GENERATING FUNCTIONS AND CENTRAL LIMIT THEOREMS

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FU Berlin, June 23, 2008

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Sums of independent random variables and powers of generating functions

Coin tossing

- $\mathbb{P}\{ct = \text{head}\} = \mathbb{P}\{ct = \text{tail}\} = \frac{1}{2}$.
- random variable $\xi = \mathbb{I}_{\{ct=\text{tail}\}} = \begin{cases} 1 & \text{if tail} \\ 0 & \text{if head} \end{cases}$
- n independent runs: $\xi_1, \xi_2, \ldots, \xi_n$, $\boxed{\mathbb{P}\{\xi_j = 1\} = \mathbb{P}\{\xi_j = 0\} = \frac{1}{2}}$.
- $\boxed{X_n = \xi_1 + \xi_2 + \cdots + \xi_n}$ ... the number of tails within n runs

$$\boxed{\mathbb{P}\{X_n = k\} = \frac{\binom{n}{k}}{2^n}}$$

Sums of independent random variables and powers of generating functions

Counting generating function

$a_n = 2^n$ ... total number of possible n-runs

$a_{n,k} = \binom{n}{k}$ ... the number of n-runs with k tails

$$A_n(u) = \sum_{k \geq 0} a_{n,k} u^k = \sum_{k \geq 0} \binom{n}{k} u^k = (1+u)^n \text{ ... counting gen. func.}$$

$$A_n(1) = \sum_{k \geq 0} a_{n,k} = a_n = (1+1)^n = 2^n$$

Sums of independent random variables and powers of generating functions

Probability generating function

$$
\begin{aligned}
\mathbb{E}\, u^{X_n} &= \sum_{k\geq 0} \mathbb{P}\{X_n = k\} \cdot u^k \\
&= \sum_{k\geq 0} \frac{1}{2^n}\binom{n}{k} \cdot u^k \\
&= \frac{(1+u)^n}{2^n} = \frac{A_n(u)}{A_n(1)}
\end{aligned}
$$

$$
\mathbb{P}\{X_n = k\} = \frac{a_{n,k}}{a_n} \quad \Longrightarrow \quad \boxed{\mathbb{E}\, u^{X_n} = \frac{A_n(u)}{A_n(1)}}
$$

Sums of independent random variables and powers of generating functions

Powers of probability generating functions

$$\mathbb{E}\, u^{\xi} = \frac{1}{2} + \frac{1}{2}u = \frac{1+u}{2}$$

$$\Longrightarrow \quad \mathbb{E}\, u^{X_n} = \mathbb{E}\, u^{\xi_1+\cdots+\xi_n}$$
$$= \mathbb{E}\left(u^{\xi_1}\cdots u^{\xi_n}\right)$$
$$= \mathbb{E}\left(u^{\xi_1}\right)\cdots\mathbb{E}\left(u^{\xi_n}\right) \qquad \xi_j \text{ independent !!!}$$
$$= \left(\frac{1+u}{2}\right)^n$$

Sums of independent random variables and powers of generating functions

General fact

$\boxed{X_n = \xi_1 + \xi_2 + \cdots + \xi_n}$, where the r.v.'s ξ_j are **iid***

$$\Longrightarrow \quad \boxed{\mathbb{E}\, u^{X_n} = \left(\mathbb{E}\, u^{\xi_1}\right)^n}$$

* **Notation.** "iid" ... independently and identically distributed

Sums of independent random variables and powers of generating functions

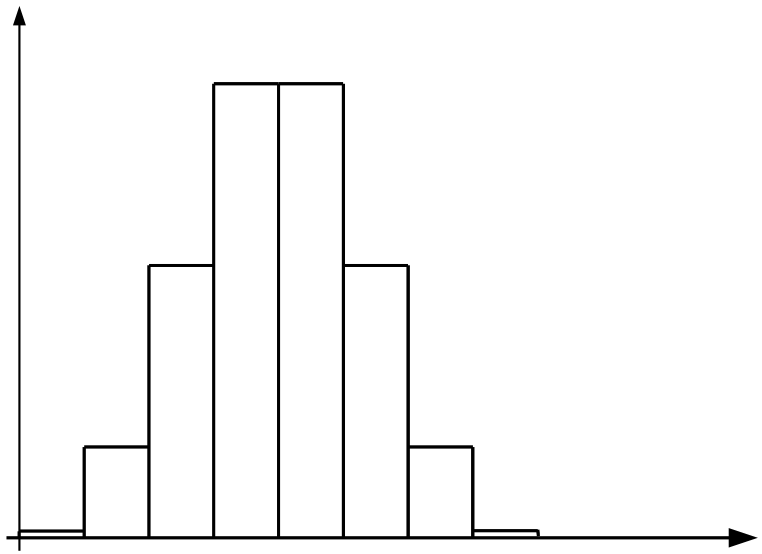
Relation to moment generating function $m_Z(v) = \mathbb{E}\, e^{vZ}$

$\mathbb{E}\,(Z^r)$ ... r-th moment of Z

$$\Longrightarrow \quad \sum_{r\geq 0} \mathbb{E}\,(Z^r)\frac{v^r}{r!} = \mathbb{E}\left(\sum_{r\geq 0}\frac{Z^r v^r}{r!}\right) = \mathbb{E}\, e^{vZ} = \mathbb{E}\, u^Z \quad \text{with } \boxed{u = e^v}.$$

A central limit theorem

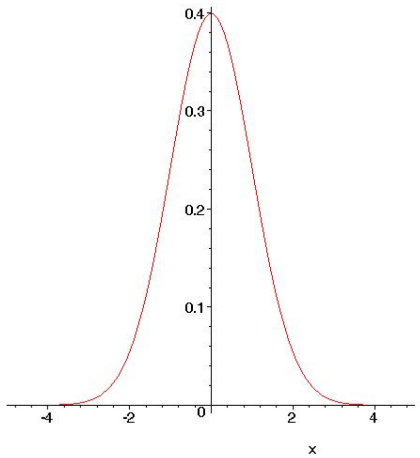
Binomial coefficients



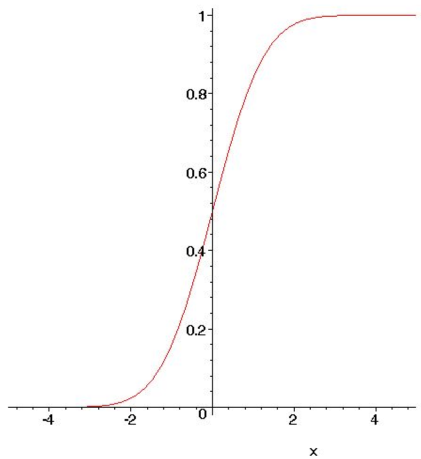
$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{2^n}{\sqrt{\pi n/2}} \exp\left(-\frac{(k-\frac{n}{2})^2}{n/2}\right) + O(2^n/n)$$

A central limit theorem

Standard normal distribution



density: $f(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}$.



normal distribution function: $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2}\, dt$

A central limit theorem

Normally distributed random variable

Definition

A random variable Z *has* **standard nomal distribution** $N(0,1)$ *if*

$$\boxed{\mathbb{P}\{Z \le x\} = \Phi(x)}.$$

A random variable Z *is* **normally distributed** *(or* **Gaussian***) with mean* μ *and variance* σ^2 *if its distribution function is given by*

$$\boxed{\mathbb{P}\{Z \le x\} = \Phi\left(\frac{x-\mu}{\sigma}\right)},$$

Notation. $\boxed{\mathcal{L}(Z) = N(\mu, \sigma^2)}$.

