A LIMITING DISTRIBUTION FOR THE NUMBER OF CUTS NEEDED TO ISOLATE THE ROOT OF A RANDOM RECURSIVE TREE

MICHAEL DRMOTA*, ALEX IKSANOV**, MARTIN MOEHLE, AND UWE ROESLER

Abstract. Let X_n be the number of cuts needed to isolate the root in a random recursive tree with n vertices. We provide a weak convergence result for X_n. The basic observation for its proof is that the probability distributions of $\{X_n : n = 2, 3, \ldots\}$ are recursively defined by $X_n \stackrel{\mathrm{d}}{=} X_{n-D_n} + 1$, $n = 2, 3, \ldots$, $X_1 = 0$, where D_n is a discrete random variable with $\mathbb{P}\{D_n = k\} = \frac{1}{k(k+1)} \frac{n}{n-1}$, $k = 1, 2, \ldots, n-1$, which is independent of $(X_2, \ldots, X_n)$. This distributional recursion was not studied previously in the sense of weak convergence.

1. Introduction

Let T be a rooted (deterministic) tree with number of vertices ≥ 2. Consider the following procedure, introduced in [7], of isolating the root of T by using successive deletions of some edges. After deleting (cutting) one edge chosen at random the tree splits into two subtrees. At the next stage only the subtree containing the root is considered and another edge of this subtree is deleted randomly etc. This recursive process stops as soon as the root has been isolated, i.e. the subtree containing the root obtained after cutting an edge consists of one vertex. We denote by $X(T)$ the (random) number of cuts needed to isolate the root of T.

In particular we will do this procedure with recursive trees. Recall that a recursive tree with n vertices is a rooted tree with vertices labelled with $1, 2, \ldots, n$ that satisfies the following property: the root is labelled with 1, and the labels of the vertices on the unique path from the root to any other vertex (labelled with $m \in \{2, 3, \ldots, n\}$) form an increasing sequence. There are $(n-1)!$ different recursive trees with n vertices, and we denote them by $T_{i,n}$, $i \in \{1, 2, \ldots, (n-1)!\}$.

A random object $\mathcal{T}_n$ is called random recursive tree with n vertices if

$$\mathbb{P}\{\mathcal{T}_n = T_{i,n}\} = ((n-1)!)^{-1}, \quad i \in \{1, 2, \ldots, (n-1)!\}.$$

The purpose of this paper is to discuss the limiting behaviour of random variables $X_n := X(\mathcal{T}_n)$ as $n \to \infty$. It is worth remarking explicitly that X_n is the

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number of cuts needed to isolate the root in a random recursive tree with n vertices. Notice that both the tree and cuts are random.

From the results of [7] and [10] it follows that the sequence $\{X_n : n = 2, 3, \ldots\}$ satisfies the following distributional recursion

$$X_n \stackrel{\mathrm{d}}{=} X_{n-D_n} + 1, \ n = 2, 3, \ldots, \quad \text{and} \quad X_1 = 0, \tag{1.1}$$

where D_n is a random variable with

$$\mathbb{P}\{D_n = k\} = \frac{1}{k(k+1)} \frac{n}{n-1}, \quad k \in \{1, 2, \ldots, n-1\},$$

that is independent of $(X_2, X_3, \ldots, X_{n-1})$.

There are other interpretations of the random variables X_n. As has already been remarked in [1] X_n can be considered as either the number of steps needed to reach the absorbing state 1 of some special Markov chains, or the number of collision events in the Bolthausen-Sznitman coalescent that take place until there is just a single block.

The main result of this paper is the following limit theorem for X_n.[1]

Theorem 1.1. *The sequence*

$$Y_n = \frac{X_n - \frac{n}{\log n} - \frac{n \log\log n}{(\log n)^2}}{\frac{n}{(\log n)^2}} = \frac{(\log n)^2}{n} X_n - \log n - \log\log n$$

converges weakly to a stable random variable Y with characteristic function

$$\varphi_Y(\lambda) = e^{i\lambda \log|\lambda| - \frac{\pi}{2}|\lambda|}.$$

The theorem will be proved in Section 2. Our basic tool is a singularity analysis of generating functions. We close this introductory section with a series of remarks.

First, equalities for distributions similar to (1.1) have received some attention in the recent literature; see, for example, formula (1) in [12], formula (2) in [8] and formula (6) in [3]. Roesler in [12] considered even more general distribution equalities and obtained a limiting result in Mallows d_2 metrics with a limiting distribution being a fixed point of an appropriate smoothing transform with extra term. For a special type of the model investigated by Gnedin [3], a clever renewal argument leads to a weak convergence result with normal limiting law. Finally, in [8] an extension of the contraction method based on mimicing a studied sequence by an accompanying sequence containing normally distributed random variables was used to establish a weak convergence result with normal limiting law. Notice, however, that the normalizations used in the three cited papers are completely different from ours, and the methods of those papers do not seem to work in the special situation of the present paper.

Weak convergence of suitably normalized $X(T_n)$ has been investigated for other classes of both random and deterministic trees T_n with n vertices; see, for example, [2, 5, 6, 9, 11].

[1]An alternative probabilistic proof of this theorem is given in the follow-up paper [4].

Let U_n be the number of cuts needed to isolate the root in a balanced binary tree by using the same edge-removal procedure as above. Curiously enough, the same normalization as in Theorem 1.1 was exploited in [5] to get an asymptotic result for U_n. Notice, however, that the corresponding limiting distribution in [5] was not stable and that while our technique is purely analytic, Svante Janson worked out a probabilistic approach based on a relation between cuttings and records in an underlying tree.

In a companion paper [1] we show that the result of Theorem 1.1 extends to a bit different recursion

$$X_n \stackrel{\mathrm{d}}{=} X_{n-D_n} + T_n,\ n = 2, 3, \ldots, \quad X_1 := 0,$$

where $\{T_n : n = 2, 3, \ldots\}$ is a sequence of independent random variables which are independent of everything else and T_n has exponential distribution with mean $n/(n-1)$. In this way, we proved that the statement of Theorem 1.1 remains true if we replace X_n by L_n or with S_n, where L_n is the total branch length of the Bolthausen-Sznitman coalescent, and S_n is the total number of mutations for that coalescent with positive mutation rate, and n is the sample size.

