# Singularities for Systems of Functional Equations

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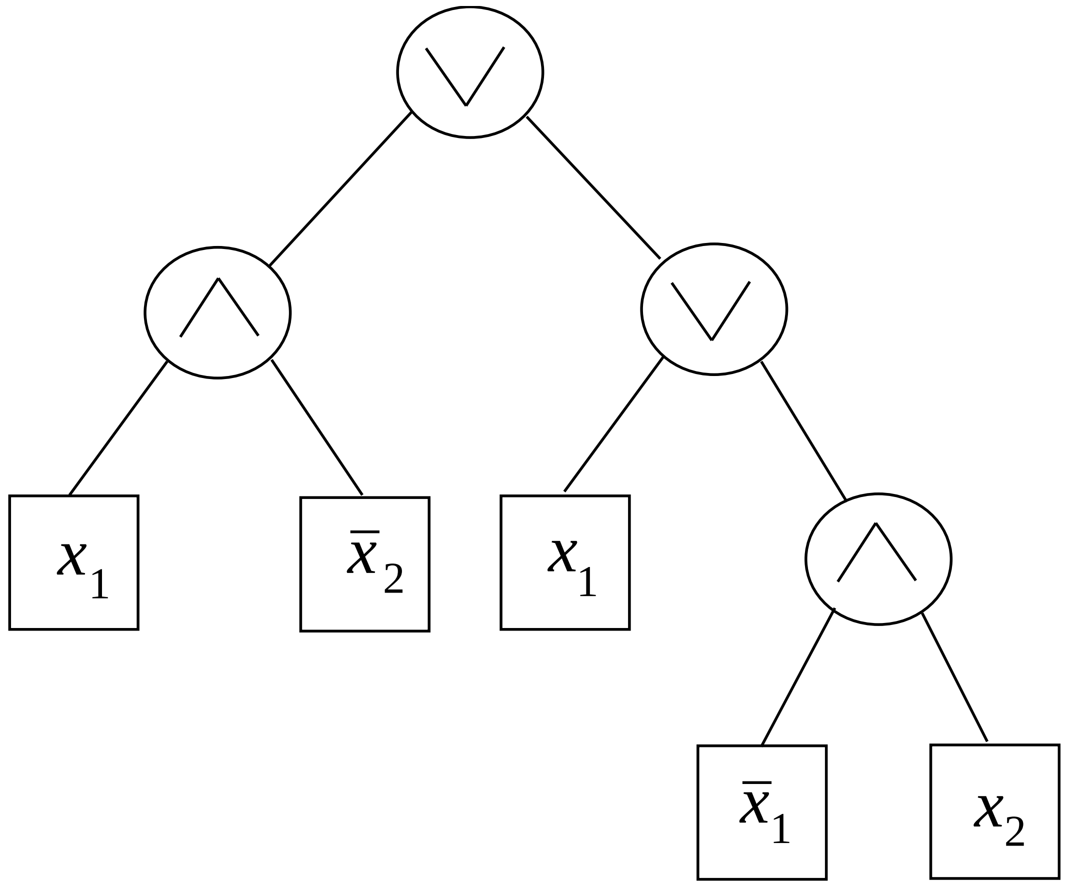
# Boolean Functions and AND/OR-Trees

**Boolean formula.** $F = (x_1 \wedge \overline{x}_2) \vee (x_1 \vee (\overline{x}_1 \wedge x_2))$

**Boolean function.** $f = x_1 \vee x_2$



# Boolean Functions and AND/OR-Trees

$B_m$ ... **binary trees** with $m$ internal nodes: $b_m = |B_m| = \frac{1}{m}\binom{2m}{m}$

$T_m$ ... **Boolean AND/OR-formulas** with $n$ literals (+ their negations)

$$t_m = |T_m| = b_m 2^m (2n)^{m+1} = \frac{2n}{m+1}(4n)^m \binom{2m}{m}.$$

$f$ ... **Boolean function** in $x_1, \ldots, x_m$

$$P_m(f) = \frac{\#\{F \in T_m : F \text{ represents } f\}}{t_m}$$

$$P(f) = \lim_{m \to \infty} P_m(f)$$

# Binary Trees

**Generating Function.** $b(z) = \sum_{m \geq 0} b_m z^m$

$$\boxed{b(z) = 1 + z\, b(z)^2}$$

$$b(z) = \frac{1 - \sqrt{1-4z}}{2z} = 2 - 2\sqrt{1-4z} + \cdots$$

$$b_m = [z^m]b(z) = \frac{1}{m}\binom{2m}{m} \sim \frac{4^m}{\sqrt{\pi}\, m^{3/2}}$$

# Binary Trees

**Generating Function.** $t(z) = \sum_{m \geq 0} t_m z^m$

$$\boxed{t(z) = 2n + 2z\, t(z)^2}$$

$$t(z) = \frac{1 - \sqrt{1 - 16nz}}{4z} = 4n - 4n\sqrt{1 - 16nz} + \cdots$$

$$t_m = [z^m]t(z) = 2^m (2n)^{m+1} \frac{1}{m} \binom{2m}{m} \sim 2n \frac{(16n)^m}{\sqrt{\pi}\, m^{3/2}}$$