A central limit theorem

Moment generating function of $N(\mu, \sigma^2)$:

$$\boxed{m_Z(v) = \mathbb{E}\, e^{vZ} = e^{\mu v - \frac{1}{2}\sigma^2 v^2}}.$$

Characteristic function of $N(\mu, \sigma^2)$:

$$\boxed{\varphi_Z(t) = \mathbb{E}\, e^{itZ} = e^{i\mu t - \frac{1}{2}\sigma^2 t^2}}.$$

Standard normal distribution: $\mu = 0$, $\sigma^2 = 1$

$$\boxed{\mathbb{E}\, e^{vZ} = e^{\frac{1}{2}v^2}}, \qquad \boxed{\mathbb{E}\, e^{itZ} = e^{-\frac{1}{2}t^2}}$$

A central limit theorem

Definition *We say, that a sequence of random variables X_n satisfies* **a central limit theorem** *with (scaling) mean μ_n and (scaling) variance σ_n^2 if*

$$\boxed{\mathbb{P}\{X_n \le \mu_n + x \cdot \sigma_n\} = \Phi(x) + o(1)}$$

as $n \to \infty$.

Example. $X_n =$ number of tails in n runs of coin tossing:

$$\mathbb{P}\{X_n \le n/2 + x \cdot \sqrt{n/4}\} = \sum_{k \le n/2 + x\cdot\sqrt{n/4}} \frac{1}{2^n}\binom{n}{k}$$

$$\sim \sum_{k \le n/2 + x\cdot\sqrt{n/4}} \frac{1}{\sqrt{\pi n/2}} \exp\left(-\frac{(k-\frac{n}{2})^2}{n/2}\right) \sim \Phi(x).$$

X_n satisfies a central limit theorem with mean $\frac{n}{2}$ and variance $\frac{n}{4}$.

Central Limit Theorem

Definition *Weak convergence:*

$$\boxed{X_n \xrightarrow{\text{d}} X} \quad :\Longleftrightarrow \quad \boxed{\lim_{n\to\infty} \mathbb{P}\{X_n \le x\} = \mathbb{P}\{X \le x\}}$$

for all points of continuity
of $F_X(x) = \mathbb{P}\{X \le x\}$

Reformulation:

X_n satisfies **a central limit theorem** with (scaling) mean μ_n and (scaling) variance σ_n^2 is the same as

$$\boxed{\frac{X_n - \mu_n}{\sigma_n} \xrightarrow{\text{d}} N(0,1)}.$$

A central limit theorem

Weak convergence via moment generating functions

$$\boxed{\lim_{n\to\infty} \mathbb{E}\, e^{vX_n} = \mathbb{E}\, e^{vX} \quad (v \in \mathbb{R}) \quad \Longrightarrow \quad X_n \xrightarrow{\mathsf{d}} X}$$

Moreover, we have convergence of all moments: $\mathbb{E}\,(X_n^r) \to \mathbb{E}\,(X^r)$.

Recall: $\mathbb{E}\, e^{vX_n} = \mathbb{E}\,((e^v)^{X_n}) = \mathbb{E}\, u^{X_n}$ for $u = e^v$.

A central limit theorem

Weak convergence via characteristic functions (Levy's Criterion)

$$\boxed{\lim_{n\to\infty} \mathbb{E}\, e^{itX_n} = \mathbb{E}\, e^{itX} \quad (t \in \mathbb{R}) \quad \Longleftrightarrow \quad X_n \xrightarrow{\mathsf{d}} X}$$

Moreover, if for all $t \in \mathbb{R}$

$$\psi(t) := \lim_{n\to\infty} \mathbb{E}\, e^{itX_n}$$

exists and $\psi(t)$ is continous at $t = 0$ then $\psi(t)$ is the characteristic function of a random variable X for which we have $X_n \xrightarrow{\mathsf{d}} X$.

Central Limit Theorem

Theorem

$\xi_1, \xi_2, \ldots$ iid, $\mathbb{E}\,\xi_i^2 < \infty$, $X_n = \xi_1 + \xi_2 + \ldots + \xi_n$

$$\Longrightarrow \quad \boxed{\frac{X_n - \mathbb{E}\,X_n}{\sqrt{\mathbb{V}\,X_n}} \xrightarrow{\text{d}} N(0,1)}$$

Remark. $\iff \ \mathbb{P}\{X_n \le \mathbb{E}\,X_n + x\sqrt{\mathbb{V}\,X_n}\} = \Phi(x) + o(1)$.

Proof

$\mu = \mathbb{E}\,\xi_i$, $\sigma^2 = \mathbb{V}\,\xi_i = \mathbb{E}\,(\xi_i^2) - (\mathbb{E}\,\xi_i)^2 \quad \Longrightarrow \mathbb{E}\,X_n = n\mu$, $\mathbb{V}\,X_n = n\sigma^2$.

Central Limit Theorem

$$\varphi_{\xi_i}(t) = \mathbb{E}\, e^{it\xi_i} = e^{it\mu - \frac{1}{2}\sigma^2 t^2\,(1+o(1))} \qquad (t \to 0)$$

$$\varphi_{X_n}(t) = \varphi_{\xi_i}(t)^n$$

$$Z_n := (X_n - \mu n)/\sqrt{\sigma^2 n}$$

$$\begin{aligned}
\implies \boxed{\varphi_{Z_n}(t)} &= \mathbb{E}\, e^{itZ_n} \\
&= e^{-it\sqrt{n}\mu/\sigma} \cdot \mathbb{E}\, \left(e^{(it/(\sqrt{n}\sigma))(\xi_1+\cdots+\xi_n)}\right) \\
&= e^{-it\sqrt{n}\mu/\sigma} \cdot \boxed{\left(\mathbb{E}\, e^{(it/(\sqrt{n}\sigma)\xi_1}\right)^n} \\
&= e^{-it\sqrt{n}\mu/\sigma} \cdot e^{it\sqrt{n}\mu/\sigma - \frac{1}{2}t^2\,(1+o(1))} \\
&= e^{-\frac{1}{2}t^2\,(1+o(1))} \to \boxed{e^{-\frac{1}{2}t^2}}.
\end{aligned}$$

+ Levy's criterion.

A central limit theorem

Quasi-Power Theorem (Hwang)

Let X_n be a sequence of random variables with the property that

$$\boxed{\mathbb{E}\, u^{X_n} = A(u)\cdot B(u)^{\lambda_n}\cdot\left(1+O\left(\frac{1}{\phi_n}\right)\right)}$$

holds uniformly in a complex neighborhood of $u=1$, $\boxed{\lambda_n\to\infty}$ and $\boxed{\phi_n\to\infty}$, and $A(u)$ and $B(u)$ are analytic functions in a neighborhood of $u=1$ with $A(1)=B(1)=1$. Set

$$\mu = B'(1) \quad \text{and} \quad \sigma^2 = B''(1)+B'(1)-B'(1)^2.$$

$$\Longrightarrow \quad \boxed{\mathbb{E}\, X_n = \mu\lambda_n + O\left(1+\lambda_n/\phi_n\right)}, \quad \boxed{\mathbb{V}\, X_n = \sigma^2\lambda_n + O\left(1+\lambda_n/\phi_n\right)},$$

$$\boxed{\frac{X_n-\mathbb{E}\,X_n}{\sqrt{\mathbb{V}\,X_n}} \xrightarrow{\text{d}} N(0,1)} \quad (\sigma^2\neq 0).$$

Bivariate generating functions

Bivariate counting generating function

$$A(x,u) = \sum_{n,k\geq 0} \binom{n}{k} u^k x^n = \sum_{n\geq 0} (1+u)^n x^n = \frac{1}{1-x(1+u)}.$$

Observation: this is a **rational function**!

