THE MAXIMUM DEGREE OF SERIES PARALLEL GRAPHS

Michael Drmota*

joint work with Omer Giménez and Marc Noy

Institut für Diskrete Mathematik und Geometrie
TU Wien

michael.drmota@tuwien.ac.at
http://www.dmg.tuwien.ac.at/drmota/

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Series-Parallel Graphs

- Series-parallel extension of a tree or forest

Series-extension:

Parallel-extension:
Series-Parallel Graphs

- $\text{Ex}(K_4)$ ... no $K_4$ as a minor
- Treewidth $\leq 2$
Series-Parallel Graphs

Theorem 1 [D. + Giménez + Noy]

\( G_n \) ... random vertex labelled SP-graph with \( n \) vertices

\( \Delta_n \) ... maximum degree of \( G_n \)

\[ \frac{\Delta_n}{\log n} \to c \quad \text{in probability} \quad \text{and} \quad \mathbb{E} \Delta_n \sim c \log n. \]
Series-Parallel Graphs

Remark 1. A corresponding result holds for 2-connected and connected SP-graphs:

\[ c \approx 3.679771 \quad \text{for 2-connected SP-graphs,} \]
\[ c \approx 3.482774 \quad \text{for connected and all SP-graphs.} \]

Remark 2. \( p_k \) ... (limiting) probability that a random vertex in a random SP-graph has degree \( k \).

\[ q^{-1} \] ... radius of convergence of \( p(w) = \sum_{k \geq 1} p_k w^k. \)

\[ \implies c = \frac{1}{\log(1/q)}. \]
Series-Parallel Graphs

Heuristically: $\Delta_n$ concentrated around level $k_0$ which satisfies $np_{k_0} \approx 1$.

- $p_k$ has “geometric” behaviour: $\log p_k \sim k \log q$ (for $0 < q < 1$)
  
  $\Rightarrow \quad \Delta_n \sim c \log n, \quad c = \frac{1}{\log(1/q)}$

  (E.g. plane trees)

- $p_k$ has “Poisson” behaviour: $p_k \sim a^k e^{-a}/k!$
  
  $\Rightarrow \quad \Delta_n \sim \frac{\log n}{\log \log n}$

  (E.g. labelled trees)
Historic Remarks

• **Gao + Wormald**: precise distribution of maximum degree in planar maps and triangulations.

• **McDiarmid + Reed**: $c \log n < \Delta_n < C' \log n$ whp for random planar graphs.

• **Bernasconi + Panagiotou + Steger**: concentration results for degree distribution (uniform up to $k \leq C' \log n$) + conjecture for max-degree of SP-graphs.
Maximum Degree

Relation to number of vertices of given degree

\[ X_n^{(k)} \quad \text{number of vertices of degree } k \text{ in } G_n. \]

\[ X_n^{(>k)} = X_n^{(k+1)} + X_n^{(k+2)} + \cdots \quad \text{number of vertices of degree } > k. \]

\[ \Delta_n \quad \text{maximum degree:} \]

\[ \Delta_n > k \iff X_n^{(>k)} > 0 \]

\[ \mathbb{P}\{\Delta_n > k\} = \mathbb{P}\{X_n^{(>k)} > 0\} \]
Maximum Degree

First moment method

$Y$ ... a discrete random variable on non-negative integers.

$$\Rightarrow \quad \mathbb{P}\{Y > 0\} \leq \min\{1, \mathbb{E}Y\}$$

Second moment method

$Y$ is a non-negative random variable with finite second moment.

$$\Rightarrow \quad \mathbb{P}\{Y > 0\} \geq \frac{(\mathbb{E}Y)^2}{\mathbb{E}(Y^2)}$$
Maximum Degree

First and second moment method

\[
\frac{\left( \mathbb{E} X_n^{(>k)} \right)^2}{\mathbb{E} (X_n^{(>k)})^2} \leq \mathbb{P}\{\Delta_n > k\} \leq \min\{1, \mathbb{E} X_n^{(>k)}\}
\]

\(X_n^{(>k)}\) ... number of vertices of degree > \(k\).
Maximum Degree

First moments

\[ p_{n,k} \quad \text{probability that a random vertex in } G_n \text{ has degree } k \]

\[ \mathbb{E} X_n^{(k)} = np_{n,k} \]

\[ \Rightarrow \mathbb{E} X_n^{(>k)} = \mathbb{E} \left( \sum_{\ell > k} X_n^{(\ell)} \right) = n \sum_{\ell > k} p_{n,\ell}. \]

Precise asymptotics for \( p_{n,k} \) are needed that are uniform in \( n \) and \( k \).
Maximum Degree

Second moments

$p_{n,k,\ell} \ldots$ probability that two different randomly selected vertices in $G_n$ have degrees $k$ and $\ell$.

$$E\left(X_n^{(k)} X_n^{(\ell)}\right) = n(n-1) p_{n,k,\ell} \quad (k \neq \ell)$$

$$
\implies E\left(X_n^{(> k)}\right)^2 = E\left(\sum_{j > k} X_n^{(j)}\right)^2 = n \sum_{\ell > k} p_{n,\ell} + n(n-1) \sum_{\ell_1, \ell_2 > k} p_{n,\ell_1,\ell_1}.
$$

Precise asymptotics for $p_{n,k,\ell}$ are needed that are uniform in $n$, $k$, and $\ell$. 
Maximum Degree

Bounds for the distribution of $\Delta_n$

$$\frac{n^2 \left( \sum_{\ell > k} p_{n,\ell} \right)^2}{n \sum_{\ell > k} p_{n,\ell} + n(n-1) \sum_{\ell_1, \ell_2 > k} p_{n,\ell_1,\ell_1}} \leq \mathbb{P}\{\Delta_n > k\} \leq \min \left\{ 1, n \sum_{\ell > k} p_{n,\ell} \right\}.$$  