In our last remark we provide more details regarding stable limiting distributions.

The characteristic function of a stable distribution ν is given by

$$\int_{\mathbb{R}} e^{itx}\nu(dx) = \exp\bigl(i\mu t - c|t|^{\alpha}(1 - i\beta\omega(t,\alpha)\,\mathrm{sgn}(t))\bigr), \quad t \in \mathbb{R},$$

where $\mu \in \mathbb{R}$, $c > 0$, $\alpha \in (0, 2]$, $|\beta| \le 1$, and

$$\omega(t,\alpha) = \begin{cases} \tan\dfrac{\pi\alpha}{2}, & \alpha \neq 1; \\ -\dfrac{2}{\pi}\log|t|, & \alpha = 1. \end{cases}$$

ν is uniquely determined by the *generating quadruple* (μ, c, α, β). It is known that ν is either degenerate, or normal, or has the Lévy spectral function of the form $L(x) = c_1|x|^{-\alpha}$ for $x < 0$, and $L(x) = -c_2x^{-\alpha}$ for $x > 0$, where $c_1, c_2 \ge 0$ and $c_1 + c_2 > 0$.[2]

The limiting distribution of Y in Theorem 1.1 corresponds to the generating quadruple $(0, \pi/2, 1, -1)$ and, thus, the distribution is a spectrally negative stable distribution with index of stability 1. The corresponding Lévy spectral function equals 0 on the positive half-line, and $L(x) = 1/|x|$ on the negative half-line.

2. Proof of Theorem 1.1

2.1. **Singular behavior of $f(s,t)$.** We begin with a simple, but crucial observation.

[2]The constants c_1, c_2 are related to (μ, c, α, β) as follows: $\beta := \frac{c_2 - c_1}{c_2 + c_1}$ and when $\alpha = 1$ we have $c = (c_1 + c_2)(\pi/2)$.

Lemma 2.1. *The generating function*

$$f(s,t) = \sum_{n=1}^{\infty} \mathbb{E}\, s^{X_n}\, t^{n-1}$$

satisfies the partial differential equation

$$\frac{f_t(s,t)}{f(s,t)} = \frac{1}{1-t+\frac{t}{\log(1-t)}\left(1-\frac{1}{s}\right)} \tag{2.1}$$

with initial condition

$$f(s,0) = 1.$$

Proof. For $n = 1,2,\ldots$ set $a_n(s) := \mathbb{E}\, s^{X_n}$ (notice that $a_1(s) := 1$) and for $n = 2,3,\ldots$ put $b_n(s) := \sum_{k=1}^{n-1} \frac{a_k(s)}{(n-k)(n-k+1)}$. From (1.1) it follows that

$$(n-1)a_n(s) = sn \sum_{k=1}^{n-1} \frac{a_k(s)}{(n-k)(n-k+1)}, \quad n = 2,3,\ldots \tag{2.2}$$

Further, from

$$\sum_{n=1}^{\infty} \frac{t^n}{n(n+1)} = 1 + \frac{\log(1-t)}{t} - \log(1-t),$$

we get

$$g(s,t) := \sum_{n=2}^{\infty} t^{n-1} b_n(s) = f(s,t)\left(1 + \frac{\log(1-t)}{t} - \log(1-t)\right)$$

and consequently

$$\sum_{n=2}^{\infty} t^{n-1} n b_n(s) = (tg(s,t))_t.$$

Finally, multiply both sides of (2.2) by t^{n-1} and sum over $n = 2,3,\ldots$ to obtain

$$tf_t(s,t) = s(f(s,t)(t + \log(1-t) - t\log(1-t)))_t$$

which after an easy manipulation leads to the result. □

Suppose that s is fixed (and reasonably close to 1). It is clear that $f(s,t)$ is singular if and only if the denominator

$$h(s,t) := 1 - t + \frac{t}{\log(1-t)}\left(1 - \frac{1}{s}\right) \tag{2.3}$$

is singular or if $h(s,t) = 0$. Obviously,

$$t = 1$$

is always a singularity that is due to the logarithmic term $\log(1-t)$.

There is, however, another zero that we denote by $t = t_0(s)$. Note that $t_0(1) = 1$. This means that both singularities coincide for $s = 1$.

Lemma 2.2. *Set $s = e^w$ and suppose that $|\arg(w)| \le \pi - \delta$ and $0 < |w| < \eta$ for some $\delta > 0$ and some sufficiently small $\eta > 0$. Then, for every w in that range, there is exactly one zero of the mapping $t \mapsto h(e^w, t)$ that is asymptotically given by*

$$t_0(e^w) = 1 - \frac{w}{\log\frac{1}{w}} + \frac{w \log\log\frac{1}{w}}{\log^2\frac{1}{w}} + O\left(\frac{w(\log\log\frac{1}{w})^2}{\log^3\frac{1}{w}}\right), \tag{2.4}$$

uniformly as $w \to 0$ and $|\arg(w)| \le \pi - \delta$.

Proof. Set $T = 1/(1-s^{-1}) = (1-e^{-w})^{-1} = \frac{1}{w}\left(1 + \frac{w}{2} + O(w^2)\right)$ as $w \to 0$. Then (2.4) is equivalent to

$$t_0(e^w) = 1 - \frac{1}{T\log T} + \frac{\log\log T}{T\log^2 T} + O\left(\frac{(\log\log T)^2}{T\log^3 T}\right). \tag{2.5}$$

as $T \to \infty$ and $|\arg(T)| \le \pi - \delta$. Note that $t = t_0(s)$ and T are linked by the equation

$$1 - t + \frac{t}{T\log(1-t)} = 0. \tag{2.6}$$

We now prove (2.5) by a simple bootstrapping procedure.