Bivariate generating functions

Rational functions

$P(x,u), Q(x,u)$ polynomials:

$$A(x,u) = \sum_{n,k \geq 0} a_{n,k}\, u^k\, x^n = \frac{P(x,u)}{Q(x,u)}$$

Assumption: factorization of denominator

$$Q(x,u) = \prod_{j=1}^{r} \left(1 - \frac{x}{\rho_j(u)}\right)$$

with

$$\boxed{|\rho_1(u)| < \max_{2 \leq j \leq r} |\rho_j(u)|} \qquad \text{for } |u-1| < \varepsilon.$$

Bivariate generating functions

Central limit theorem for rational functions

Suppose that $A(x,u) = \sum a_{n,k}\, u^k\, x^n$ with $a_{n,k} \ge 0$ is **rational** and satisfies the assumptions from above.

Let X_n be a sequence of random variables with

$$\mathbb{P}\{X_n = k\} = \frac{a_{n,k}}{a_n}$$

with $a_n = \sum_k a_{n,k}$.

Then X_n satisfies a **central limit theorem** with

$$\mu_n = -n\frac{\rho_1'(1)}{\rho_1(1)} \quad \text{and} \quad \sigma_n^2 = n\left(-\frac{\rho_1''(1)}{\rho_1(1)} - \frac{\rho_1'(1)}{\rho_1(1)} + \frac{\rho_1'(1)^2}{\rho_1(1)^2}\right).$$

Bivariate generating functions

Proof

Partial fraction decomposition:

$$A(x,u) = \frac{C_1(u)}{1 - x/\rho_1(u)} + \cdots + \frac{C_r(u)}{1 - x/\rho_r(u)}$$

$$\Longrightarrow \quad A_n(u) = \sum_{k \ge 0} a_{n,k}\, u^k = C_1(u)\rho_1(u)^{-n} + \cdots + C_r(u)\rho_r(u)^{-n} \sim C_1(u)\rho_1(u)^{-n}$$

$$\Longrightarrow \quad \boxed{\mathbb{E}\, u^{X_n} = \frac{A_n(u)}{A_n(1)} \sim \frac{C_1(u)}{C_1(1)} \left(\frac{\rho_1(1)}{\rho_1(u)}\right)^n}$$

$\Longrightarrow$ **central limit theorem**.

Bivariate generating functions

Integer compositions

$3 = 1+1+1 = 2+1 = 1+2 = 3$ ... 4 compositions of 3.

$a_n =$ number of compositions of n, $A(x) = \sum a_n x^n$:

$$\boxed{A(x) = 1 + A(x)(x + x^2 + x^3 + \cdots) = 1 + A(x)\frac{x}{1-x}}.$$

$$\implies \quad A(x) = \frac{1}{1 - \frac{x}{1-x}} = \frac{1-x}{1-2x}$$

$$\implies \quad \boxed{a_n = 2^{n-1}}$$

Bivariate generating functions

Integer compositions

$a_{n,k}$ = number of integer composition of n with k summands

$A(x,u) = \sum a_{n,k} u^k x^n$:

$$\boxed{A(x,u) = 1 + uA(x,u)(x + x^2 + x^3 + \cdots) = 1 + A(x,u)\frac{xu}{1-x}}.$$

$$\Longrightarrow \quad A(x,u) = \frac{1}{1 - \frac{xu}{1-x}} = \frac{1-x}{1-x(1+u)}$$

$\Longrightarrow$ **central limit theorem** with $\mu_n = \frac{n}{2}$ and $\sigma^2 = \frac{n}{4}$.

Bivariate generating functions

Systems of linear equations

Suppose, that several generating functions

$$A_1(x,u) = \sum_{n,k} a_{1;n,k} u^k x^n, \ldots, A_r(x,u) = \sum_{n,k} a_{r;n,k} u^k x^n$$

satisfy a **linear system of equations**.

Then all generating functions $A_j(x,u)$ are rational and a **central limit theorem** for corresponding random variables is **expected**.

Bivariate generating functions

Meromorphic functions

The function $A(x,u)$ is meromorphic in x when u is considered as a parameter and there exists a dominant root $\rho_1(u)$ such that (locally)

$$\boxed{A(x,u) = \frac{C(x,u)}{1 - \frac{x}{\rho_1(u)}}}$$

$$\Longrightarrow \quad A_n(u) \sim C(\rho_1(u),u) \cdot \rho_1(u)^{-n}$$

$$\Longrightarrow \quad \boxed{\mathbb{E}\, u^{X_n} \sim \frac{C(\rho_1(u),u)}{C(\rho_1(1),1)} \left(\frac{\rho_1(1)}{\rho_1(u)}\right)^n}$$

$\Longrightarrow$ **central limit theorem**.

Bivariate generating functions

Number of cycles in permutations

$p_{n,k}$ = number of permutations of $\{1, 2, \ldots, n\}$ with k cycles

$$\widehat{P}(x,u) = \sum_{n,k \geq 0} p_{n,k} \cdot u^k \cdot \frac{x^n}{n!} = e^{u \cdot \log \frac{1}{1-x}} = \frac{1}{(1-x)^u}$$

Remark: $p_{n,k} = (-1)^{n-k} s_{n,k}$, where $s_{n,k}$ are the **Stirling number of the first kind**.

Excursion: Singularity Analysis

Lemma 1 *Suppose that*

$$\boxed{y(x) = (1 - x/x_0)^{-\alpha}}.$$

Then

$$\boxed{y_n = (-1)^n \binom{-\alpha}{n} x_0^{-n} = \frac{n^{\alpha-1}}{\Gamma(\alpha)} x_0^{-n} + \mathcal{O}\left(n^{\alpha-2}\right) x_0^{-n}.}$$

Remark: This asymptotic expansion is uniform in α if α varies in a compact region of the complex plane.

Excursion: Singularity Analysis

Lemma 2 (Flajolet and Odlyzko) *Let*

$$y(x) = \sum_{n\ge 0} y_n\, x^n$$

be analytic in a region

$$\Delta = \{x : |x| < x_0 + \eta,\ |\arg(x - x_0)| > \delta\},$$

$x_0 > 0$, $\eta > 0$, $0 < \delta < \pi/2$.