“Master Theorem” Suppose that

$$p_{n,k} \sim c k^\alpha q^k$$

$$p_{n,k,\ell} \sim p_{n,k} p_{n,\ell} \sim c^2 (k\ell)^\alpha q^{k+\ell}$$

$$\implies \frac{\Delta_n}{\log n} \rightarrow \frac{1}{\log(1/q)} \quad \text{in probability}$$
Maximum Degree

**Remark 1** More precisely we need

\[ p_{n,k} \sim c k^\alpha q^k \] uniformly for \( k \leq C \log n \)

and

\[ p_{n,k} = O(\bar{q}^k) \] uniformly for all \( n, k \geq 0 \)

for some \( q \) and \( \bar{q} \) with \( 0 < q \leq \bar{q} < 1 \)

(and similar conditions for \( p_{n,k,\ell} \)).

**Remark 2** (Thanks to Kosta Panagiotou)
The relations for \( p_{n,k,\ell} \) can be replaced by proper estimates for the covariance of \( X_n^{(k)} X_n^{(\ell)} \). For example, if \( G_n \) has **many small blocks** whp then the **degrees** of two independently chosen vertices will be **almost independent** since they will be in different blocks whp.
Series-Parallel Graphs

Generating functions

\[ b_{n,m} \] ... number of 2-connected labelled series-parallel graphs with \( n \) vertices and \( m \) edges, \( b_n = \sum_m b_{n,m} \)

\[ B(x, y) = \sum_{n,m} b_{n,m} \frac{x^n}{n!} y^m \]

\[ c_{n,m} \] ... number of connected labelled series-parallel graphs with \( n \) vertices and \( m \) edges, \( c_n = \sum_m c_{n,m} \)

\[ C(x, y) = \sum_{n,m} c_{n,m} \frac{x^n}{n!} y^m \]

\[ g_{n,m} \] ... number of labelled series-parallel graphs with \( n \) vertices and \( m \) edges, \( g_n = \sum_m g_{n,m} \)

\[ G(x, y) = \sum_{n,m} g_{n,m} \frac{x^n}{n!} y^m \]
Series-Parallel Graphs

Generating functions

\[ G(x, y) = e^{C(x, y)} \]

\[ \frac{\partial C(x, y)}{\partial x} = \exp \left( \frac{\partial B}{\partial x} \left( x \frac{\partial C(x, y)}{\partial x}, y \right) \right), \]

\[ \frac{\partial B(x, y)}{\partial y} = \frac{x^2}{2} \frac{1 + D(x, y)}{1 + y} = \frac{x^2}{2} e^{S(x, y)} \]

\[ D(x, y) = (1 + y) e^{S(x, y)} - 1, \]

\[ S(x, y) = (D(x, y) - S(x, y)) xD(x, y). \]
Series-Parallel Graphs

**Series-parallel networks**: series-parallel extension of an edge

Series-extension:  

Parallel-extension:

There are always two **poles** \((0, \infty)\) coming from the original two vertices.
Series-Parallel Graphs

Series-parallel networks

Parallel decomposition of a Series-parallel network:

Series decomposition of a series-parallel network
Series-Parallel Graphs

Series-parallel networks

d_{n,m} \ldots \text{number of SP-networks with } n + 2 \text{ vertices and } m \text{ edges}

s_{n,m} \ldots \text{number of series SP-networks } n + 2 \text{ vertices and } m \text{ edges}

\[ D(x, y) = \sum_{n,m} d_{n,m} \frac{x^n}{n!} y^m, \quad S(x, y) = \sum_{n,m} s_{n,m} \frac{x^n}{n!} y^m, \]

\[ D(x, y) = e^{S(x,y)} - 1 + ye^{S(x,y)} \]
\[ = (1 + y)e^{S(x,y)} - 1, \]

\[ S(x, y) = (D(x, y) - S(x, y))xD(x, y) \]
Series-Parallel Graphs

2-connected SP-graphs

A SP-network network with non-adjacent poles (which is counted by $e^{S(x,y)}$) is obtained by distinguishing, orienting and then deleting any edge of an arbitrary 2-connected series-parallel graph:

\[
\frac{\partial B(x, y)}{\partial y} = \frac{x^2}{2} e^{S(x,y)} = \frac{x^2}{2} \frac{1 + D(x, y)}{1 + y}
\]
Series-Parallel Graphs

Connected SP-graphs

\[ \frac{\partial C(x, y)}{\partial x} = \exp \left( \frac{\partial B}{\partial x} \left( x \frac{\partial C(x, y)}{\partial x}, y \right) \right) \]

All SP-graphs

\[ G(x, y) = e^{C(x, y)} \]
Series-Parallel Graphs

Asymptotic enumeration

\[ b_n = b \cdot \rho_1^{-n} n^{-\frac{5}{2}} n! \left(1 + O\left(\frac{1}{n}\right)\right), \]
\[ c_n = c \cdot \rho_2^{-n} n^{-\frac{5}{2}} n! \left(1 + O\left(\frac{1}{n}\right)\right), \]
\[ g_n = g \cdot \rho_2^{-n} n^{-\frac{5}{2}} n! \left(1 + O\left(\frac{1}{n}\right)\right), \]

\[ \rho_1 = 0.1280038..., \]
\[ \rho_2 = 0.11021..., \]
\[ b = 0.0010131..., \]
\[ c = 0.0067912..., \]
\[ g = 0.0076388... \]
Series-Parallel Graphs