# Boolean Functions and AND/OR-Trees

**Lemma**

$f$ ... Boolean function in $x_1, \ldots, x_n$

$$t_m(f) = \#\{F \in T_m : F \text{ represents } f\}, \quad t_f(z) = \sum_{m \ge 0} t_m(f)\, z^n$$

Suppose that $t_f(z)$ has radius of convergence $1/(16n)$ and a local expansion in terms of $\sqrt{1-16nz}$ of the form

$$\boxed{t_f(z) = \alpha_f - \beta_f \sqrt{1-16nz} + \cdots}$$

[+ some technical conditions] then

$$\boxed{P(f) = \lim_{m\to\infty} \frac{t_m(f)}{t_m} = \frac{\beta_f}{4n}}.$$

# Boolean Functions and AND/OR-Trees

**Lemma** [Chauvin+Flajolet+Gardy+Gittenberger]

$f$ ... Boolean function in $x_1, \ldots, x_n$

$$\boxed{t_f(z) = 1_{[f \text{ literal}]} + z \sum_{g,h: g \vee h = f} t_g(z) t_h(z) + z \sum_{g,h: g \wedge h = f} t_g(z) t_h(z)}$$

**Remark.** This is a system of $2^{2^n}$ equations.

# Boolean Functions and AND/OR-Trees

**Example.** $n = 1$: $True$, $False$, $x_1$, $\overline{x}_1$

$$t_{True}(z) = 2zt_{x_1}t_{\overline{x}_1} + 2zt_{True}(z)t(z)$$

$$t_{True}(z) = 2zt_{x_1}t_{\overline{x}_1} + 2zt_{False}(z)t(z)$$

$$t_{x_1}(z) = 1 + 2zt_{x_1}(z)^2 + 2zt_{x_1}(z)t_{True}(z) + 2zt_{x_1}(z)t_{False}(z)$$

$$t_{\overline{x}_1}(z) = 1 + 2zt_{\overline{x}_1}(z)^2 + 2zt_{\overline{x}_1}(z)t_{True}(z) + 2zt_{\overline{x}_1}(z)t_{False}(z)$$

[$t(z)$ abbreviates $t(z) = t_{True}(z) + t_{False}(z) + t_{x_1}(z) + t_{\overline{x}_1}(z)$]

This system can be solved explicitly. For example, one gets:

$$\begin{aligned} t_{x_1}(z) &= -\frac{1}{8z}\left(1 + \sqrt{1-16z} - \sqrt{2 + 16z + 2\sqrt{1-16z}}\right) \\ &= 2(\sqrt{3}-1) + 2(1/\sqrt{3}-1)\sqrt{1-16z} + \cdots \end{aligned}$$

# Boolean Functions and AND/OR-Trees

**Example.** $n = 1$: $True$, $False$, $x_1$, $\overline{x}_1$

$$P(True) = P(False) = \frac{1}{2\sqrt{3}} = 0.28867513...$$

$$P(x_1) = P(\overline{x}_1) = \frac{\sqrt{3}-1}{2\sqrt{3}} = 0.21132486...$$

# A Single Functional Equation

**Theorem** (Bender, Canfield, Meir & Moon)

Suppose that $y(z)$ satisfies $\boxed{y(z) = \Phi(z, y(z))}$, where $\Phi(z, y)$ has a power series expansion at $(0, 0)$ with non-negative coefficients, $\Phi_{yy}(z, y) \neq 0$, and $\Phi_z(z, y) \neq 0$

Let $z_0 > 0$, $y_0 > 0$ (inside the region of convergence) satisfy the system of equations:

$$\boxed{y_0 = \Phi(z_0, y_0), \quad 1 = \Phi_y(z_0, y_0)}.$$

Then there exists analytic function $g(z) and h(z)$ such that locally

$$\boxed{y(z) = g(z) - h(z)\sqrt{1 - \frac{z}{z_0}}},$$

where $g(z_0) = y_0$ and $h(z_0) \neq 0$.

# A Single Functional Equation

**Example.** $y(z) = 1 + zy(z)^2$, $\boxed{\Phi(z,y) = 1 + zy^2}$.

$$y_0 = 1 + z_0 y_0^2, \quad 1 = 2 z_0 y_0$$

$$z_0 = \frac{1}{4}, \quad y_0 = 2$$

$$g(z_0) = 2, \quad h(z_0) = 2$$

$$\boxed{y(z) = 2 - 2\sqrt{1 - 4z} + \cdots}$$

# A Single Functional Equation

**The case $\Phi_{yy}(z,y) = 0$.**

$$y = \Phi(z,0) + \Phi_y(z,0)y$$

$$\boxed{y(z) = \frac{\Phi(z,0)}{1 - \Phi_y(z,0)}}$$

$$1 = \Phi_y(z_0,0) \quad \Longrightarrow \quad 1 - \Phi_y(z,0) = K(z)(1 - z/z_0)$$

$$\boxed{y(z) = \frac{\Phi(z,0)}{K(z)(1 - z/z_0)}}$$

$\Longrightarrow$ **Polar singularity**

# A Single Functional Equation

**Idea of the Proof.**

Set $F(z,y) = \Phi(z,y) - y$. Then we have

$$\begin{aligned} F(z_0, y_0) &= 0 \\ F_y(z_0, y_0) &= 0 \\ F_z(z_0, y_0) &\neq 0 \\ F_{yy}(z_0, y_0) &\neq 0. \end{aligned}$$