Suppose that for some real α

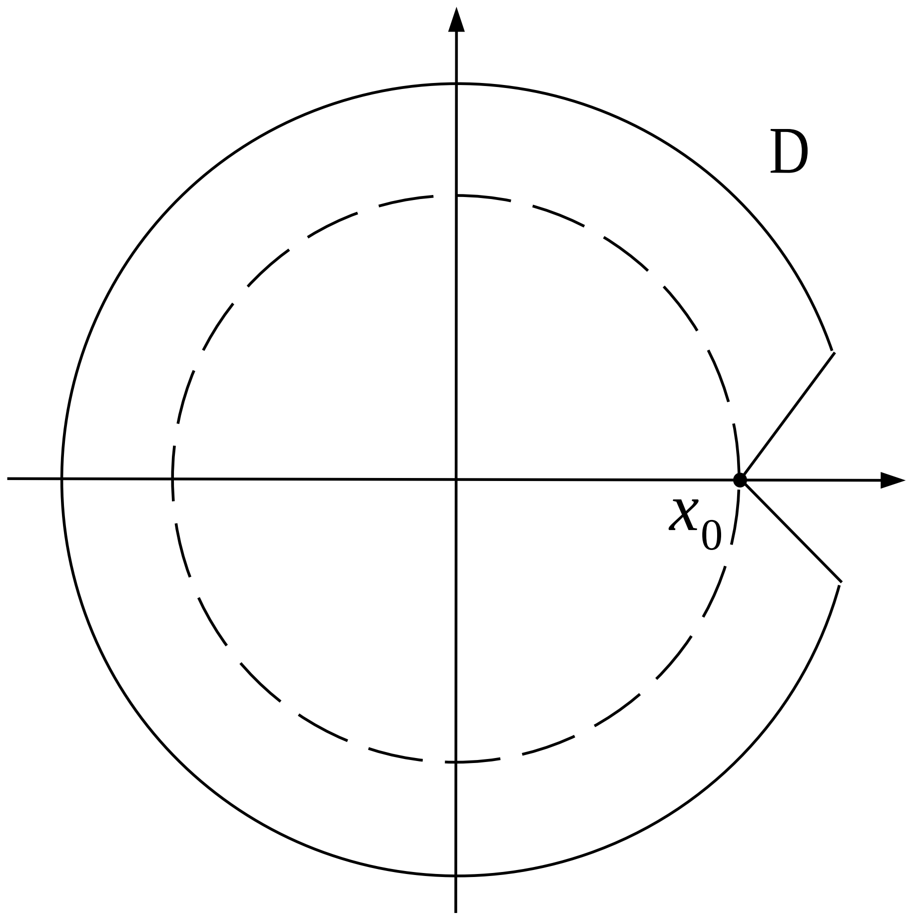
$$\boxed{y(x) = \mathcal{O}\left((1 - x/x_0)^{-\alpha}\right) \qquad (x \in \Delta).}$$

Then

$$\boxed{y_n = \mathcal{O}\left(x_0^{-n} n^{\alpha-1}\right).}$$

Excursion: Singularity Analysis

Δ-region



Bivariate generating functions

Number of cycles in permutations (continued)

$$\widehat{P}(x,u) = e^{u \log \frac{1}{1-x}} = \frac{1}{(1-x)^u}$$

$$\Longrightarrow \quad p_n(u) = \sum_{k \geq 0} p_{n,k} u^k$$

$$\sim n! \frac{n^{u-1}}{\Gamma(u)}$$

$$= n! \frac{e^{(u-1)\log n}}{\Gamma(u)}$$

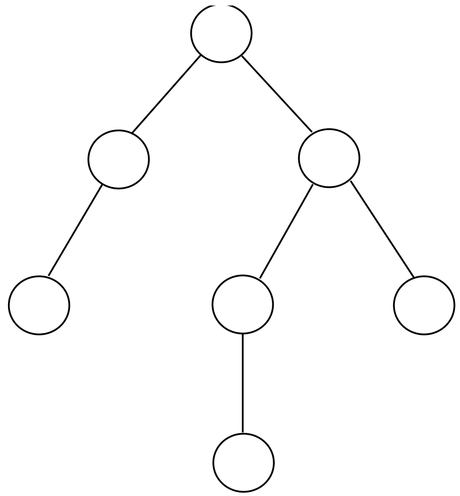
$$\Longrightarrow \quad \boxed{\mathbb{E}\, u^{X_n} \sim \frac{1}{\Gamma(u)} (e^{u-1})^{\log n}}$$

$\Longrightarrow$ **central limit theorem** with $\mu_n = \log n$ and $\sigma_n^2 = \log n$.

Generalization: Exp-Log-Schemes: $F(x,u) = e^{h(u) \log \frac{1}{1-x} + R(x,u)}$.

Bivariate generating functions

Catalan trees g_n = number of Catalan trees of size n.



$$\boxed{G(x) = x(1 + G(x) + G(x)^2 + \cdots) = \frac{x}{1 - G(x)}}$$

$$G(x) = \frac{1 - \sqrt{1-4x}}{2} \quad \Longrightarrow \quad \boxed{g_n = \frac{1}{n}\binom{2n-2}{n-1}}.$$

(Catalan numbers)

Bivariate generating functions

Catalan trees with singularity analysis

$$G(x) = \frac{1-\sqrt{1-4x}}{2} = \frac{1}{2} - \frac{1}{2}\sqrt{1-4x}$$

$$\implies \quad g_n \sim -\frac{1}{2} \cdot \frac{4^n n^{-3/2}}{\Gamma(-\frac{1}{2})} = \frac{4^{n-1}}{\sqrt{\pi} \cdot n^{3/2}}$$

Bivariate generating functions

Number of leaves of Catalan trees

$g_{n,k}$ = number of Catalan trees of size n with k leaves.

$$\boxed{G(x,u) = xu + x(G(x,u) + G(x,u)^2 + \cdots = xu + \frac{xG(x,u)}{1 - G(x,u)}}$$

$$\Longrightarrow \quad G(x,u) = \frac{1}{2}\left(1 + (u-1)x - \sqrt{1 - 2(u+1)x + (u-1)^2x^2}\right)$$

$$\Longrightarrow \quad \boxed{G(x,u) = g(x,u) - h(x,u)\sqrt{1 - \frac{x}{\rho(u)}}}$$

for certain analytic function $g(x,u), h(x,u)$, and $\rho(u)$.

Bivariate generating functions

Application of singularity analysis

Considering u as a parameter we get

$$G_n(u) = \sum_{k \geq 0} g_{n,k} u^k \sim \frac{h(\rho(u), u) \cdot \rho(u)^{-n} \cdot n^{-3/2}}{2\sqrt{\pi}}$$

$$\Longrightarrow \quad \boxed{\mathbb{E}\, u^{X_n} = \frac{G_n(u)}{G_n(1)} \sim \frac{h(\rho(u), u)}{h(\rho(1), 1)} \left(\frac{\rho(1)}{\rho(u)}\right)^n}$$

$\Longrightarrow$ **central limit theorem** with $\mu_n = \frac{n}{2}$ and $\sigma_n^2 = \frac{n}{8}$

Bivariate generating functions

Cayley trees

$r_{n,k}$ = number of Cayley trees of size n with k leaves

$$R(x,u) = \sum_{n,k \geq 0} r_{n,k}\, u^k \frac{x^n}{n!}$$

$$\Longrightarrow \quad \boxed{R(x,u) = xe^{R(x,u)} + x(u-1)}$$

$$\Longrightarrow \quad ?????$$

Functional equations

Catalan trees: $G(x,u) = xu + xG(x,u)/(1 - G(x,u))$

Cayley trees: $R(x,u) = xe^{R(x,u)} + x(u-1)$

Recursive structure leads to functional equation for gen. func.:

$$\boxed{\boxed{A(x,u) = \Phi(x,u,A(x,u))}}$$

Functional equations

Linear functional equation: $\Phi(x, u, a) = \Phi_0(x, u) + a\Phi_1(x, u)$

$$\Longrightarrow \quad A(x, u) = \frac{\Phi_0(x, u)}{1 - \Phi_1(x, u)}$$

Usually techniques similar to those used for rational resp. meromorphic functions work and prove a **central limit theorem**.