Asymptotic enumeration

\[ D(x, y) = (1 + y) \exp \left( \frac{xD(x, y)^2}{1 + xD(x, y)} \right) - 1 = \Phi(x, y, D(x, y)) \]

\[ \implies D(x, y) = g(x, y) - h(x, y) \sqrt{1 - \frac{x}{\rho(y)}}, \]

with \( \rho(1) = \rho_1 = 0.12800 \ldots \).
Series-Parallel Graphs

Asymptotic enumeration

\[ \frac{\partial B(x, y)}{\partial y} = \frac{x^2}{2} \frac{1 + D(x, y)}{1 + y} \]

\[ = g_2(x, y) - h_2(x, y) \sqrt{1 - \frac{x}{\rho(y)}} \]

\[ \Rightarrow \quad B(x, y) = g_3(x, y) + h_3(x, y) \left(1 - \frac{x}{\rho(y)}\right)^{\frac{3}{2}} \]

\[ \Rightarrow \quad b_n \sim b \cdot \rho(1)^{-n} n^{-\frac{5}{2}} n! \]
Series-Parallel Graphs

Asymptotic enumeration \((C' := \frac{\partial}{\partial x} C')\)

\[ C'(x, y) = e^{B'(xC'(x,y), y)}, \quad v(x, y) = xC'(x, y), \quad \Phi(x, y, v) = xe^{B'(v, y)} \]

\[ v(x, y) = \Phi(x, y, v(x, y)) \]

\[ v(x, y) = xC'(x, y) = g_4(x, y) - h_4(x, y) \sqrt{1 - \frac{x}{\rho_2(y)}} \]

with \(\rho_2(1) = 0.11021\ldots\) (Note that \(v(\rho) = 0.1279695\ldots < \rho_1 !!!\))

\[ C(x, y) = g_5(x, y) + h_5(x, y) \left(1 - \frac{x}{\rho_2(y)}\right)^{\frac{3}{2}}. \]

\[ c_n \sim c \rho_2^{-n} n^{-\frac{5}{2}n!} \]
Series-Parallel Graphs

Asymptotic enumeration

\[ C(x, y) = g_5(x, y) + h_5(x, y) \left(1 - \frac{x}{\rho(y)}\right)^{\frac{3}{2}} \]

\[ \implies G(x, y) = e^{C(x,y)} = g_6(x, y) + h_6(x, y) \left(1 - \frac{x}{\rho_2(y)}\right)^{\frac{3}{2}}. \]

\[ \implies g_n \sim g \cdot \rho_2^{-n} n^{-\frac{5}{2}} n! \]
Root Degree

Random vertex versus root vertex

\( G_n \) ... random vertex labelled SP-graph with \( n \) vertices

\( G_n^\bullet \) ... random vertex labelled SP-graph with \( n \) vertices, where one vertex is distinguished (= root)

\[ p_{n,k} = \text{probability that a random vertex in } G_n \text{ has degree } k \]

\[ = \text{probability that the root in } G_n^\bullet \text{ has degree } k \]
Root Degree

Generating functions

\[ b_{n,k}^\bullet \ldots \text{ number of rooted 2-connected labelled series-parallel graphs with } n \text{ vertices and root-degree } k. \]

\[ B^\bullet(x, w) = \sum_{n,k} b_{n,k}^\bullet \frac{x^n}{n!} w^k \]

\[ c_{n,k}^\bullet \ldots \text{ number of rooted connected labelled series-parallel graphs with } n \text{ vertices and root-degree } k. \]

\[ C^\bullet(x, w) = \sum_{n,k} c_{n,k}^\bullet \frac{x^n}{n!} w^k \]

\[ g_{n,k}^\bullet \ldots \text{ number of rooted labelled series-parallel graphs with } n \text{ vertices and root-degree } k. \]

\[ G^\bullet(x, w) = \sum_{n,k} g_{n,k}^\bullet \frac{x^n}{n!} w^k \]
Root Degree

Computation of $p_{n,k}$

\[
p_{n,k} = \frac{g_{n,k}}{n g_n} = \frac{[x^n w^k] G^\bullet(x, w)}{[x^n] G^\bullet(x, 1)}
\]
Root Degree

Generating functions

\[ G^\bullet(x, w) = C^\bullet(x, w) e^{C(x)}, \]
\[ C^\bullet(x, w) = e^{B^\bullet(xC'(x), w)}, \]
\[ w \frac{\partial}{\partial w} B^\bullet(x, w) = \sum_{k \geq 1} kB_k(x) w^k = xwe^{S^\bullet(x, w)}, \]
\[ D^\bullet(x, w) = (1 + w)e^{S^\bullet(x, w)} - 1, \]
\[ S^\bullet(x, w) = (D^\bullet(x, w) - S^\bullet(x, w))xD(x, 1). \]
Root Degree

Series-parallel networks

\(d_{n,k}^\bullet\) … number of SP-networks with \(n+2\) vertices, where the first pole has degree \(k\)

\(s_{n,m}^\bullet\) … number of series SP-networks \(n+2\) vertices, where the first pole has degree \(k\)

\[
D^\bullet(x, y) = \sum_{n,k} d_{n,k}^\bullet \frac{x^n}{n!} w^k, \quad S^\bullet(x, y) = \sum_{n,k} s_{n,k}^\bullet \frac{x^n}{n!} w^k,
\]

\[
D^\bullet(x, w) = (1 + w)e^{S^\bullet(x, w)} - 1,
\]

\[
S^\bullet(x, w) = (D^\bullet(x, w) - S^\bullet(x, w))xD(x, 1)
\]
Root Degree

2-connected SP-graphs

A SP-network with non-adjacent poles (which is counted by $e^{S(x,w)}$) is obtained by distinguishing, orienting and then deleting any edge of an arbitrary 2-connected series-parallel graph:

\[
w \frac{\partial}{\partial w} B^\bullet(x, w) = \sum_{k \geq 1} k B_k(x) w^k = x e^{S(x,w)},
\]

\[
= \frac{1 + D^\bullet(x, w)}{1 + w}
\]
Root Degree

Connected SP-graphs

\[ C^\bullet(x, w) = e^{B^\bullet(xC'(x), w)} \]