Weierstrass preparation theorem implies that there exist analytic functions $H(z,y)$, $p(z)$, $q(z)$ with $H(z_0,y_0) \neq 0$, $p(z_0) = q(z_0) = 0$ and

$$F(z,y) = H(z,y)\Big((y-y_0)^2 + p(z)(y-y_0) + q(z)\Big).$$

# A Single Functional Equation

$$F(z,y)=0 \quad \Longleftrightarrow \quad (y-y_0)^2+p(z)(y-y_0)+q(z)=0.$$

Consequently

$$\begin{aligned} y(z) &= y_0 - \frac{p(z)}{2} \pm \sqrt{\frac{p(z)^2}{4} - q(z)} \\ &= \boxed{g(z) - h(z)\sqrt{1-\frac{z}{z_0}}}, \end{aligned}$$

where we write

$$\frac{p(z)^2}{4} - q(z) = K(z)(z-z_0)$$

which is again granted by the Weierstrass preparation theorem and we set

$$g(z) = y_0 - \frac{p(z)}{2} \quad \text{and} \quad h(z) = \sqrt{-K(z)z_0}.$$

# A Single Functional Equation

**Variation of the Theorem.** $u$ denotes an **additional parameter**.

Suppose that $y(z;u)$ satisfies $\boxed{y(z;u) = \Phi(z, y(z;u); u)}$, where $\Phi(z, y; u)$ has a power series expansion at $(0, 0)$ with non-negative coefficients, $\Phi_{yy}(z, y; u) \neq 0$, and $\Phi_z(z, y; u) \neq 0$

Let $z_0(u) > 0$, $y_0(u) > 0$ (inside the region of convergence) satisfy the system of equations:

$$\boxed{y_0(u) = \Phi(z_0(u), y_0(u)), \quad 1 = \Phi_y(z_0(u) y_0(u))}.$$

Then there exists analytic function $g(z;u), h(z;u)$ such that locally

$$\boxed{y(z;u) = g(z;u) - h(z;u)\sqrt{1 - \frac{z}{z_0(u)}}},$$

where $g(z_0;u) = y_0(u)$ and $h(z_0(u)) \neq 0$.

# Systems of functional equations

**Positive System.**

Suppose, that several generating functions $y_1(z), \ldots, y_r(z)$ satisfy a **system of equations**

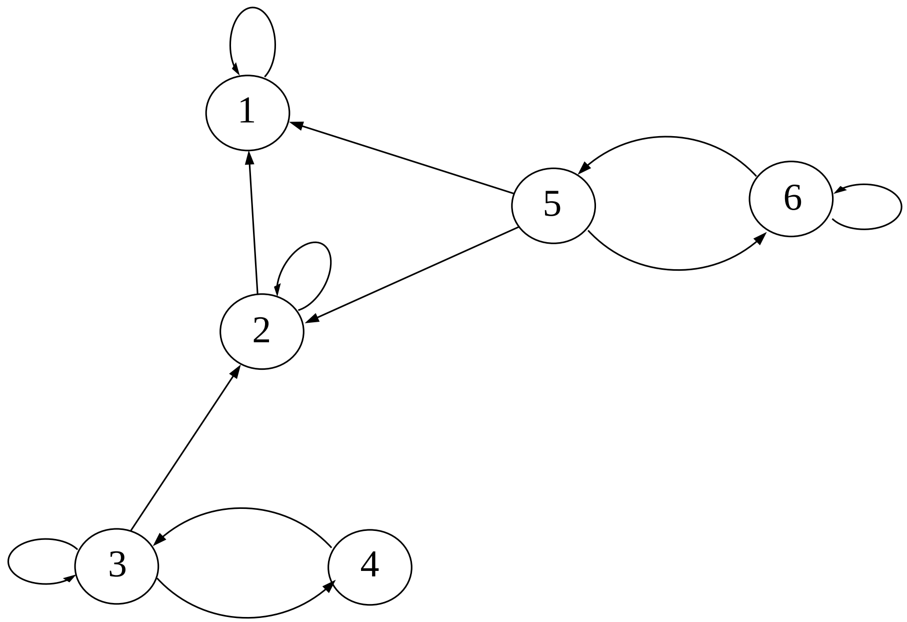
$$\boxed{y_j(z) = \Phi_j(z, y_1(z), \ldots, y_r(z))},$$

where $\Phi_j(z, y_1, \ldots, y_r)$ has as a power series expansion at $(0, 0)$ in $y_1, \ldots, y_r$. If these coefficients are **non-negative coefficients** (for all $j$) then we call it **positive system**.