Functional equations

Non-linear functional equations: $\Phi_{aa}(x,u,a) \neq 0$.

Suppose that $\boxed{A(x,u) = \Phi(x,u,A(x,u))}$, where $\Phi(x,u,a)$ has a power series expansion at $(0,0,0)$ with non-negative coefficients and $\Phi_{aa}(x,u,a) \neq 0$.

Let $x_0 > 0$, $a_0 > 0$ (inside the region of convergence) satisfy the system of equations:

$$\boxed{a_0 = \Phi(x_0, 1, a_0), \quad 1 = \Phi_a(x_0, 1, a_0)}.$$

Then there exists analytic function $g(x,u), h(x,u)$, and $\rho(u)$ such that locally

$$\boxed{A(x,u) = g(x,u) - h(x,u)\sqrt{1 - \frac{x}{\rho(u)}}}.$$

Functional equations

Idea of the Proof.

Set $F(x,u,a) = \Phi(x,u,a) - a$. Then we have

$$\begin{aligned} F(x_0,1,a_0) &= 0 \\ F_a(x_0,1,a_0) &= 0 \\ F_x(x_0,1,a_0) &\neq 0 \\ F_{aa}(x_0,1,a_0) &\neq 0. \end{aligned}$$

Weierstrass preparation theorem implies that there exist analytic functions $H(x,u,a)$, $p(x,u)$, $q(x,u)$ with $H(x_0,1,a_0) \neq 0$, $p(x_0,1) = q(x_0,1) = 0$ and

$$F(x,u,a) = H(x,u,a)\Big((a-a_0)^2 + p(x,u)(a-a_0) + q(x,u)\Big).$$

Functional equations

$$F(x,u,a)=0 \quad \Longleftrightarrow \quad (a-a_0)^2+p(x,u)(a-a_0)+q(x,u)=0.$$

Consequently

$$\begin{aligned} A(x,u) &= a_0-\frac{p(x,u)}{2}\pm\sqrt{\frac{p(x,u)^2}{4}-q(x,u)} \\ &= \boxed{g(x,u)-h(x,u)\sqrt{1-\frac{x}{\rho(u)}}}, \end{aligned}$$

where we write

$$\frac{p(x,u)^2}{4}-q(x,u)=K(x,u)(x-\rho(u))$$

which is again granted by the Weierstrass preparation theorem and we set

$$g(x,u)=a_0-\frac{p(x,u)}{2} \quad \text{and} \quad h(x,u)=\sqrt{-K(x,u)\rho(u)}.$$

Functional equations

A central limit theorem for functional equations

Suppose that $\boxed{A(x,u) = \Phi(x,u,A(x,u))}$, where $\Phi(x,u,a)$ has a power series expansion at $(0,0,0)$ with non-negative coefficients and $\Phi_{aa}(x,u,a) \neq 0$ (+ *minor* technical conditions). Set

$$\mu = \frac{x_0 \Phi_x(x_0,1,a_0)}{\Phi(x_0,1,a_0)} \quad \text{and} \quad \sigma^2 = \text{``}long\ formula''.$$

Then then random variable X_n defined by $\mathbb{P}\{X_n = k\} = a_{n,k}/a_n$ satisfies a **central limit theorem** with

$$\mu_n = n\mu \quad \text{and} \quad \sigma_n^2 = n\sigma^2.$$

Functional equations

Number of leaves in Cayley trees $(R(x) = xe^{T(x)})$

$$R(x,u) = xe^{R(x,u)} + x(u-1)$$

$$x_0 = \frac{1}{e}, \quad r_0 = R(x_0) = 1.$$

$\Longrightarrow$ **central limit theorem** with $\mu_n = \frac{1}{e}\, n$ and $\sigma^2 = \frac{e-2}{e^2}\, n$.

Functional equations

Systems of functional equations

Suppose, that several generating functions

$$A_1(x,u) = \sum_{n,k} a_{1;n,k} u^k x^n, \ \ldots \ , A_r(x,u) = \sum_{n,k} a_{r;n,k} u^k x^n$$

satisfy a **system of non-linear equations**

$$\boxed{A_j(x,u) = \Phi_j(x,u,A_1(x,u),\ldots,A_r(x,u))},$$

where $\Phi_j(x,u,a_1,\ldots,a_r)$ is non-linear in $a_1,\ldots,a_r$ for some j and has a power series expansion at $(0,0,0)$ with non-negative coefficients (for all j).

Let $x_0 > 0$, $\mathbf{a}_0 = (a_{0,0},\ldots,a_{r,0}) > 0$ (inside the region of convergence) satisfy the system of equations: ($\mathbf{\Phi} = (\Phi_1,\ldots,\Phi_r)$)

$$\boxed{\mathbf{a}_0 = \mathbf{\Phi}(x_0,1,\mathbf{a}_0), \quad 0 = \det(\mathbb{I} - \mathbf{\Phi}_{\mathbf{a}}(x_0,1,\mathbf{a}_0)}.$$

Functional equations

Suppose further, that the **dependency graph** of the system $\mathbf{a} = \Phi(x, u, \mathbf{a})$ is **strongly connected**.

Then there exists analytic function $g_j(x,u), h_j(x,u)$, and $\rho(u)$ (that is **independent of** j) such that locally

$$\boxed{A_j(x,u) = g_j(x,u) - h_j(x,u)\sqrt{1 - \frac{x}{\rho(u)}}}.$$

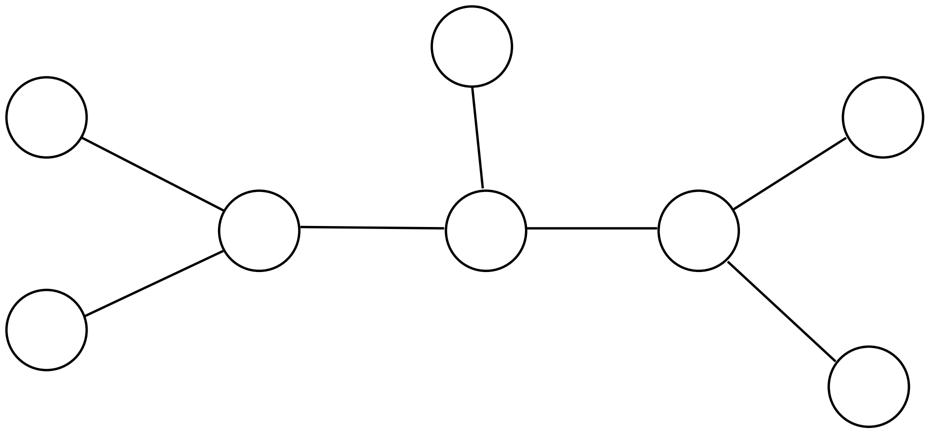
If $\boxed{A(x,u) = \sum_{n,k} a_{n,k} x^n u^k = F(x,u,A_1(x,u),\ldots,A_j(x,u))}$ (for some analytic function F satisfying certain conditions) then then random variable X_n defined by $\boxed{\mathbb{P}\{X_n = k\} = a_{n,k}/a_n}$ satisfies a **central limit theorem** with

$$\mu_n = n\mu \quad \text{and} \quad \sigma_n^2 = n\sigma^2,$$

where μ and σ^2 can be computed.