All SP-graphs

\[ G^\bullet(x, w) = C^\bullet(x, w)e^{C(x)} \]
Degree Distribution

Theorem 2 [D. + Giménez + Noy]

Let $p_{n,k}$ be the probability that a random vertex in a random 2-connected, connected or unrestricted series-parallel graph with $n$ vertices has degree $k$. Then the limit

$$p_k := \lim_{n \to \infty} p_{n,k}$$

exists. The probability generating function

$$p(w) = \sum_{k \geq 1} p_k w^k$$

can be computed explicitly and we have asymptotically

$$p_k \sim c q^k k^{-3/2}.$$
Degree Distribution

For 2-connected series-parallel graphs the series $p(w) = \sum_{k \geq 1} p_k w^k$ is given by:

$$p(w) = \frac{B_1(1, w)}{B_1(1, 1)},$$

where $B_1(y, w)$ is given by the following procedure ...
Degree Distribution

\[
\frac{E_0(y)^3}{E_0(y) - 1} = \left( \log \frac{1 + E_0(y)}{1 + R(y)} - E_0(y) \right)^2,
\]

\[
R(y) = \frac{\sqrt{1 - 1/E_0(y)} - 1}{E_0(y)},
\]

\[
E_1(y) = -\left( \frac{2R(y)E_0(y)^2(1 + R(y)E_0(y))^2}{(2R(y)E_0(y) + R(y)^2E_0(y)^2)^2 + 2R(y)(1 + R(y)E_0(y))} \right)^{\frac{1}{2}},
\]

\[
D_0(y, w) = (1 + yw)e^{\frac{R(y)E_1(y)D_0(y, w)}{1 + R(y)E_0(y)}} D_0(y, w) - 1,
\]

\[
D_1(y, w) = \frac{(1 + D_0(y, w))^{\frac{R(y)E_1(y)D_0(y, w)}{1 + R(y)E_0(y)}}}{1 - (1 + D_0(y, w))^{\frac{R(y)E_0(y)D_0(y, w)}{1 + R(y)E_0(y)}}},
\]

\[
B_0(y, w) = \frac{R(y)D_0(y, w)}{1 + R(y)E_0(y)} - \frac{R(y)^2E_0(y)D_0(y, w)^2}{2(1 + R(y)E_0(y))},
\]

\[
B_1(y, w) = \frac{R(y)D_1(y, w)}{1 + R(y)E_0(y)} - \frac{R(y)^2E_0(y)D_0(y, w)D_1(y, w)}{1 + R(y)E_0(y)} - \frac{R(y)^2E_1(y)D_0(y, w)(1 + D_0(y, w)/2)}{(1 + R(y)E_0(y))^2}.
\]
Degree Distribution

Remark 3 [D.＋Giménez＋Noy] $X_n^{(k)}$ satisfies a central limit theorem with

$$
\mathbb{E} X_n^{(k)} \sim \mu_k n \quad \text{and} \quad \mathbb{V} X_n^{(k)} \sim \sigma_k^2 n.
$$

Remark. $\mu_k = p_k$. 
Asymptotic Analysis

We know

\[ p_k = \lim_{n \to \infty} p_{n,k} \sim c q^k k^{-\frac{3}{2}} \]

We need (uniformly for \( k \leq C \log n \))

\[ p_{n,k} \sim c q^k k^{-\frac{3}{2}}. \]

The goal is to extend Theorem 2 to a bivariate asymptotics.
Asymptotic Analysis

Series-parallel networks

\[ D(x, 1) = 2 \exp \left( \frac{x D(x, 1)^2}{1 + x D(x, 1)} \right) - 1 = \Phi(x, D(x, 1)) \]

\[ \implies D(x, 1) = g_1(x) - h_1(x) \sqrt{1 - \frac{x}{\rho_1}}, \]

with \( \rho_1 = 0.12800 \ldots \)

[Repetition of the previous case with \( y = 1 \)].
Asymptotic Analysis

Series-parallel networks

\[ D^\bullet(x, w) = 2 \exp \left( \frac{xD(x, 1)D^\bullet(x, w)}{1 + xD^\bullet(x, w)} \right) - 1 = \Phi(x, w, D(x, 1), D^\bullet(x, w)) \]

\[ \Rightarrow D^\bullet(x, w) = g_2(x, w, D(x, 1)) - h_2(x, w, D(x, 1)) \sqrt{1 - \frac{w}{\rho(x, D(x, 1))}}, \]

with

\[ \rho(x, D(x, 1)) = \bar{g}(x) - \bar{h}(x) \sqrt{1 - \frac{x}{\rho_1}} \]
Asymptotic Analysis

2-connected SP-graphs

\[
\frac{\partial B^\bullet(x, w)}{\partial w} = \frac{1 + D^\bullet(x, w)}{1 + w} D^\bullet(x, w) = g_3(x, w, D(x, 1)) - h_3(x, w, D(x, 1)) \sqrt{1 - \frac{w}{\rho(x, D(x, 1))}}
\]

\[
B^\bullet(x, w) = g_4(x, w, D(x, 1)) + h_4(x, w, D(x, 1)) \left(1 - \frac{w}{\rho(x, D(x, 1))}\right)^{\frac{3}{2}}
\]

\[
= G(x, w) + H(x, w) (1 - y(x)w)^{\frac{3}{2}}
\]

with\[
y(x) = \rho(x, D(x, 1))^{-1} = g(x) - h(x) \sqrt{1 - x/\rho_1},
\]
\[
G(x, w) = g_4(x, w, D(x, 1)) = G_1(x, w) - G_2(x, w) \sqrt{1 - x/\rho_1},
\]
\[
H(x, w) = h_4(x, w, D(x, 1)) = H_1(x, w) - H_2(x, w) \sqrt{1 - x/\rho_1}.
\]
Asymptotic Analysis