# Systems of functional equations

## Dependency Graph.

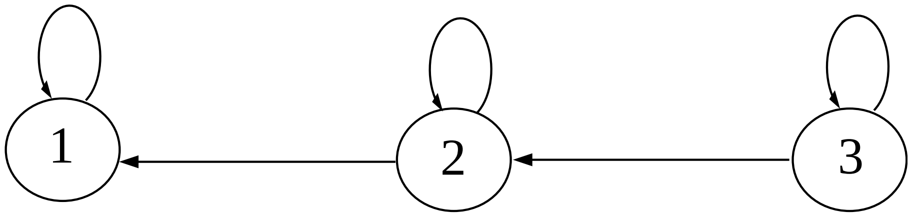
$$
\begin{aligned}
y_1 &= \Phi_1(z, y_1, y_2, y_5)\\
y_2 &= \Phi_2(z, y_2, y_3, y_5)\\
y_3 &= \Phi_3(z, y_3, y_4)\\
y_4 &= \Phi_4(z, y_3)\\
y_5 &= \Phi_5(z, y_6)\\
y_6 &= \Phi_6(z, y_5, y_6)
\end{aligned}
$$



# Systems of functional equations

## Example

$$y_1 = z(e^{y_2} + y_1)$$
$$y_2 = z(1 + 2y_2y_3)$$
$$y_3 = z(1 + y_3^2)$$




$$y_1(z) = \frac{z}{1-z}\exp\left(\frac{z}{\sqrt{1-4z^2}}\right)$$
$$y_2(z) = \frac{z}{\sqrt{1-4z^2}}$$
$$y_3(z) = \frac{1-\sqrt{1-4z^2}}{2z}$$

# Systems of functional equations

**Dependency graph:** $G_{\mathbf{\Phi}} = (V, E)$

$V$ ... vertex set $= \{y_1, y_2, \ldots, y_r\}$
$E$ ... (directed) edge set:

$$\begin{aligned} (y_i, y_j) \in E :&\Longleftrightarrow y_j(z) \text{ depends on } y_i(z) \\ &\Longleftrightarrow \Phi_j \text{ depends on } y_i \\ &\Longleftrightarrow \frac{\partial \Phi_j}{\partial y_i} \neq 0. \end{aligned}$$

**Stongly connected dependency graphs.**

$$\begin{aligned} G_{\mathbf{\Phi}} \text{ is strongly connected} &\Longleftrightarrow \mathbf{\Phi_y} := \left( \frac{\partial \Phi_j}{\partial y_i} \right) \text{ irreducible} \\ &\Longleftrightarrow \text{no subsystem can be solved} \\ &\qquad \text{before the whole system} \end{aligned}$$

# Systems of functional equations

A digraph $G$ is **strongly connected** if each pair of vertices $(v_1, v_2)$ is connected by a (directed) path.

A **positive** matrix $A = (a_{i,j})$, i.e. $a_{i,j} \geq 0$, is **irreducible** if for every pair of indiced $(i_1, i_2)$ there exists an integer $m$ such that $a_{i_1,i_2}^{(m)} > 0$, where $A^m = (a_{i,j}^{(m)})$.

A digraph $G$ is **strongly connected** if and only if its adjacency matrix $A(G)$ is **irreducible**.

# Systems of functional equations

**Perron-Frobenius theory.**

Every **positive irreducibe** matrix $A = (a_{i,j})$ has a real positive eigenvalue $r(A)$ with the property that all other eigenvalues have modulus $\le r(A)$. Furthermore, $r(A)$ is a **simple eigenvalue**.

If $B < A$, that is $b_{i,j} \le a_{i,j}$ for all pairs $(i, j)$ but $A \ne B$, then $\boxed{r(B) < r(A)}$.

In particular if $B$ is a **submatrix** of $A$, then we also have $\boxed{r(B) < r(A)}$.

# Systems of functional equations

**Theorem** [D., Lalley, Woods]

Suppose that $\mathbf{y} = \mathbf{\Phi}(z, \mathbf{y})$ is a **positive** and **non-linear** system. Suppose further, that the **dependency graph** of the system $\mathbf{y} = \mathbf{\Phi}(z, \mathbf{y})$ is **strongly connected**.

Let $z_0 > 0$, $\mathbf{y}_0 = (y_{0,0}, \ldots, y_{r,0}) > 0$ (inside the region of convergence) satisfy the system of equations: $(\mathbf{\Phi} = (\Phi_1, \ldots, \Phi_r))$

$$\boxed{\mathbf{y}_0 = \mathbf{\Phi}(z_0, \mathbf{y}_0), \quad 0 = \det(\mathbf{I} - \mathbf{\Phi}_{\mathbf{y}}(z_0, \mathbf{y}_0))}$$

such that all eigenvalues of $\mathbf{\Phi}_{\mathbf{y}}(z_0, \mathbf{y}_0)$ have modulus $\le 1$.