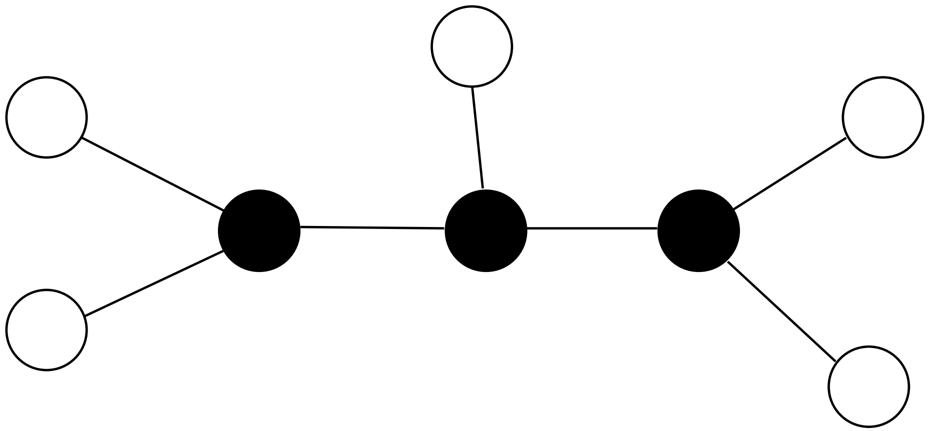
Patterns in Trees

Pattern $\mathcal{M}$

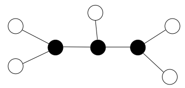


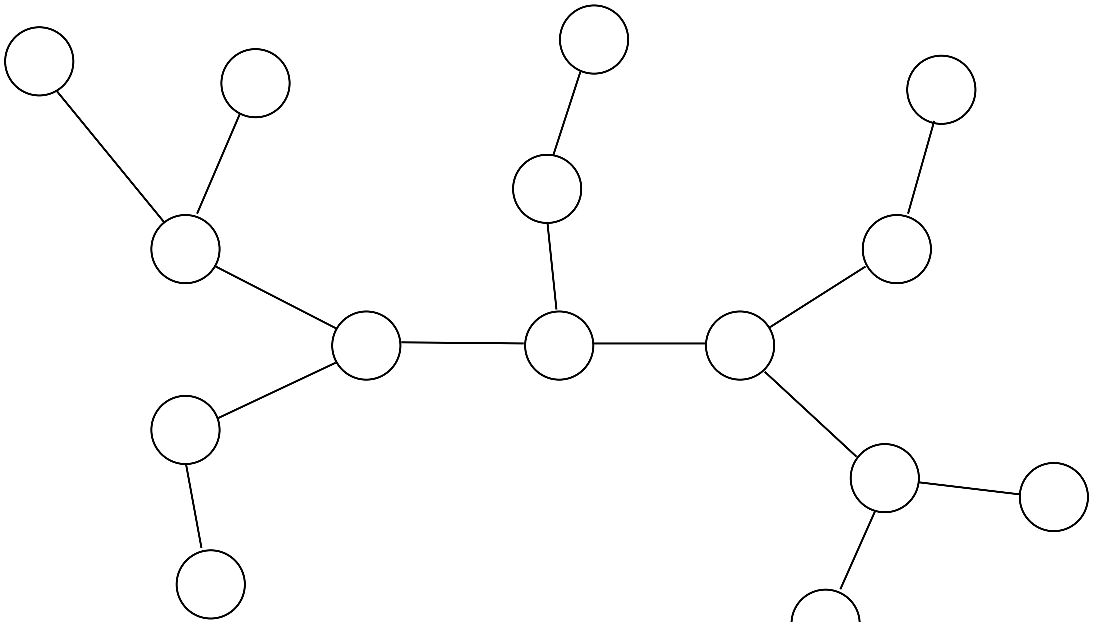
Patterns in Trees

Pattern $\mathcal{M}$

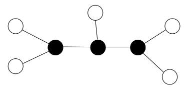


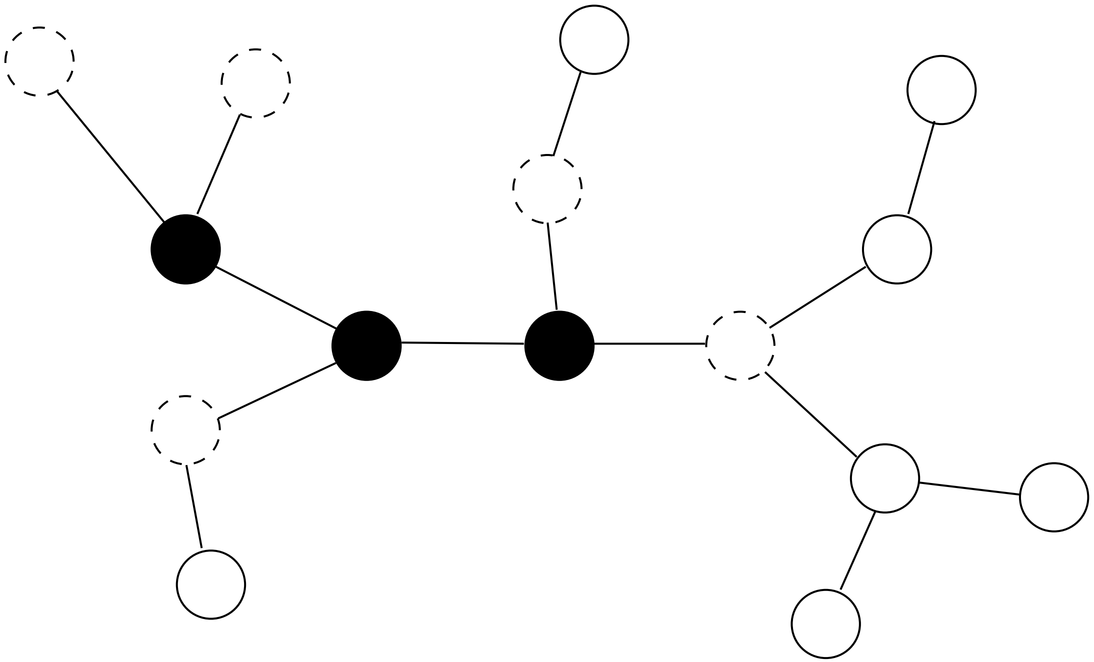
Patterns in Trees

Occurence of a pattern $\mathcal{M}$ 

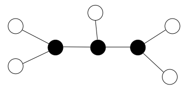


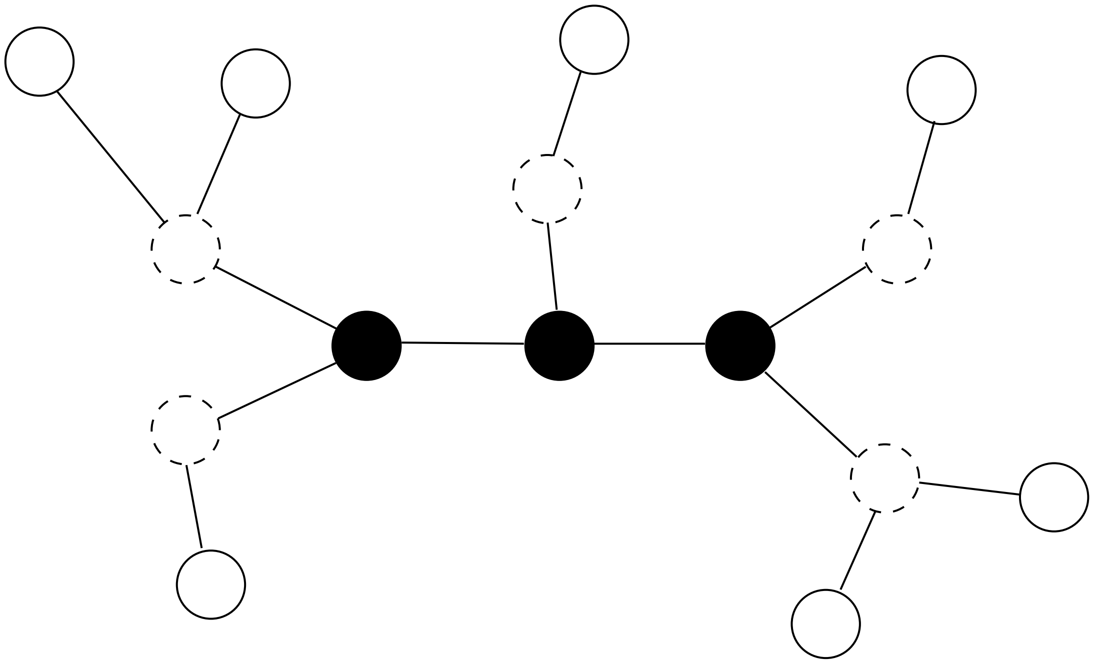
Patterns in Trees

Occurence of a pattern $\mathcal{M}$ 

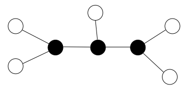


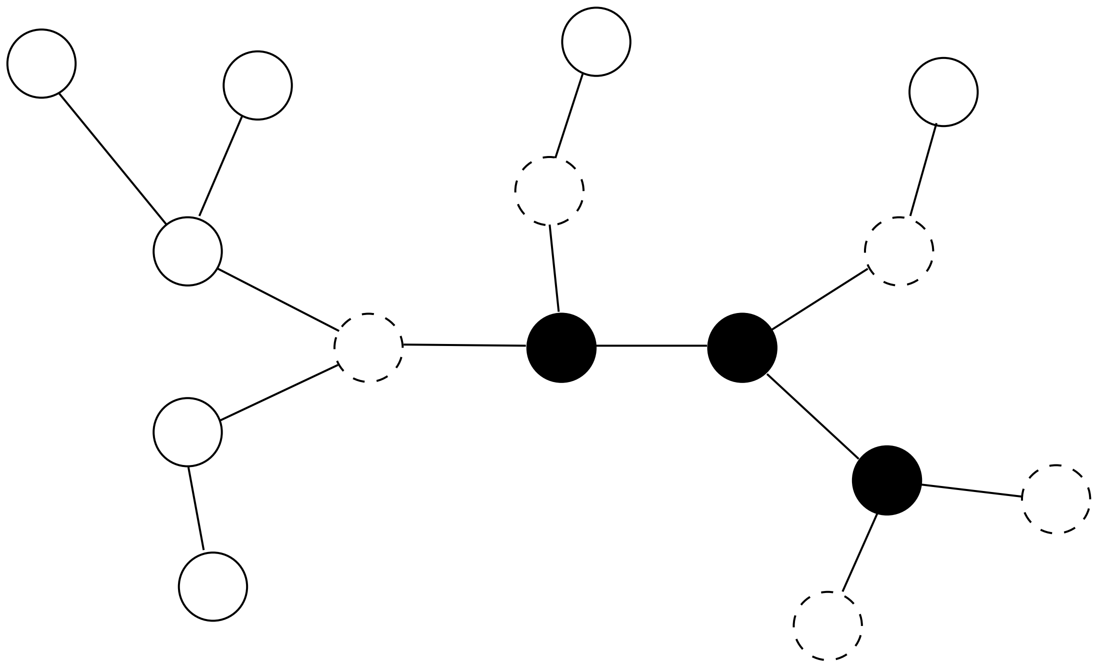
Patterns in Trees

Occurence of a pattern $\mathcal{M}$ 