Connected SP-graphs

\[ C^\bullet(x, w) = e^{B^\bullet(xC'(x), w)} = G(x, w) + H(x, w) \left(1 - \bar{y}(x)w\right)^{3/2} \]

with

\[ \bar{y}(x) = y(xC'(x)) = \bar{g}(x) - \bar{h}(x)\sqrt{1 - x/\rho_2}, \]

\[ \bar{G}(x, w) = \bar{G}_1(x, w) - \bar{G}_2(x, w)\sqrt{1 - x/\rho_2}, \]

\[ \bar{H}(x, w) = \bar{H}_1(x, w) - \bar{H}_2(x, w)\sqrt{1 - x/\rho_2}. \]
Asymptotic Analysis

Lemma 1

\[ f(x, w) = \sum_{n,k \geq 0} f_{n,k} x^n w^k \]

\[ = G(x, w) + H(x, w) \left( 1 - y(x) w \right)^{\frac{3}{2}}, \]

where

\[ y(x) = g(x) - h(x) \sqrt{1 - x/x_0}, \]

\[ G(x, w) = G_1(x, w) - G_2(x, w) \sqrt{1 - x/x_0}, \]

\[ H(x, w) = H_1(x, w) - H_2(x, w) \sqrt{1 - x/x_0}. \]

with analytic functions \( g, h, G_1, G_2, H_1, H_2 \)

(\(+\) some technical conditions)

\[ f_{n,k} = \frac{3h(x_0) H(x_0, 0, 1/g(x_0))}{8\pi} g(x_0)^{k-1} x_0^{-n} k^{-\frac{3}{2}} n^{-\frac{3}{2}} \left( 1 + O \left( \frac{1}{k} \right) \right) \]

uniformly for \( k \leq C \log n \) (for any constant \( C > 0 \)) and

\[ f_{n,k} = O \left( (g(x_0) + \varepsilon)^k \rho^n n^{-\frac{3}{2}} \right). \]
Asymptotic Analysis

Application

\[ B^\bullet(x, w) = G(x, w) + H(x, w) (1 - y(x)w)^{\frac{3}{2}}, \]

\[ \iff \quad \frac{b_{n,k}}{n!} \sim c_1 q^k x_0^{-n} k^{-\frac{3}{2}} n^{-\frac{3}{2}}. \]

with \( q = g(x_0) < 1. \)

\[ \frac{b_n}{n!} \sim bx_0^{-n} n^{-\frac{5}{2}} \quad \text{(from above)} \]

\[ \iff \quad p_{n,k} = \frac{b_{n,k}}{nb_n} \sim c q^k k^{-\frac{3}{2}} \]
Double Rooting

Generating Functions

\[ G^{\bullet\bullet}(x, w, t) = e^{C(x)}G^{\bullet}(x, w)G^{\bullet}(x, t) + e^{C(x)}C^{\bullet\bullet}(x, w, t), \]
\[ C^{\bullet\bullet}(x, w, t) = \frac{x}{(xC'(x))'} \frac{\partial}{\partial x} C^{\bullet}(x, w) \frac{\partial}{\partial x} C^{\bullet}(x, t) \]
\[ + B^{\bullet\bullet}(xC'(x), w, t)C^{\bullet}(x, w)C^{\bullet}(x, t), \]
\[ w \frac{\partial}{\partial w} B^{\bullet\bullet}(x, w, t) = wte^{S_1(x, w, t)} + we^{S(x, w)}S_2(x, w, t), \]
\[ D_1(x, w, t) = (1 + wt)e^{S_1(x, w, t)} - 1, \]
\[ S_1(x, w, t) = x(D^{\bullet}(x, w) - S^{\bullet}(x, w))D^{\bullet}(x, t), \]
\[ D_2(x, w, t) = (1 + wt)e^{S_2(x, w, t)}, \]
\[ S_2(x, w, t) = x(D_2(x, w, t) - S_2(x, w, t))D_2(x, 1) \]
\[ + x(D_1(x, w, t) - S_1(x, w, t))D^{\bullet}(x, t) \]
\[ + x(D^{\bullet}(x, w) - S^{\bullet}(x, w))D_2(x, 1, t). \]
Asymptotic Analysis

\[ B^{\bullet\bullet}(x, w, t) = \frac{G(x, w, t) + H(x, w, t)W + I(x, w, t)T + J(x, w, t)WT}{\sqrt{1 - x/\rho_1}} \]

with the abbreviations

\[ W = \sqrt{1 - y(x)w} \quad \text{and} \quad T = \sqrt{1 - y(x)t} \]

and with

\[ y(x)g(x) - h(x)\sqrt{1 - x/\rho_1}, \]
\[ G(x, w, t) = G_1(x, w, t) - G_2(x, w, t)\sqrt{1 - x/\rho_1}, \]
\[ H(x, w, t) = H_1(x, w, t) - H_2(x, w, t)\sqrt{1 - x/\rho_1}, \]
\[ I(x, w, t) = I_1(x, w, t) - I_2(x, w, t)\sqrt{1 - x/\rho_1}, \]
\[ J(x, w, t) = J_1(x, w, t) - J_2(x, w, t)\sqrt{1 - x/\rho_1}. \]

The analytic behaviour of \( C^{\bullet\bullet}(x, w, t) \) is of the same kind.
Asymptotic Analysis