Then there exists analytic function $g_j(z), h_j(z) \ne 0$ such that locally

$$\boxed{y_j(z) = g_j(z) - h_j(z)\sqrt{1 - \frac{z}{z_0}}}.$$

# Systems of functional equations

**Example.**

$$t_{True}(z) = 2zt_{x_1}t_{\overline{x}_1} + 2zt_{True}(z)t(z)$$

$$t_{True}(z) = 2zt_{x_1}t_{\overline{x}_1} + 2zt_{False}(z)t(z)$$

$$t_{x_1}(z) = 1 + 2zt_{x_1}(z)^2 + 2zt_{x_1}(z)t_{True}(z) + 2zt_{x_1}(z)t_{False}(z)$$

$$t_{\overline{x}_1}(z) = 1 + 2zt_{\overline{x}_1}(z)^2 + 2zt_{\overline{x}_1}(z)t_{True}(z) + 2zt_{\overline{x}_1}(z)t_{False}(z)$$

[$t(z)$ abbreviates $t(z) = t_{True}(z) + t_{False}(z) + t_{x_1}(z)$]

# Systems of functional equations

**Linear Systems.**

$$\mathbf{y} = \mathbf{\Phi}(z, \mathbf{0}) + \mathbf{\Phi}_{\mathbf{y}}(z, \mathbf{0}) \cdot \mathbf{y}$$

$$\boxed{\mathbf{y}(z) = (\mathbf{I} - \mathbf{\Phi}_{\mathbf{y}}(z, \mathbf{0}))^{-1} \mathbf{\Phi}(z, \mathbf{0})}.$$

If $\mathbf{\Phi}_{\mathbf{y}}(z_0, \mathbf{0})$ is **irreducible**, has eigenvalue 1 and all other eigenvalues have modulus $\le 1$ then

$$\det(\mathbf{I} - \mathbf{\Phi}_{\mathbf{y}}(z, \mathbf{0})) = (1 - z/z_0) K(z)$$

and consequently all functions $y_j(z)$ have a **polar singularity** of order 1 at $z = z_0$.

**Conclusion.** In a positive irreducible system we have either a common polar singularity or a squareroot singularity.

# Systems of functional equations

**Idea of the proof** (reduction to a single equation)

$\mathbf{y} = (y_1, \ldots, y_r) = (y_1, \overline{\mathbf{y}})$, $\mathbf{\Phi} = (\Phi_1, \ldots, \Phi_r) = (\Phi_1, \overline{\mathbf{\Phi}})$

$$\mathbf{y} = \mathbf{\Phi}(\mathbf{y}, z) \quad \Longleftrightarrow \quad \begin{array}{ll} y_1 & = \Phi_1(y_1, \overline{\mathbf{y}}, z), \\ \overline{\mathbf{y}} & = \overline{\mathbf{\Phi}}(y_1, \overline{\mathbf{y}}, z) \end{array}$$

The second system has dominant eigenvalue $< 1$
$\Longrightarrow \overline{\mathbf{y}} = \overline{\mathbf{y}}(z, \boxed{y_1})$ is **analytic**

Insertion of this analytic solution into the first equation:

$$\boxed{y_1 = \Phi_1((y_1, \overline{\mathbf{y}}(z, y_1), z) = G(y_1, z)}$$

leads to **single equation**.

# Systems of functional equations

**Existence of limiting probabilities for Boolean functions.**
[Chauvin+Flajolet+Gardy+Gittenberger]

The system

$$t_f(z) = 1_{[f \text{ literal}]} + z \sum_{g,h: g \vee h = f} t_g(z) t_h(z) + z \sum_{g,h: g \wedge h = f} t_g(z) t_h(z)$$

has a strongly connected dependency graph. The common radius of convergence is $z_0 = 1/(16n)$. Consequently we have

$$t_f(z) = \alpha_f - \beta_f \sqrt{1 - 16nz} + \cdots$$

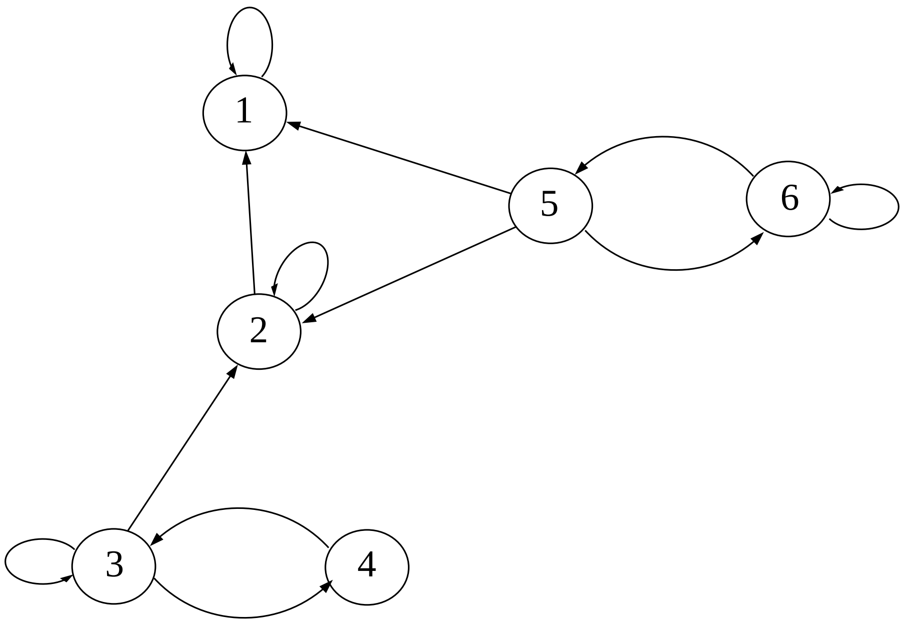
and the limiting probabilities $P(f) = \beta_f/(4n)$ exist.

