat{X}_n(0) = 0\} = 1$. Furthermore it can be shown that for polygonal functions like $\hat{X}_n(t)$ it suffices to establish the second condition of this theorem only for the vertices of the polygon (use the ideas of [**?**, p.86]), i.e. we have to prove that

$$P\left\{\left|\hat{X}_n\left(\frac{i}{n}\right) - \hat{X}_n\left(\frac{j}{n}\right)\right| \geq \varepsilon\right\} \leq \frac{K}{\varepsilon^\beta}\left|\frac{i-j}{n}\right|^\alpha,$$

where $K > 0$, $\beta \geq \alpha > 1$, holds for all $n \geq 1, 0 \leq i, j \leq n, \varepsilon > 0$. Therefore we have to set up the GF corresponding to the bivariate distributions of $\hat{X}_n(t)$:

$$\begin{aligned}
B_{k_1k_2}(z, u_1, u_2) &= \sum_{l=0}^{\min(k_1,k_2)-1} B_{k_1lk_2}(z, u_1, u_2) \\
&= \varphi_0^p z^2 u_1 u_2 \phi_2(z, u_1u_2, u_2) \sum_{l=0}^{\min(k_1,k_2)-1} \phi_1(z, u_1u_2, 1)^l \\
&\quad \times \phi_1(z, u_1u_2, u_2)^{k_1-1-l}\phi_1(z, u_2, 1)^{k_2-1-l} \\
&= \varphi_0^p z^2 u_1 u_2 \phi_2(z, u_1u_2, u_2)\phi_1(z, u_1u_2, u_2)^{k_1-1} \\
&\quad \times \phi_1(z, u_2, 1)^{k_2-1}\frac{1 - q(z, u_1u_2, u_2)^{\min(k_1,k_2)}}{1 - q(z, u_1u_2, u_2)},
\end{aligned}$$

where

$$q(z, u, v) = \frac{\phi_1(z, u, 1)}{\phi_1(z, u, v)\phi_1(z, v, 1)}.$$

Then

$$P\left\{\left|\hat{X}_n\left(\frac{\lfloor\mu_1 n\rfloor}{n}\right) - \hat{X}_n\left(\frac{\lfloor\mu_2 n\rfloor}{n}\right)\right| \geq \varepsilon\right\} = \frac{1}{a_n}[z^n u_1^{m_1} u_2^{m_2}] \sum_{\substack{k,l\geq 1\\ |k-l|\geq\lfloor\varepsilon\sqrt{n}\rfloor}} B_{kl}(z, u_1, u_2).$$

Therefore we have to get estimates for the expression

$$\frac{1}{1-q}\sum_{\substack{k,l\geq 0\\ |k-l|\geq\lfloor\varepsilon\sqrt{n}\rfloor}} x^k y^l - \frac{1}{1-q}\sum_{\substack{k,l\geq 0\\ |k-l|\geq\lfloor\varepsilon\sqrt{n}\rfloor}} x^k y^l q^{\min(k,l)+1}$$

where we used the abbreviations

$$x = \phi_1(z, u_1u_2, u_2), \quad y = \phi_1(z, u_2, 1), \quad xyq = \phi_1(z, u_1u_2, 1).$$

Splitting this sum yields

$$\begin{aligned}
S_1 &= \sum_{k\geq\lfloor\varepsilon\sqrt{n}\rfloor}\sum_{l<k-\lfloor\varepsilon\sqrt{n}\rfloor} x^k y^l &&= \frac{x^{1+\lfloor\varepsilon\sqrt{n}\rfloor}}{(1-x)(1-xy)} \\
S_2 &= \sum_{k\geq 0}\sum_{l\geq k+\lfloor\varepsilon\sqrt{n}\rfloor} x^k y^l &&= \frac{y^{\lfloor\varepsilon\sqrt{n}\rfloor}}{(1-y)(1-xy)} \\
S_3 &= q\sum_{k\geq\lfloor\varepsilon\sqrt{n}\rfloor}\sum_{l<k-\lfloor\varepsilon\sqrt{n}\rfloor} x^k (qy)^l &&= \frac{qx^{1+\lfloor\varepsilon\sqrt{n}\rfloor}}{(1-x)(1-xyq)} \\
S_4 &= q\sum_{k\geq 0}\sum_{l\geq k+\lfloor\varepsilon\sqrt{n}\rfloor} (xq)^k y^l &&= \frac{qy^{\lfloor\varepsilon\sqrt{n}\rfloor}}{(1-y)(1-xyq)}.
\end{aligned}$$