From

$$\frac{1}{T} = -\frac{1}{t}(1-t)\log(1-t)$$

it follows that, if T is large then t has to be close to 1 and $1 - t \sim 1/(T\log T)$. Hence, starting with the a-priori bound $t = 1 - 1/(T\log T)\cdot(1 + o(1))$ we first get

$$\log(1-t) = -\log T\left(1 + \frac{\log\log T}{\log T}\right)\left(1 + o\left(\frac{1}{\log T}\right)\right)$$

and thus

$$\begin{aligned} t &= 1 - \frac{1}{T\log T}\left(1 + \frac{\log\log T}{\log T}\right)^{-1}\left(1 + o\left(\frac{1}{\log T}\right)\right) \\ &= 1 - \frac{1}{T\log T} + \frac{\log\log T}{T\log^2 T} + o\left(\frac{1}{T\log^2 T}\right). \end{aligned}$$

After a second bootstrapping step one arrives finally at (2.5). □

We will be only interested in the case where

$$w = iv$$

is purely imaginary, that is, v is real, and we also assume that it is small: $|v| < \eta$.

For this case we will provide the structure of the singularity of $f(s,t)$ for fixed $s \ne 1$.[3]

[3]For the proof of Theorem 1.1 we will not need Lemma 2.3 but the refined version Lemma 2.4. However, Lemma 2.3 provides a first insight into the behavior of $f(s,t)$ and makes the arguments in the proof of Lemma 2.4 more transparent.

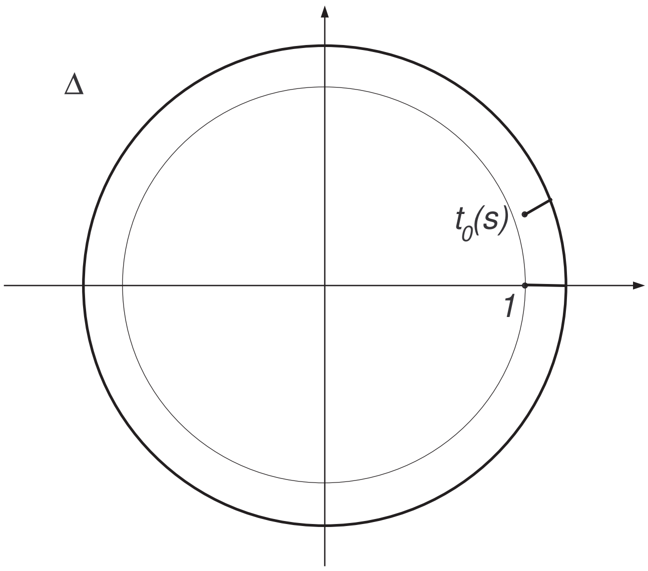

FIGURE 1. Region of analyticity of $f(s,t)$.

Lemma 2.3. *Suppose that $s = e^w = e^{iv}$ for some fixed real number $0 < |v| < \eta$. Then $f(s,t)$ has singularities at $t = 1$ and $t = t_0(s)$, where $t_0(s)$ is not real and there exists $\kappa > 0$ such that $f(s,t)$ has an analytic continuation to the range*

$$\Delta = \{t \in \mathbb{C} : |t| < 1+\kappa,\ |\arg(1-t)| < \pi,\ |\arg(1-t/t_0(s))| < \pi\}.$$

Further, $f(s,1)$ is finite and we have for $t \to 1$, $t \in \Delta$:

(2.7)

$$f(s,t) = f(s,1)\exp\left(-\frac{(1-t)\log(1-t)-(1-t)}{1-\frac{1}{s}} + O\left((1-t)^2\log^2(1-t)\right)\right)$$

Moreover, there exist $\varepsilon > 0$ and continuous functions $C_0(s,t)$, $C_1(s)$ and positive reals δ, κ such that

$$f(s,t) = C_0(s,t)\left(1-\frac{t}{t_0(s)}\right)^{C_1(s)} \tag{2.8}$$

holds for $|t-t_0(s)| < \varepsilon$ and $t \in \Delta$.

Proof. By (2.1) it follows that $f(s,t)$ is given by

$$f(s,t) = \exp\left(\int_0^t \frac{1}{h(s,\tau)}\,d\tau\right)$$

where $h(s,t)$ is defined in (2.3), and the integral $\int_0^t$ can be chosen in an arbitrary way as long as the path of integration is contained in the analyticity range of $1/h(s,t)$. If $s = e^w = e^{iv}$ for some fixed real number $0 < |v| < \eta$ and if η is sufficiently small then $h(s,t) \neq 0$ for $t \in \Delta$ and $\log(1-t)$ is well defined, too. Thus, $f(s,t)$ is surely analytic for $t \in \Delta$.

We will first show that the integral

$$\int_0^1 \frac{1}{h(s,\tau)}\,d\tau$$

is finite. If $s \neq 1$ then there exists $\varepsilon > 0$ such that

$$\left|\frac{\tau}{\log(1-\tau)}\left(1-\frac{1}{s}\right)\right| > 2|1-\tau|$$

for $|1-\tau| \leq \varepsilon$ and $\tau \in \Delta$. Hence we have with certain constants C, C' (depending on s)

$$\begin{aligned}
\int_0^1 \frac{1}{h(s,\tau)}\,d\tau &= \int_0^{1-\varepsilon} \frac{1}{h(s,\tau)}\,d\tau + \int_{1-\varepsilon}^1 \frac{1}{h(s,\tau)}\,d\tau \\
&= C + \int_{1-\varepsilon}^1 \frac{\log(1-\tau)}{\tau\left(1-\frac{1}{s}\right)\left(1+\frac{(1-\tau)\log(1-\tau)}{\tau\left(1-\frac{1}{s}\right)}\right)}\,d\tau \\
&= C + \frac{1}{1-\frac{1}{s}}\int_{1-\varepsilon}^1 \log(1-\tau)\left(1+O\left((1-\tau)\log(1-\tau)\right)\right)\,d\tau \\
&= C + C' < \infty.
\end{aligned}$$

These calculations also show that

$$\begin{aligned}
\int_t^1 \frac{1}{h(s,\tau)}\,d\tau &= \frac{1}{1-\frac{1}{s}}\int_t^1 \log(1-\tau)\left(1+O\left((1-\tau)\log(1-\tau)\right)\right)\,d\tau \\
&= \frac{(1-t)\log(1-t)-(1-t)}{1-\frac{1}{s}} + O\left((1-t)^2\log^2(1-t)\right).
\end{aligned}$$

Of course, this proves (2.7) for $t \to 1$ and $t \in \Delta$.