Lemma 2

\[ f(x, w, t) = \sum_{n,k,\ell} f_{n,k,\ell} x^n w^k t^\ell \]

\[ = \frac{G(x, w, t) + H(x, w, t) W + I(x, w, t) T + J(x, w, t) WT}{\sqrt{1 - x/x_0}} \]

with the abbreviations \( W = \sqrt{1 - y(x) w} \) and \( T = \sqrt{1 - y(x) t} \), where

\[ y(x) g(x) - h(x) \sqrt{1 - x/x_0}, \]

\[ G(x, w, t) = G_1(x, w, t) - G_2(x, w, t) \sqrt{1 - x/x_0}, \]

\[ H(x, w, t) = H_1(x, w, t) - H_2(x, w, t) \sqrt{1 - x/x_0}, \]

\[ I(x, w, t) = I_1(x, w, t) - I_2(x, w, t) \sqrt{1 - x/x_0}, \]

\[ J(x, w, t) = J_1(x, w, t) - J_2(x, w, t) \sqrt{1 - x/x_0} \]

with analytic functions \( g, h, G_1, G_2, H_1, H_2, I_1, I_2, J_1, J_2 \)

(\(+\) some technical conditions)
Asymptotic Analysis

Lemma 2 (cont.)

\[ f_{n,k,\ell} \sim \frac{J\left(x_0, 0, \frac{1}{g(x_0)}, \frac{1}{g(x_0)}\right)}{4\pi^{3/2}} g(x_0)^{k+\ell} x_0^{-n} (k\ell)^{-\frac{3}{2} n^{-\frac{1}{2}}} \]

uniformly for \( k, \ell \leq C \log n \) (for any constant \( C > 0 \)) and

\[ f_{n,k,\ell} = O\left((g(x_0) + \varepsilon)^{k+\ell} x_0^{-n} n^{-\frac{1}{2}}\right). \]

uniformly for all \( n, k, \ell \geq 0 \) for every \( \varepsilon > 0 \).

Remark This proves \( p_{n,k,\ell} \sim c^2 q^{k+\ell} (k\ell)^{-\frac{3}{2}} \).
Proof of Lemma 1

1. Singularity Analysis
(following Flajolet-Odlyzko)

Suppose that
\[ y(x) = (1 - x/x_0)^{-\alpha}. \]
Then
\[ y_n = [x^n] y(x) = (-1)^n \binom{-\alpha}{n} x_0^n = \frac{n^{\alpha-1}}{\Gamma(\alpha)} x_0^n + O(n^{\alpha-2} x_0^n). \]
Proof of Lemma 1

1. Singularity Analysis

Cauchy's formula:

\[ (-1)^n \binom{-\alpha}{n} x_0^n = \frac{1}{2\pi i} \int_\gamma (1-x/x_0)^{-\alpha} x^{-n-1} \, dx. \]

\[ \gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4: \]

\[ \gamma_1 = \left\{ x = x_0 \left(1 - \frac{i + (\log n)^2 - t}{n}\right) : 0 \leq t \leq (\log n)^2 \right\}, \]

\[ \gamma_2 = \left\{ x = x_0 \left(1 - \frac{1}{n} e^{-i\phi}\right) : -\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2} \right\}, \]

\[ \gamma_3 = \left\{ x = x_0 \left(1 + \frac{i + t}{n}\right) : 0 \leq t \leq (\log n)^2 \right\}, \]

and \( \gamma_4 \) is a circular arc centred at the origin and making \( \gamma \) a closed curve.
1. Singularity Analysis

Path of integration
1. Singularity Analysis

Substitution for \( x \in \gamma_1 \cup \gamma_2 \cup \gamma_3 \):

\[
x/x_0 = 1 + \frac{t}{n} \implies x^{-n-1} = e^{-t} \left( 1 + O \left( \frac{t^2}{n} \right) \right)
\]

With Hankel’s integral representation for \( 1/\Gamma(\alpha) \)

\[
\frac{1}{2\pi i} \int_{\gamma_1 \cup \gamma_2 \cup \gamma_3} (1 - x/x_0)^{-\alpha} x^{-n-1} \, dx = \frac{n^{\alpha-1} x_0^n}{2\pi i} \int_{H} (-t)^{-\alpha} e^{-t} \, dt \\
+ \frac{n^{\alpha-2} x_0^n}{2\pi i} \int_{H} (-t)^{-\alpha} e^{-t} \cdot O \left( t^2 \right) \, dt \\
= n^{\alpha-1} \frac{1}{\Gamma(\alpha)} x_0^n + O \left( n^{\alpha-2} x_0^n \right).
\]

\( H = \{ t \mid |t| = 1, \Re t \leq 0 \} \cup \{ t \mid 0 < \Re t \leq \log^2 n, \Im t = \pm 1 \} \):
1. Singularity Analysis

Remark

\[ x \in \gamma_1 \cup \gamma_2 \cup \gamma_3 \implies \frac{1}{n} \leq \left| 1 - \frac{x}{x_0} \right| \leq \frac{(\log n)^2}{n} \]
Asymptotic Analysis

**Lemma 1** (the same as before)

\[
f(x, w) = \sum_{n,k \geq 0} f_{n,k} x^n w^k
\]

\[
= \left[ G(x, w) + H(x, w) \left(1 - y(x)w\right)^{\frac{3}{2}} \right],
\]

where

\[
y(x) = g(x) - h(x) \sqrt{1 - x/x_0},
\]

\[
G(x, w) = G_1(x, w) - G_2(x, w) \sqrt{1 - x/x_0},
\]

\[
H(x, w) = H_1(x, w) - H_2(x, w) \sqrt{1 - x/x_0}.
\]

with analytic functions \(g, h, G_1, G_2, H_1, H_2\)