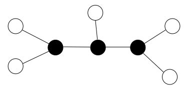


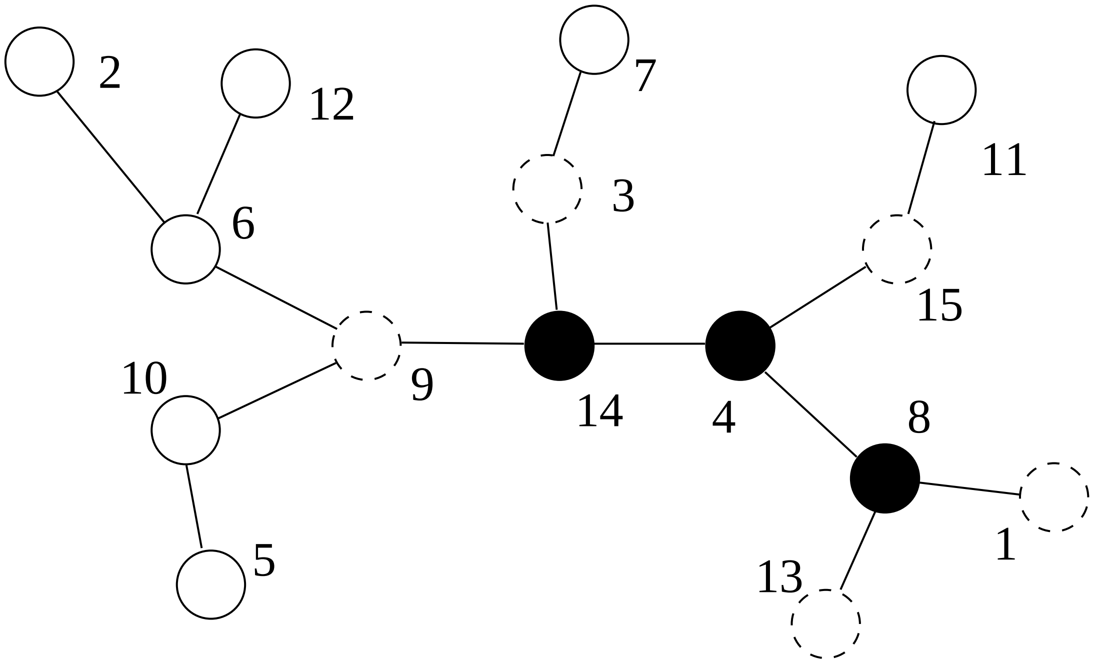
Patterns in Trees

Occurence of a pattern $\mathcal{M}$ 



Patterns in Trees

Occurence of a pattern $\mathcal{M}$  in a labelled tree



Patterns in Trees

Cayley's formula

$r_n = n^{n-1}$ ... number of **rooted** labelled trees with n nodes

$t_n = n^{n-2}$ ... number of labelled trees with n nodes

Generating functions

$$R(x) = \sum_{n\geq 1} r_n \frac{x^n}{n!}:$$

$$\boxed{R(x) = xe^{R(x)}}$$

$$t(x) = \sum_{n\geq 1} t_n \frac{x^n}{n!}:$$

$$\boxed{T(x) = R(x) - \frac{1}{2}R(x)^2}$$

Patterns in Trees

Theorem

$\mathcal{M}$ ... be a given finite tree.