Now suppose that t is close to $t_0(s)$. Then there exists $\varepsilon > 0$ such that

$$\begin{aligned}
h(s,\tau) &= h(s,t_0(s)) + h_t(s,t_0(s))(\tau - t_0(s)) + O((\tau - t_0(s))^2) \\
&= h_t(s,t_0(s))(\tau - t_0(s)) + O((\tau - t_0(s))^2).
\end{aligned}$$

for $|\tau - t_0(s)| \le \varepsilon |t_0(s)|$. Consequently we get with certain constants C_1', C_1'', C_1''' (that depend on s)

$$\begin{aligned}
\int_0^t \frac{1}{h(s,\tau)}\,d\tau &= \int_0^{t_0(s)(1-\varepsilon)} \frac{1}{h(s,\tau)}\,d\tau + \int_{t_0(s)(1-\varepsilon)}^t \frac{1}{h(s,\tau)}\,d\tau \\
&= C_1' + \int_{t_0(s)(1-\varepsilon)}^t \frac{1}{h_t(s,t_0(s))(\tau - t_0(s)) + O((\tau-t_0(s))^2)}\,d\tau \\
&= C_1' + \int_{t_0(s)(1-\varepsilon)}^t \frac{1}{h_t(s,t_0(s))(\tau - t_0(s))}\left(1 + O(|\tau - t_0(s)|)\right)\,d\tau \\
&= C_1' + \frac{1}{h_t(s,t_0(s))}\left(\log(t_0(s)-t) - \log(t_0(s)\varepsilon)\right) \\
&+ \int_{t_0(s)(1-\varepsilon)}^t O(1)\,d\tau \\
&= C_1' - C_1'' + \frac{1}{h_t(s,t_0(s))}\log(t_0(s)-t) + C_1''' - O(|t-t_0(s)|).
\end{aligned}$$

This directly proves (2.8) with $C_1(s) = 1/h_t(s,t_0(s))$. $\square$

In the next step we have to be a little bit more precise. We introduce a parameter n and will derive asymptotic representations for $f(s,t)$ if $s = s(n)$ tends to 1 and t is very close (depending on n) to 1 respectively to $t_0(s)$.

Lemma 2.4. *Let $s = s(n) = e^{i\lambda n^{-1}(\log n)^A}$ for a real number $\lambda \ne 0$ and some $A > 1$. Then we have $f(s,1) = e^{\log n - (A-1)\log\log n + O(1)}$ and*

$$f(s,t) = f(s,1)\exp\Bigg(- \frac{(1-t)\log(1-t) - (1-t)}{i\lambda n^{-1}(\log n)^A} + O\left(\frac{|1-t|^2(\log|1-t|)^2}{n^{-2}(\log n)^{2A}}\right) + O\left(|1-t|\log|1-t|\right)\Bigg) \tag{2.9}$$

uniformly for $|t-1| \le n^{-1}(\log n)^{A-1}$ and $t \in \Delta$.

Moreover, if $|t - t_0(s)| \le n^{-1}(\log n)^{A-1}$ and $t \in \Delta$ then

$$f(s,t) = \left(\frac{1}{1 - t/t_0(s)}\right)^{1 + \frac{1}{\log n} + O\left(\frac{\log\log n}{(\log n)^2}\right)} \exp\left(C - (A-1)\frac{\log\log n}{\log n} + O\left(\frac{1}{\log n}\right)\right) \tag{2.10}$$

for some constant C.

Proof. The proof is a variation of the proof of Lemma 2.3. First we note that $1 - \frac{1}{s}$ and $t_0(s)$ are asymptotically given by

$$1 - \frac{1}{s} = \frac{i\lambda\,(\log n)^A}{n}\left(1 + O\left(\frac{(\log n)^A}{n}\right)\right)$$

and

$$t_0(s) = 1 - \frac{i\lambda\,(\log n)^{A-1}}{n}\left(1 + (A-1)\frac{\log\log n}{\log n} + O\left(\frac{1}{\log n}\right)\right) \tag{2.11}$$

Next we determine the asymptotic behavior of $f(s,1)$.

Set $c(n) := 1 - n^{-1}(\log n)^{A-1}$ and divide the integral into two parts:

$$\int_0^1 \frac{1}{h(s,\tau)}\,d\tau = \int_0^{c(n)} \frac{1}{h(s,\tau)}\,d\tau + \int_{c(n)}^1 \frac{1}{h(s,\tau)}\,d\tau.$$

For the first interval $0 \le \tau \le c(n)$ we have

$$\begin{aligned}\frac{1}{h(s,\tau)} &= \frac{1}{1-\tau} + \frac{-\frac{\tau}{\log(1-\tau)}\left(1-\frac{1}{s}\right)}{(1-\tau)\left(1-\tau+\frac{\tau}{\log(1-\tau)}\left(1-\frac{1}{s}\right)\right)}\\ &= \frac{1}{1-\tau} + O\left(\left(1-\frac{1}{s}\right)\frac{\tau}{(1-\tau)^2\log(1-\tau)}\right).\end{aligned}$$

Thus,

$$\begin{aligned}\int_0^{c(n)} \frac{1}{h(s,\tau)}\,d\tau &= \int_0^{c(n)} \frac{1}{1-\tau}\,d\tau\\ &\quad + O\left(\frac{(\log n)^A}{n}\int_0^{c(n)} \frac{\tau}{(1-\tau)^2\log(1-\tau)}\,d\tau\right)\\ &= \log n - (A-1)\log\log n\\ &\quad + O\left(\frac{(\log n)^A}{n}\frac{1}{c(n)\log(1/c(n))}\right)\\ &= \log n - (A-1)\log\log n + O(1).\end{aligned}$$

For the second interval $c(n) \le \tau \le 1$ we use the (trivial) bound

$$|h(s,\tau)| \ge \frac{|\lambda|(\log n)^A}{2n}\frac{1}{\log\frac{1}{1-\tau}}$$

and obtain

$$\begin{aligned}\left|\int_{c(n)}^1 \frac{1}{h(s,\tau)}\,d\tau\right| &\le \frac{2n}{|\lambda|(\log n)^A}\int_{c(n)}^1 \log\frac{1}{1-\tau}\,d\tau\\ &= O\left(\frac{n}{(\log n)^A}c(n)\log\frac{1}{c(n)}\right)\\ &= O\left(1\right).\end{aligned}$$

Of course, this implies $f(s,1) = e^{\log n-(A-1)\log\log n+O(1)}$ as proposed.