(+ some technical conditions)

\[
\Rightarrow f_{n,k} = \frac{3h(x_0)H(x_0, 0, 1/g(x_0))}{8\pi} g(x_0)^{k-1} x_0^{-n} k^{-\frac{3}{2}} n^{-\frac{3}{2}} \left(1 + O\left(\frac{1}{k}\right)\right)
\]

uniformly for \(k \leq C \log n\) (for any constant \(C > 0\)) and

\[
f_{n,k} = O \left((g(x_0) + \varepsilon)^k \rho^{-n} n^{-\frac{3}{2}}\right).
\]
Proof of Lemma 1

2. Cauchy’s formula

\[ f_{n,k} = \frac{1}{(2\pi i)^2} \int_{\gamma} \int_{\Gamma} \frac{f(x,w)}{x^{n+1}w^{k+1}} \, dx \, dw \]

Integration with respect to \( x \): \( \gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4 \), where

\[
\gamma_1 = \left\{ x = x_0 \left(1 - \frac{i + (\log n)^2 - t}{n}\right) : 0 \leq t \leq (\log n)^2 \right\},
\]

\[
\gamma_2 = \left\{ x = x_0 \left(1 - \frac{1}{n} e^{-i\phi}\right) : -\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2} \right\},
\]

\[
\gamma_3 = \left\{ x = x_0 \left(1 + \frac{i + t}{n}\right) : 0 \leq t \leq (\log n)^2 \right\},
\]

and \( \gamma_4 \) is a circular arc centred at the origin and making \( \gamma \) a closed curve.
2. Cauchy’s formula

Integration with respect to $w$: $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$, where

$$\Gamma_1 = \left\{ w = w_0 \left(1 - \frac{i + (\log k)^2 - r}{k}\right) : 0 \leq s \leq (\log k)^2 \right\},$$

$$\Gamma_2 = \left\{ w = w_0 \left(1 - \frac{1}{k} e^{-i\psi}\right) : -\frac{\pi}{2} \leq \psi \leq \frac{\pi}{2} \right\},$$

$$\Gamma_3 = \left\{ w = w_0 \left(1 + \frac{i + s}{w}\right) : 0 \leq s \leq (\log k)^2 \right\},$$

and $\Gamma_4$ is a circular arc centred at the origin and making $\Gamma$ a closed curve.

$(w_0 = 1/g(x_0))$
2. Cauchy’s formula

Remark

\[ x \in \gamma_1 \cup \gamma_2 \cup \gamma_3 \text{ and } w \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3: \]

\[ \frac{1}{n} \leq \left| 1 - \frac{x}{x_0} \right| \leq \frac{(\log n)^2}{n} \quad \text{and} \quad \frac{1}{k} \leq \left| 1 - \frac{w}{w_0} \right| \leq \frac{(\log k)^2}{k} \]

For \( k \leq C \log n \) we thus have

\[ X = \sqrt{1 - \frac{x}{x_0}} \quad \text{is much smaller than} \quad W = 1 - \frac{w}{w_0} \]
Proof of Lemma 1

3. Local expansion around the singularity

\[ y(x) = g(x) - h(x)\sqrt{1 - x/x_0} \]
\[ = g(x_0) - h(x_0)X + O(X^2) \]
\[ w = w_0 + w - w_0 = w_0(1 - W) \]
\[ 1 - y(x)w = W + h(x_0)w_0X + O(X^2) \]
\[
(1 - y(x)w)^{3/2} = \left( W + h(x_0)w_0X + O(X^2) \right)^{3/2}
\]
\[ = W^{3/2} \left( 1 + \frac{(3/2)h(x_0)w_0X}{W} + O \left( \frac{X^2}{W} \right) \right) \]
\[ = W^{3/2} + \frac{3}{2}h(x_0)w_0X W^{1/2} + O \left( X^2 W^{1/2} \right) \]
3. Local expansion around the singularity

\[ X W^{1/2} = \left(1 - \frac{x}{x_0}\right)^{\frac{1}{2}} \left(1 - \frac{w}{w_0}\right)^{\frac{1}{2}} \]

... Cauchy integration provides the asymptotic leading term

\[ \frac{1}{4\pi x_0^{n-1} w_0^{k-2}} n^{-\frac{3}{2}} k^{-\frac{3}{2}} \]
Random Planar Graphs

Conjecture for maximum degree $\Delta_n$

\[
\frac{\Delta_n}{\log n} \rightarrow \frac{1}{\log(1/q)} \quad \text{in probability}
\]

and

\[
\mathbb{E} \Delta_n \sim \frac{\log n}{\log(1/q)}
\]

where $q = 0.6734506...$ appear in the asymptotics of $p_k \sim c k^{-\frac{1}{2}} q^k$;

$1/\log(1/q) = 2.529464248...$
Random Planar Graphs

Degree Distribution

**Theorem [D. + Giménez + Noy]**

Let $p_{n,k}$ be the probability that a random vertex in a random planar graph $\mathcal{R}_n$ has degree $k$. Then the limit

$$p_k := \lim_{n \to \infty} p_{n,k}$$

exists. The probability generating function

$$p(w) = \sum_{k \geq 1} p_k w^k$$

can be explicitly computed; $p_k \sim c k^{-\frac{1}{2}} q^k$ for some $c > 0$ and $0 < q < 1$.