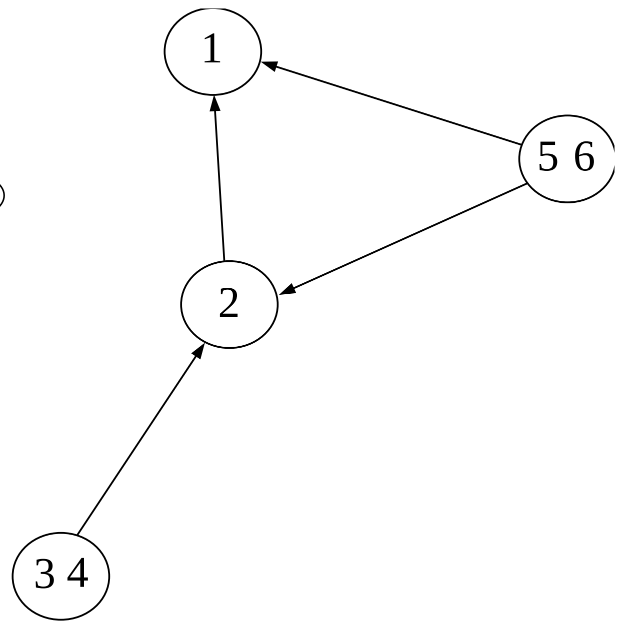
# General Dependency Graphs

## Dependency Graph and Reduced Dependency Graph

$$
\begin{aligned}
y_1 &= \Phi_1(z, y_1, y_2, y_5) \\
y_2 &= \Phi_2(z, y_2, y_3, y_5) \\
y_3 &= \Phi_3(z, y_3, y_4) \\
y_4 &= \Phi_4(z, y_3) \\
y_5 &= \Phi_5(z, y_6) \\
y_6 &= \Phi_6(z, y_5, y_6)
\end{aligned}
$$

$$
\begin{aligned}
y_1 &= \Phi_1(z, y_1, y_2, (y_5, y_6)) \\
y_2 &= \Phi_2(z, y_2, (y_3, y_4), (y_5, y_6)) \\
(y_3, y_4) &= (\Phi_3, \Phi_4)(z, y_3, y_4) \\
(y_5, y_6) &= (\Phi_5, \Phi_6)(z, y_5, y_6)
\end{aligned}
$$



# General Depencency Graphs

**Theorem**

Suppose that $\mathbf{y} = \mathbf{\Phi}(z, \mathbf{y})$ is a **positive** and **non-linear** system of entire functions such that there is a unique solution $(y_1(z), \ldots, y_r(z))$ that is analytic at $z = 0$.

Then all functions $y_j(z)$ have **non-negative coefficients** and a **finite radius of convergence** $\rho_j$.

**(A)** If $\boxed{\dfrac{\partial^2 \Phi_j}{\partial y_j^2} \neq 0}$ (for all $j$) then for every $j$ there exists an integer $k_j \geq 1$ such that locally

$$\boxed{y_j(z) = a_{0,j} + a_{1,j}(1 - z/\rho_j)^{1/2^{k_j}} + a_{2,j}(1 - z/\rho_j)^{2/2^{k_j}} + \ldots}.$$

# General Depencency Graphs

**Theorem (cont.)**

**(B)** If we just have the condition that for all pairs $(i, j)$ with $\frac{\partial \Phi_j}{\partial y_i} \neq 0$ there exists $k$ with $\boxed{\dfrac{\partial^2 \Phi_j}{\partial y_i y_k} \neq 0}$ then for every $j$ we either have

$$\boxed{y_j(z) = a_{0,j} + a_{1,j}(1 - z/\rho_j)^{2^{-k_j}} + a_{2,j}(1 - z/\rho_j)^{2 \cdot 2^{-k_j}} + \ldots}$$

for an integer $k_j \geq 1$ or

$$\boxed{y_j(z) = \frac{a_{-1,j}}{(1 - z/\rho_j)^{2^{-k_j}}} + a_{0,j} + a_{1,j}(1 - z/\rho_j)^{2^{-k_j}} + \ldots}.$$

for an integer $k_j \geq 0$.