Moreover, since

$$\begin{aligned}\int_1^t \frac{1}{h(s,\tau)}\,d\tau &= \frac{1}{1-\frac{1}{s}}\int_1^t \log(1-\tau)\,d\tau \\ &\quad + O\left(\frac{1}{\left(1-\frac{1}{s}\right)^2}\int_1^t |1-\tau|(\log|1-\tau|)^2\,d\tau\right) \\ &= -\frac{(1-t)\log(1-t)-(1-t)}{i\lambda n^{-1}(\log n)^A}\left(1+O\left(\frac{(\log n)^A}{n}\right)\right) \\ &\quad + O\left(\frac{|1-t|^2(\log|1-t|)^2}{n^{-2}(\log n)^{2A}}\right)\end{aligned}$$

for $|t-1| \le |\lambda|(2n)^{-1}(\log n)^{A-1}$ and $t \in \Delta$ we also obtain an asymptotic representation for

$$f(s,t) = \exp\left(\int_0^1 \frac{1}{h(s,\tau)}\,d\tau + \int_1^t \frac{1}{h(s,\tau)}\,d\tau\right)$$

in that range.

Next we will analyze $f(s,t)$ for t close to $t_0(s)$. Again we proceed in the way similar to the proof of Lemma 2.3. We will first provide an asymptotic representation for $f(s,t_0(s)(1-n^{-1}))$. For this purpose we have to calculate the integral

$$\int_0^{t_0(s)(1-n^{-1})} \frac{1}{h(s,\tau)}\,d\tau.$$

To this end we use the straight line connection 0 and $t_0(s)(1-n^{-1})$.

We use the substitution $\tau = t_0(s)(1-\rho)$, $\frac{1}{n} \le \rho \le 1$, and consider (with help of the abbreviation $\sigma := 1-\frac{1}{s}$) the ratio

$$\begin{aligned}\frac{h(s,\tau)}{t_0(s)-\tau} &= \frac{1-t_0(s)(1-\rho)+\sigma\frac{t_0(s)(1-\rho)}{\log(1-t_0(s)(1-\rho)}}{t_0(s)\rho} \\ &= \frac{h(s,t_0(s))+t_0(s)\rho-\sigma\frac{t_0(s)\rho}{\log(1-t_0(s))}+\sigma t_0(s)(1-\rho)\left(\frac{1}{\log(1-t_0(s)(1-\rho))}-\frac{1}{\log(1-t_0(s))}\right)}{t_0(s)\rho}\end{aligned}$$

$$= 1-\sigma\frac{1}{\log(1-t_0(s))}-\sigma\frac{(1-\rho)\log\left(1+\frac{\rho t_0(s)}{1-t_0(s)}\right)}{\rho\log(1-t_0(s))\log(1-t_0(s)(1-\rho))}. \tag{2.12}$$

Since

$$\sigma\frac{1}{\log(1-t_0)} = -i\frac{\lambda(\log n)^{A-1}}{n}\left(1+(A-1)\frac{\log\log n}{\log n}+O\left(\frac{1}{\log n}\right)\right)$$

is very small, we just have to concentrate on the last term. If $0 \le \rho \le n^{-1}(\log n)^{A-1}$ then we can directly expand the expression in (2.12) and obtain

$$\sigma \frac{(1-\rho)\log\left(1+\frac{\rho t_0(s)}{1-t_0(s)}\right)}{\rho\log(1-t_0(s))\log(1-t_0(s)(1-\rho))}$$
$$= \frac{1}{\log n}\left(1+2(A-1)\frac{\log\log n}{\log n}+O\left(\frac{1}{\log n}\right)+O\left(\frac{\rho n}{(\log n)^{A-1}}\right)\right).$$

This gives (in that range)

$$\frac{1}{h(s,\tau)} = \frac{1}{\rho}\left(1+\frac{1}{\log n}+O\left(\frac{\log\log n}{(\log n)^2}\right)\right)+O\left(\frac{n}{(\log n)^A}\right).$$

If ρ is larger, say $n^{-1}(\log n)^{A-1} \le \rho \le \frac{1}{2}$, then we cannot expand the logarithm. However, observe that

$$\sigma \frac{\log\left(1+\frac{\rho t_0(s)}{1-t_0(s)}\right)}{\rho\log(1-t_0(s))} = -\frac{\log\left(1+\frac{\rho t_0(s)}{1-t_0(s)}\right)}{\frac{\rho t_0(s)}{1-t_0(s)}}\left(1+O\left(\frac{\log\log n}{\log n}\right)\right)$$

and even more easily

$$\log(1-t_0(s)(1-\rho)) = -\log n\left(1-\frac{\log\left(1+\frac{\rho t_0(s)}{1-t_0(s)}\right)}{\log n}+O\left(\frac{\log\log n}{\log n}\right)\right).$$

Thus, if we use the substitution $\xi = \rho\, n(\log n)^{-(A-1)}$ then the *crucial term* can be rewritten as

$$\sigma \frac{(1-\rho)\log\left(1+\frac{\rho t_0(s)}{1-t_0(s)}\right)}{\rho\log(1-t_0(s))\log(1-t_0(s)(1-\rho))} = \frac{\log(1-i\xi/\lambda)\left(1-\xi n^{-1}(\log n)^{A-1}\right)}{-i\xi\lambda^{-1}\log n\left(1+\frac{\log(1-i\xi/\lambda)}{\log n}+O\left(\frac{\log\log n}{\log n}\right)\right)}$$

and $1/h(s,\tau)$ rewrites to

$$\frac{1}{h(s,\tau)} = \frac{n}{\xi(\log n)^{A-1}}\left(1-\frac{\log(1-i\xi/\lambda)\left(1-\xi n^{-1}(\log n)^{A-1}\right)}{-i\xi\lambda^{-1}\log n\left(1+\frac{\log(1-i\xi/\lambda)}{\log n}+O\left(\frac{\log\log n}{\log n}\right)\right)}\right)^{-1}.$$

Note that the range $\rho \ge n^{-1}(\log n)^{A-1}$ corresponds to the case $\xi \ge 1$.