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<th>$p_1$</th>
<th>$p_2$</th>
<th>$p_3$</th>
<th>$p_4$</th>
<th>$p_5$</th>
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<td>0.0861805</td>
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Random Planar Graphs

Counting Generating Functions

\[ G(x, y) = \exp(C(x, y)), \]
\[ \frac{\partial C(x, y)}{\partial x} = \exp\left(\frac{\partial B}{\partial x}\left(x \frac{\partial C(x, y)}{\partial x}, y\right)\right), \]
\[ \frac{\partial B(x, y)}{\partial y} = \frac{x^2}{2} \frac{1 + D(x, y)}{1 + y}, \]
\[ M(x, D) = \log\left(\frac{1 + D}{1 + y}\right) - \frac{xD^2}{1 + xD}, \]
\[ M(x, y) = x^2y^2 \left(\frac{1}{1 + xy} + \frac{1}{1 + y} - 1 - \frac{(1 + U)^2(1 + V)^2}{(1 + U + V)^3}\right), \]
\[ U = xy(1 + V)^2, \]
\[ V = y(1 + U)^2. \]
Random Planar Graphs

Asymptotic enumeration of planar graphs

\[ b_n = b \cdot \rho_1^{-n} n^{-\frac{7}{2}} n! \left(1 + O \left(\frac{1}{n}\right)\right), \]
\[ c_n = c \cdot \rho_2^{-n} n^{-\frac{7}{2}} n! \left(1 + O \left(\frac{1}{n}\right)\right), \]
\[ g_n = g \cdot \rho_2^{-n} n^{-\frac{7}{2}} n! \left(1 + O \left(\frac{1}{n}\right)\right) \]

\[ \rho_1 = 0.03819..., \]
\[ \rho_2 = 0.03672841..., \]
\[ b = 0.3704247487... \cdot 10^{-5}, \]
\[ c = 0.4104361100... \cdot 10^{-5}, \]
\[ g = 0.4260938569... \cdot 10^{-5} \]
Random Planar Graphs

Generating functions for the degree distribution of planar graphs

\[ C^\bullet = \frac{\partial C}{\partial x} \quad \text{GF, where one vertex is marked} \]

\( w \) ... additional variable that counts the degree of the marked vertex

Generating functions:

\[ G^\bullet(x, y, w) \quad \text{all rooted planar graphs} \]
\[ C^\bullet(x, y, w) \quad \text{connected rooted planar graphs} \]
\[ B^\bullet(x, y, w) \quad \text{2-connected rooted planar graphs} \]
\[ T^\bullet(x, y, w) \quad \text{3-connected rooted planar graphs} \]
Random Planar Graphs

\[ G^\bullet(x, y, w) = \exp(C(x, y, 1)) \, C^\bullet(x, y, w), \]
\[ C^\bullet(x, y, w) = \exp(B^\bullet(xC^\bullet(x, y, 1), y, w)) , \]
\[ w \frac{\partial B^\bullet(x, y, w)}{\partial w} = xyw \exp \left( S(x, y, w) + \frac{1}{x^2D(x, y, w)} \right) \left( T^\bullet \left( x, D(x, y, 1), \frac{D(x, y, w)}{D(x, y, 1)} \right) \right) - 1 \]
\[ D(x, y, w) = (1 + yw) \exp \left( S(x, y, w) + \frac{1}{x^2D(x, y, w)} \right) \times \]
\[ \times T^\bullet \left( x, D(x, y, 1), \frac{D(x, y, w)}{D(x, y, 1)} \right) \left( T^\bullet \left( x, D(x, y, 1), \frac{D(x, y, w)}{D(x, y, 1)} \right) \right) - 1 \]
\[ S(x, y, w) = xD(x, y, 1) \, (D(x, y, w) - S(x, y, w)) , \]
\[ T^\bullet(x, y, w) = \frac{x^2y^2w^2}{2} \left( \frac{1}{1 + wy} + \frac{1}{1 + xy} - 1 - \right. \]
\[ \left. \quad (u + 1)^2 \left( -w_1(u, v, w) + (u - w + 1)\sqrt{w_2(u, v, w)} \right) \right) , \]
\[ u(x, y) = xy(1 + v(x, y))^2, \quad v(x, y) = y(1 + u(x, y))^2. \]
Degree Distribution

with polynomials $w_1 = w_1(u, v, w)$ and $w_2 = w_2(u, v, w)$ given by

$$w_1 = -uvw^2 + w(1 + 4v + 3uv^2 + 5v^2 + u^2 + 2u + 2v^3 + 3u^2v + 7uv) + (u + 1)^2(u + 2v + 1 + v^2),$$

$$w_2 = u^2v^2w^2 - 2wuv(2u^2v + 6uv + 2v^3 + 3uv^2 + 5v^2 + u^2 + 2u + 4v + 1) + (u + 1)^2(u + 2v + 1 + v^2)^2.$$
Random Planar Graphs

Singular structure of $B^*(x, 1, w)$

$$\frac{\partial B^*(x, 1, w)}{\partial w} = K(X, W) + \sqrt{L(X, W)}$$

$$X = \sqrt{1 - \frac{x}{x_0}}, \quad W = 1 - \frac{w}{w_0}$$

$$L(X, W) = X^3h_1(W) + X^2Wh_2(X, W) + 0 + W^3h_4(W)$$
Random Planar Graphs

Lemma 1.2

\[ f(x, w) = \sum_{n,k \geq 0} f_{n,k} x^n w^k = K(X, W) + \sqrt{L(X, W)}, \]

where \( X = \sqrt{1 - x/x_0} \) and \( W = 1 - w/w_0 \) and

\[ L(X, W) = X^3 h_1(W) + X^2 W h_2(X, W) + 0 + W^3 h_4(W) \]

with analytic functions \( K, h_1, h_2, h_4 \)

(+ some technical conditions)

\[ \Rightarrow f_{n,k} = c x_0^{-n} w_0^{-k} k^\frac{1}{2} n^{-\frac{5}{2}} \left( 1 + O \left( \frac{1}{k} \right) \right) \]
Random Planar Graphs

Work in progress...

• Generating functions for double rooting

• Singular structure of generating functions

• Lemma 2.2
Thank You for Your Attention!