# General Depencency Graphs

**(Counter-)Example.**

$$y_1 = z(e^{y_2} + y_1)$$
$$y_2 = z(1 + 2y_2y_3)$$
$$y_3 = z(1 + y_3^2)$$

$$y_1(z) = \frac{z}{1-z} \exp\left(\frac{z}{\sqrt{1-4z^2}}\right)$$
$$y_2(z) = \frac{z}{\sqrt{1-4z^2}}$$
$$y_3(z) = \frac{1-\sqrt{1-4z^2}}{2z}$$

- $\Phi_3 = z(1 + y_3^2)$ ... satisfies (A)
- $(\Phi_2, \Phi_3) = (z(1 + 2y_2y_3), z(1 + y_3^2))$ ... satisfies (B)
- $(\Phi_1, \Phi_2, \Phi_3) = (z(e^{y_2} + y_1), .., ..)$ **does not satisfy** (B)

# General Depencency Graphs

**Two equations for case (A)**

$$y_1 = \Phi_1(z, y_1, y_2)$$
$$y_2 = \Phi_2(z, y_2)$$

$$\Longrightarrow \quad y_2(z) = g_2(z) - h_2(z)\sqrt{1 - z/\rho_2},$$
$$y_1(z, y_2) = g_1(z, y_2) - h_1(z, y_2)\sqrt{1 - z/\rho(y_2)}$$

$$\begin{aligned} \Longrightarrow \quad y_1(z) &= y_1(z, y_2(z)) \\ &= g_1(z, y_2(z)) - h_1(z, y_2(z))\sqrt{1 - z/\rho(y_2(z))} \\ &= g_1(z, y_2(z)) - h_1(z, y_2(z))\rho(y_2(z))^{-1/2}\sqrt{\rho(y_2(z)) - z} \end{aligned}$$

**3 cases:** (1) $\rho(y_2(\rho_2)) > \rho_2$ (2) $\rho(y_2(\rho_2)) = \rho_2$ (3) $\rho(y_2(\rho_2)) < \rho_2$

# General Depencency Graphs

**Case (1).** $\boxed{\rho(y_2(\rho_2)) > \rho_2}$

$$
\begin{aligned}
g_1(z, y_2(z)) &= g_1\left(z, g_2(z) - h_2(z)\sqrt{1 - z/\rho_2}\right) \\
&= g_1(\rho_2, g_2(\rho_2)) - g_{1,y}(\rho_2, g_2(\rho_2))h_2(\rho_2)\sqrt{1 - z/\rho_2}) + \cdots \\
h_1(z, y_2(z)) &= h_1(\rho_2, g_2(\rho_2)) - h_{1,y}(\rho_2, g_2(\rho_2))h_2(\rho_2)\sqrt{1 - z/\rho_2} + \cdots \\
\rho(y_2(z)) - z &= \rho(y_2(\rho_2)) - \rho_2 - \rho'(y_2(\rho_2))h_2(\rho_2)\sqrt{1 - z/\rho_2} + \cdots \\
\sqrt{\rho(y_2(z)) - z} &= \sqrt{\rho(y_2(\rho_2)) - \rho_2 - \rho'(y_2(\rho_2))h_2(\rho_2)\sqrt{1 - z/\rho_2}) + \cdots} \\
&= \sqrt{\rho(y_2(\rho_2)) - \rho_2} - c_1\sqrt{1 - z/\rho_2} + \cdots
\end{aligned}
$$

$$
\begin{aligned}
\Longrightarrow \quad y_1(z) &= g_1(z, y_2(z)) - h_1(z, y_2(z))\rho(y_2(z))^{-1/2}\sqrt{\rho(y_2(z)) - z} \\
&= c_0 - c_1\sqrt{1 - z/\rho_2} + \cdots
\end{aligned}
$$

# General Depencency Graphs

**Case (2).** $\boxed{\rho(y_2(\rho_2)) = \rho_2}$

$$\begin{aligned}
\rho(y_2(z)) - z &= \rho(y_2(\rho_2)) - \rho_2 - \rho'(y_2(\rho_2))h_2(\rho_2)\sqrt{1 - z/\rho_2}) + \cdots \\
&= c_1'\sqrt{1 - z/\rho_2} + \cdots \\
\sqrt{\rho(y_2(z)) - z} &= \sqrt{c_1'\sqrt{1 - z/\rho_2}) + \cdots} \\
&= \sqrt{c_1'}(1 - z/\rho_2)^{1/4} + c_2'(1 - z/\rho_2)^{3/4} + \cdots
\end{aligned}$$

$$\Longrightarrow \quad \begin{aligned}
y_1(z) &= g_1(z, y_2(z)) - h_1(z, y_2(z))\rho(y_2(z))^{-1/2}\sqrt{\rho(y_2(z)) - z} \\
&= c_0 + c_1(1 - z/\rho_2)^{1/4} + c_2\sqrt{1 - z/\rho_2} + \cdots
\end{aligned}$$