We now divide the integral $\int_0^{t_0(s)(1-1/n)}$ into three parts and obtain with help of the above asymptotic representations

$$\begin{aligned}
\int\limits_0^{t_0(s)(1-1/n)} \frac{1}{h(s,\tau)}\,d\tau &= \int\limits_0^{t_0(s)/2} \frac{1}{h(s,\tau)}\,d\tau + \int\limits_{t_0(s)/2}^{t_0(s)(1-n^{-1}(\log n)^{A-1})} \frac{1}{h(s,\tau)}\,d\tau \\
&\quad + \int\limits_{t_0(s)(1-n^{-1}(\log n)^{A-1})}^{t_0(s)(1-1/n)} \frac{1}{h(s,\tau)}\,d\tau \\
&= \int\limits_0^{t_0(s)/2} \frac{1}{h(s,\tau)}\,d\tau \\
&\quad + t_0(s)\frac{(\log n)^{A-1}}{n} \int\limits_1^{n/(2(\log n)^{A-1})} \frac{1}{h(s,t_0(s)\cdot(1-\xi n^{-1}(\log n)^{A-1}))}\,d\xi \\
&\quad + t_0(s) \int\limits_{n^{-1}}^{n^{-1}(\log n)^{A-1}} \frac{1}{h(s,t_0(s)\cdot(1-\rho))}\,d\rho.
\end{aligned}$$

Obviously, the first integral

$$I_1 = \int\limits_0^{t_0(s)/2} \frac{1}{h(s,\tau)}\,d\tau$$

represents an analytic function in s, and thus

$$I_1 = C_2' + O\left(\frac{(\log n)^A}{n}\right).$$

for some constant C_2'.

The third integral can be easily computed asymptotically:

$$\begin{aligned}
I_3 &= t_0(s) \int\limits_{n^{-1}}^{n^{-1}(\log n)^{A-1}} \frac{1}{h(s,t_0(s)\cdot(1-\rho))}\,d\rho \\
&= \int\limits_{n^{-1}}^{n^{-1}(\log n)^{A-1}} \left(\frac{1}{\rho}\left(1+\frac{1}{\log n}+O\left(\frac{\log\log n}{(\log n)^2}\right)\right)+O\left(\frac{n}{(\log n)^A}\right)\right)d\rho \\
&= (A-1)\log\log n\left(1+\frac{1}{\log n}+O\left(\frac{\log\log n}{(\log n)^2}\right)\right)+O\left(\frac{1}{\log n}\right).
\end{aligned}$$

The second integral is the most interesting one. Here we have

$$
\begin{aligned}
I_2 &= t_0(s)\frac{(\log n)^{A-1}}{n}\int\limits_1^{n/(2(\log n)^{A-1})}\frac{1}{h(s,t_0(s)\cdot(1-\xi n^{-1}(\log n)^{A-1}))}\,d\xi \\
&= \int\limits_1^{n/(2(\log n)^{A-1})}\frac{1}{\xi}\left(1-\frac{\log(1-i\xi/\lambda)\left(1-\xi n^{-1}(\log n)^{A-1}\right)}{-i\xi\lambda^{-1}\log n\left(1+\frac{\log(1-i\xi/\lambda)}{\log n}+O\left(\frac{\log\log n}{\log n}\right)\right)}\right)^{-1}d\xi \\
&= \int\limits_1^{n/(2(\log n)^{A-1})}\frac{1}{\xi}\,d\xi+\int\limits_1^{n/(2(\log n)^{A-1})}\frac{1}{\xi}\frac{\frac{\log(1-i\xi/\lambda)\left(1-\xi n^{-1}(\log n)^{A-1}\right)}{-i\xi\lambda^{-1}\log n\left(1+\frac{\log(1-i\xi/\lambda)}{\log n}+O\left(\frac{\log\log n}{\log n}\right)\right)}}{1-\frac{\log(1-i\xi/\lambda)(1-\xi n^{-1}(\log n)^{A-1})}{-i\xi\lambda^{-1}\log n\left(1+\frac{\log(1-i\xi/\lambda)}{\log n}+O\left(\frac{\log\log n}{\log n}\right)\right)}}\,d\xi \\
&= \log\frac{n}{2(\log n)^{A-1}}+I_2'
\end{aligned}
$$

A trivial estimate shows that I_2' is bounded:

$$
I_2' = O\left(\frac{1}{\log n}\int_1^\infty\frac{\log|1-i\xi/\lambda|}{\xi^2}\,d\xi\right)=O\left(\frac{1}{\log n}\right).
$$

Hence, we have shown that (for some constant C)

$$
\begin{aligned}
f(s,t_0(s)(1-n^{-1})) &= \exp\left(\log n + C + (A-1)\frac{\log\log n}{\log n}+O((\log n)^{-1})\right) \\
&= n\,e^C\left(1+(A-1)\frac{\log\log n}{\log n}+O\left(\frac{1}{\log n}\right)\right).
\end{aligned}
$$

Finally, if t is close to $t_0(s)$ we just have to evaluate the integral

$$
\int_{t_0(s)(1-1/n)}^{t}\frac{1}{h(s,\tau)}\,d\tau.
$$

If we assume that $|t-t_0(s)|\le n^{-1}(\log n)^{A-1}$ then we can use the approximation

$$
\begin{aligned}
\frac{1}{h(s,\tau)} &= \frac{1}{1-\tau/t_0(s)}\left(1+\frac{1}{\log n}+2(A-1)\frac{\log\log n}{(\log n)^2}+O\left(\frac{1}{(\log n)^2}\right)\right) \\
&\quad + O\left(\frac{n}{(\log n)^A}\right).
\end{aligned}
$$

to obtain the asymptotic relation

$$
\begin{aligned}
&\int_{t_0(s)(1-1/n)}^{t}\frac{1}{h(s,\tau)}\,d\tau \\
&= \left(\log\frac{1}{t_0(s)-t}-\log n\right)\left(1+\frac{1}{\log n}+2(A-1)\frac{\log\log n}{(\log n)^2}+O\left(\frac{1}{(\log n)^2}\right)\right) \\
&+O\left(\frac{1}{\log n}\right)
\end{aligned}
$$