# General Depencency Graphs

**Case (3).** $\boxed{\rho(y_2(\rho_2)) < \rho_2}$

There exists $\rho_1 < \rho_2$ with $\rho(y_2(\rho_1)) = \rho_1$:

$$\begin{aligned}
\rho(y_2(z)) - z &= \rho(y_2(\rho_1)) - \rho_1 + \rho'(y_2(\rho_1))y_2'(\rho_1)(z - \rho_1) \\
&= c_1''(\rho_1 - z) + \cdots \\
\sqrt{\rho(y_2(z)) - z} &= \sqrt{c_1''}\sqrt{\rho_1 - z} + \cdots \\
&= \sqrt{c_1''\rho_1}\sqrt{1 - z/\rho_1} + \cdots
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow \quad y_1(z) &= g_1(z, y_2(z)) - h_1(z, y_2(z))\rho(y_2(z))^{-1/2}\sqrt{\rho(y_2(z)) - z} \\
&= c_0 - c_1\sqrt{1 - z/\rho_1} + \cdots
\end{aligned}$$

with $\rho_1 < \rho_2$.

**Remark.** It is important that $\boxed{\lim_{u\to\infty} \rho(u) = 0}$. This is assured by conditions (A) or (B).

# Infinite Systems

**Infinite linear systems.** $\mathbf{y} = A(z)\mathbf{y} + \mathbf{b}(z) \Longrightarrow \boxed{\mathbf{y}(\mathbf{z}) = (\mathbf{I} - A(z))^{-1}\mathbf{b}(z)}$

**Example.**

$$\begin{aligned} y_1 &= 1 + zy_2 \\ y_j &= z(y_{j-1} + y_{j+1}) \end{aligned}$$

$$\Longrightarrow \quad \boxed{y_j(z) = \frac{1}{z}\left(\frac{1 - \sqrt{1 - 4z^2}}{2z}\right)^j}$$

$$A = \begin{pmatrix} 0 & z & 0 & 0 & \cdots \\ z & 0 & z & 0 & \cdots \\ 0 & z & 0 & z & \\ 0 & 0 & z & \ddots & \ddots \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix}$$

# Infinite Systems

**Compact operator $A(z)$.** $\quad \mathbf{y} = A(z)\mathbf{y} + \mathbf{b}(z)$

$A(z)$ ... irreducible (and compact in a proper $\ell^p$-space)
$r(A(z))$ ... spectral radius of $A(z)$.

$r(A(z_0)) = 1 \Longrightarrow$ resolvent $(x\mathbf{I} - \mathbf{A(z_0)})^{-\mathbf{1}}$ has a simple pole
$\Longrightarrow \mathbf{y(z)} = (\mathbf{I} - A(z))^{-1}\mathbf{b}(z)$ has a **simple pole** at $z = z_0$.

This is the same situation as in the finite dimensional case

# Infinite Systems

**Theorem** [Lalley, Morgenbesser]

Suppose that $\mathbf{y} = (y_j)_{j\ge 1} = \mathbf{\Phi}(z,\mathbf{y})$ is a **positive**, **non-linear**, **infinite** and **irreducibe** system such that $\mathbf{\Phi}_{\mathbf{y}}(z,\mathbf{y})$ is **compact**.

Let $z_0 > 0$, $\mathbf{y}_0 = (y_{0,0},\ldots,y_{r,0}) > 0$ (inside the region of convergence) satisfy the system of equations: ($\mathbf{\Phi} = (\Phi_1,\ldots,\Phi_r)$)

$$\boxed{\mathbf{y}_0 = \mathbf{\Phi}(z_0,\mathbf{y}_0), \quad r(\mathbf{\Phi}_{\mathbf{y}}(z_0,\mathbf{y}_0)) = 1}.$$

Then there exists analytic function $g_j(z), h_j(z) \ne 0$ such that locally

$$\boxed{y_j(z) = g_j(z) - h_j(z)\sqrt{1-\frac{z}{z_0}}}.$$

with $g_j(z_0) = (\mathbf{y}_0)_j$ and $h_j(z_0) \ne 0$.

# Infinite Systems

A linear operator $A$ is **compact** is the image of a bounded set is relative compact.
[**Informally**, an infinite matrix $A$ is compact if it can be **well approximated** by finite dimensional matrices.]

An infinite matrix $A = (a_{i,j})$ is **irreducible** if for every pair of indiced $(i_1, i_2)$ there exists an integer $m$ such that $a^{(m)}_{i_1,i_2} > 0$, where $A^m = (a^{(m)}_{i,j})$.

An **infinite**, **irreducible**, **positive** and **compact** matrix $A = (a_{i,j})$ has a dominant positive real eigenvalue $\boxed{r(A)}$ (the **spectral radius**) that is **isolated** and **simple**.

# Infinite Systems

**Lemma**

$A = (a_{i,j})_{i,j\geq 1}$ ... positive, irreducibe, compact
$B = (a_{i+1,j+1})_{i,j\geq 1}$ [i.e., first column are row are deleted]

$$\Longrightarrow \quad \boxed{r(B) < r(A)}$$

**Remark.** With the help of this property the proof is precisely the same as in the finite dimensional case. It is possible to reduce the infinite system to a single equation.