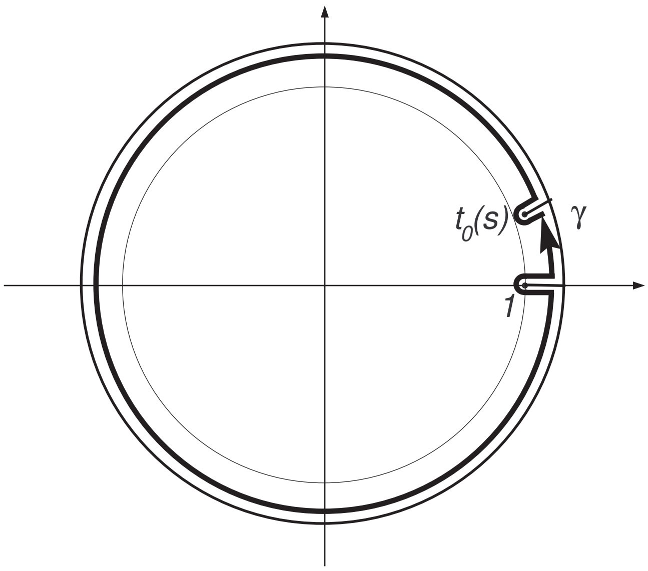

FIGURE 2. Path of integration.

which directly implies

$$\int_0^t \frac{1}{h(s,\tau)}\,d\tau = \log\frac{1}{1-t/t_0(s)}\left(1+\frac{1}{\log n}+2(A-1)\frac{\log\log n}{(\log n)^2}+O\left(\frac{1}{(\log n)^2}\right)\right)$$
$$+(C-1)-(A-1)\frac{\log\log n}{\log n}+O\left(\frac{1}{\log n}\right).$$

This finally gives (2.10). ☐

2.2. **A Cauchy integral.** The next step is to extract the coefficient of t^{n-1} of the power series $f(s,t)=\sum \mathbb{E}\, s^{X_n}\, t^{n-1}$ in order to get an asymptotic representation for

$$\mathbb{E}\, s^{X_n} = \frac{1}{2\pi i}\int_\gamma f(s,t)\,\frac{dt}{t^n}, \tag{2.13}$$

where we assume that

$$s = s(n) = e^{i\lambda n^{-1}(\log n)^A}$$

for some real $\lambda \neq 0$ and for the special choice $A=2$; γ is a curve within the region of analyticity of $f(s,t)$ with winding number 1 around the origin.

Lemma 2.5. *Suppose that $s=s(n)=e^{i\lambda n^{-1}(\log n)^2}$ for some real $\lambda\neq 0$. Then*

$$\mathbb{E}\, s^{X_n} = t_0(s)^{-n}\left(1+O\left(\frac{1}{\log n}\right)\right),$$

where $t_0(s)\neq 1$ satisfies $h(s,t_0(s))=0$ and is asymptotically given by (2.11).

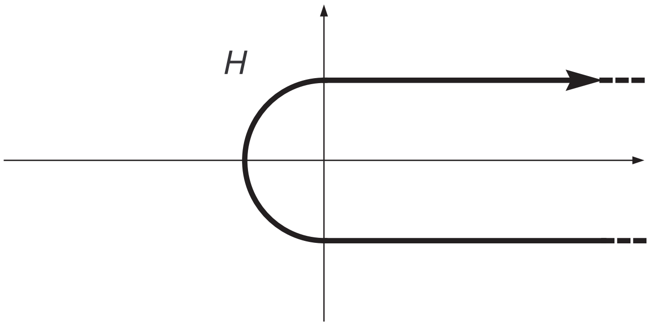

FIGURE 3. Hankel contour.

Proof. We will use (2.13) with path of integration γ that is depicted in Figure 2. This means that the singularities $t = 1$ and $t = t_0(s)$ are *surrounded* by a so-called Hankel-contour (see Figure 3) and the remaining path is a circle of radius $R > 1$.

More precisely we choose $R = 1 + n^{-1}(\log n)$ and let

$$H = \{v \in \mathbb{C} : |v| = 1,\ \Re(v) \leq 0\} \cup \{v \in \mathbb{C} : 0 \leq \Re(v) \leq (\log n),\ \Im(v) = \pm 1\}$$

denote the *major part* of a Hankel contour.

First, let us consider the integral around the singularity $t = 1$, where we use the substitution $t = 1 + v/n$ and v varies on H:

$$J_1 := \frac{1}{2\pi i} \int_{v \in H} f\left(s, 1 + \frac{v}{n}\right) \left(1 + \frac{v}{n}\right)^{-n} \frac{dv}{n}.$$

Since

$$-(1-t)\log(1-t) = -v\frac{\log n}{n}\left(1 + O\left(\frac{\log\log n}{\log n}\right)\right)$$

we get from (2.9)

$$\left|f\left(s, 1 + \frac{v}{n}\right)\right| = O\left(\frac{n}{\log n}\right)$$

for $v \in H$. Furthermore,

$$\left(1 + \frac{v}{n}\right)^{-n} = e^{-v}\left(1 + \frac{|v|^2}{n}\right).$$

Consequently

$$\begin{aligned} J_1 &= O\left(\frac{1}{\log n} + \frac{1}{\log n}\int_0^{\log n} e^{-v}\,dv\right) \\ &= O\left(\frac{1}{\log n}\right). \end{aligned}$$

Next we consider the integral

$$J_2 := \frac{t_0(s)^{-n+1}}{2\pi i} \int_{v \in H} f\left(s, t_0(s)\left(1 + \frac{v}{n}\right)\right) \left(1 + \frac{v}{n}\right)^{-n} \frac{dv}{n}.$$

By (2.10) we have for $v \in H$:

$$f\left(s, t_0(s)\left(1+\frac{v}{n}\right)\right) = n^{1+\frac{1}{\log n}+O\left(\frac{\log\log n}{(\log n)^2}\right)}(-v)^{-1-\frac{1}{\log n}+O\left(\frac{\log\log n}{(\log n)^2}\right)}e^{C+O\left(\frac{\log\log n}{\log n}\right)}.$$

Thus,

$$\begin{aligned} J_2 &= e^{C+O\left(\frac{\log\log n}{\log n}\right)} n^{\frac{1}{\log n}} \frac{t_0(s)^{-n+1}}{2\pi i}\int_{v\in H} v^{-1-\frac{1}{\log n}} e^{-v}\, dv \\ &= e^{C+1} t_0(s)^{-n} \frac{1}{\Gamma(1+(\log n)^{-1})}\left(1+O\left(\frac{1}{\log n}\right)\right) \\ &= e^{C+1} t_0(s)^{-n}\left(1+O\left(\frac{1}{\log n}\right)\right), \end{aligned}$$

where we have used Hankel's integral representation of $1/F(s)$, in particular we have

$$\frac{1}{2\pi i}\int_{v\in H}(-v)^{-s}e^{-v}\, dv = \frac{1}{\Gamma(s)} + O\left(\frac{1}{n\,(\log n)^{\Re(s)}}\right)$$

Finally, we have to estimate the integral over the circle $|t| = R = 1 + n^{-1}(\log n)$. Since, $h(s,t)$ has no zeros different from $t = 1$ and $t = t_0(s)$ it follows that $f(s,t)$ is uniformly bounded for $|t| = R$ and $|t-1| \ge \varepsilon$ (for any $\varepsilon > 0$) Thus, the integral

$$J_3 := \frac{1}{2\pi i}\int\limits_{|t|=R,\,|t-1|\ge\varepsilon} f(s,t)t^{-n}\, dt$$

is bounded by

$$J_3 = O\left(R^{-n}\right) = O\left(\frac{1}{n}\right).$$

If $|t| = R$ and $|t-1| \le \varepsilon$ then we need a slight (but easy) extension of Lemma 2.4 to show that (in that range)

$$f(s,t) = O\left(\frac{n}{\log n}\right).$$

We leave the details to the reader. Hence, the integral

$$J_4 := \frac{1}{2\pi i}\int\limits_{|t|=R,\,|t-1|\le\varepsilon} f(s,t)t^{-n}\, dt$$

is bounded by

$$J_4 = O\left(\frac{n}{\log n}R^{-n}\right) = O\left(\frac{1}{\log n}\right).$$

Of course, this completes the proof of Lemma 2.5. We just note that by continuity $C = -1$ since $t_0(1) = 1$ and $\mathbb{E}\, 1^{X_n} = 1$. □

2.3. **A final step.** We are ready to complete the proof of Theorem 1.1. In a final step we consider the normalized random variable

$$Y_n = \frac{X_n - a_n}{b_n},$$

where

$$a_n = \frac{n}{\log n} + \frac{n \log\log n}{(\log n)^2}$$

and

$$b_n = \frac{n}{(\log n)^2}.$$

Note that b_n^2 is not asymptotically equal to the variance of X_n (see Panholzer [10]).

The characteristic function of Y_n is given by

$$\begin{aligned}\varphi_{Y_n}(\lambda) &= \mathbb{E}\, e^{i\lambda Y_n}\\ &= \mathbb{E}\, e^{i\lambda (X_n - a_n)/b_n}\\ &= e^{-i\lambda a_n/b_n} \mathbb{E}\, e^{i\lambda X_n/b_n}.\end{aligned}$$

Obviously, Theorem 1.1 follows from the next lemma which will be proved by applying Lemma 2.5.

Lemma 2.6. *For every fixed real* $\lambda \neq 0$ *we have*

$$\varphi_{Y_n}(\lambda) = e^{i\lambda \log|\lambda| - \frac{\pi}{2}|\lambda|} + O\left(\frac{(\log\log n)^2}{\log n}\right).$$

Proof. For $s = s(n) = e^{i\lambda n^{-1}(\log n)^2}$ we have

$$\begin{aligned}t_0(s) &= 1 - \frac{i\lambda(\log n)^2}{n \log \frac{n}{i\lambda(\log n)^2}} + \frac{i\lambda(\log n)^2 \log\log \frac{n}{i\lambda(\log n)^2}}{n\left(\log \frac{n}{i\lambda(\log n)^2}\right)^2} + O\left(\frac{(\log\log n)^2}{n\log n}\right)\\ &= 1 - \frac{i\lambda \log n}{n} - \frac{i\lambda \log\log n}{n} - \frac{i\lambda \log(i\lambda)}{n} + O\left(\frac{(\log\log n)^2}{n\log n}\right).\end{aligned}$$

Hence, by applying Lemma 2.5 we get

$$\mathbb{E}\, e^{i\lambda X_n/b_n} = e^{i\lambda\log n + i\lambda \log\log n + i\lambda\log(i\lambda)} + O\left(\frac{(\log\log n)^2}{\log n}\right).$$

Further note that

$$e^{-i\lambda a_n/b_n} = e^{-i\lambda \log n - i\lambda\log\log n}$$

and that

$$e^{i\lambda\log(i\lambda)} = e^{i\lambda\log|\lambda| - \frac{\pi}{2}|\lambda|}.$$

This proves that

$$\varphi_{Y_n}(\lambda) = e^{i\lambda \log|\lambda| - \frac{\pi}{2}|\lambda|} + O\left(\frac{(\log\log n)^2}{\log n}\right)$$

which completes the proof of Lemma 2.6. □

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Michael Drmota, Institute of Discrete Mathematics and Geometry, Technical University of Vienna, Wiedner Hauptstrasse 8–10/104, A-1040 Vienna, Austria

E-mail address: michael.drmota@tuwien.ac.at

Alex Iksanov, Faculty of Cybernetics, National T. Shevchenko University of Kiev, Kiev 01033, Ukraine

E-mail address: iksan@unicyb.kiev.ua

Martin Moehle, Mathematical Institute, University of Duesseldorf, Universitaetsstr. 1, 40225 Duesseldorf, Germany

E-mail address: moehle@math.uni-duesseldorf.de

Uwe Roesler, Mathematisches Seminar, Christian-Albrechts-Universitaet zu Kiel, Ludewig-Meyn-Str. 4, D-24098 Kiel

E-mail address: roesler@math.uni-kiel.de