X_n ... number of occurences of of $\mathcal{M}$ in a labelled tree of size n

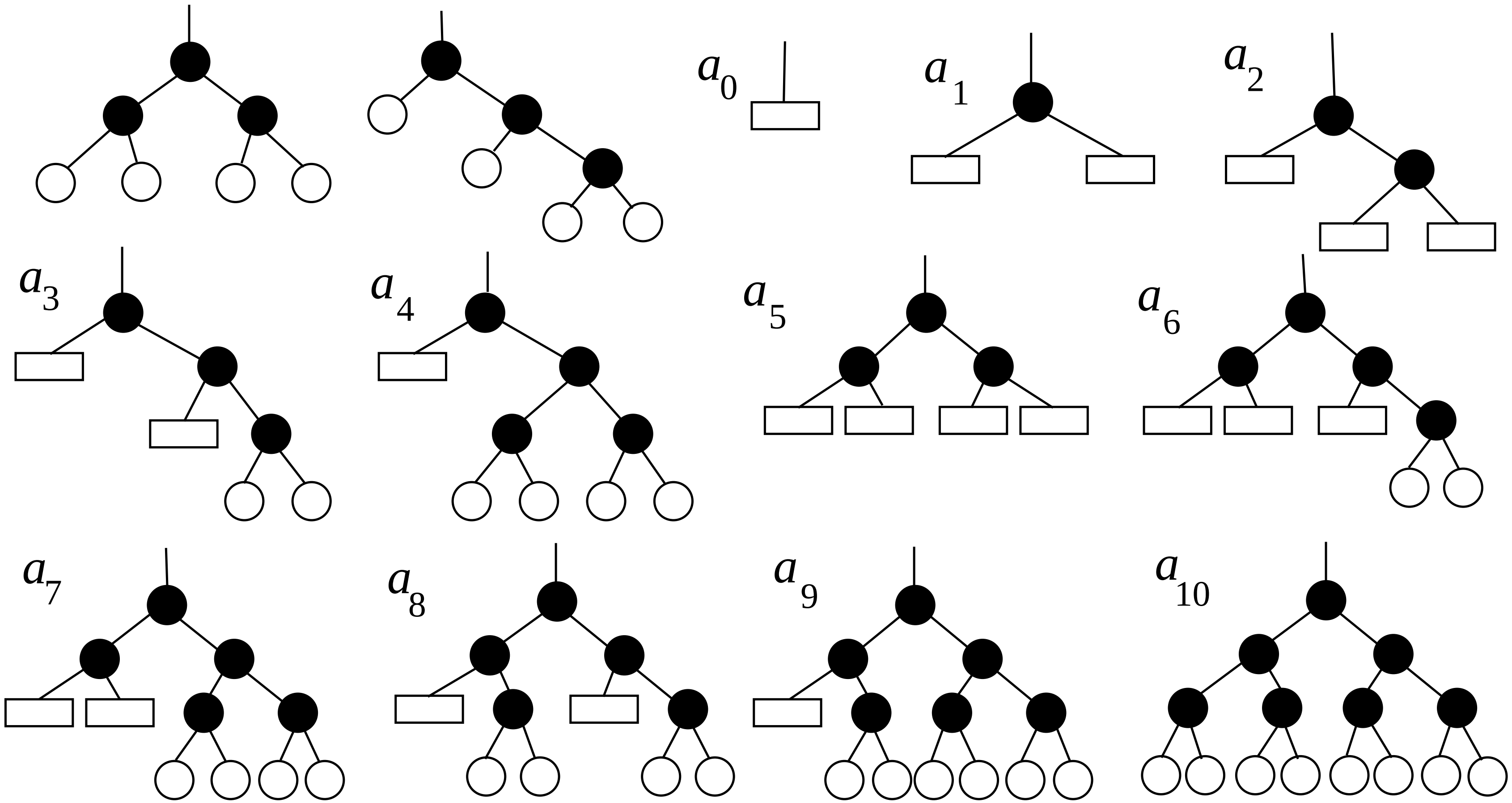
$\Longrightarrow$ X_n satisfies a **central limit theorem** with

$$\mathbb{E}\, X_n \sim \mu n \quad \text{and} \quad \mathbb{V}\, X_n \sim \sigma^2 n.$$

$\mu > 0$ and $\sigma^2 \geq 0$ depend on the pattern $\mathcal{M}$ and can be computed explicitly and algorithmically and can be represented as polynomials (with rational coefficients) in $1/e$.

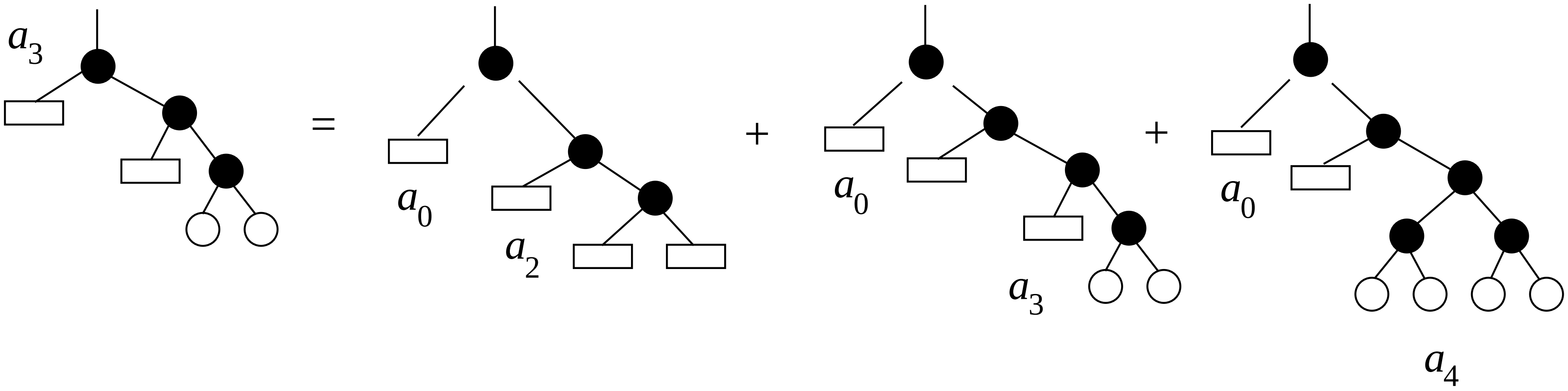
Patterns in Trees

Partition of trees in classes ( ... out-degree different from 2)



Patterns in Trees

Recurrences $\boxed{A_3 = xA_0A_2 + xA_0A_3 + xA_0A_4}$

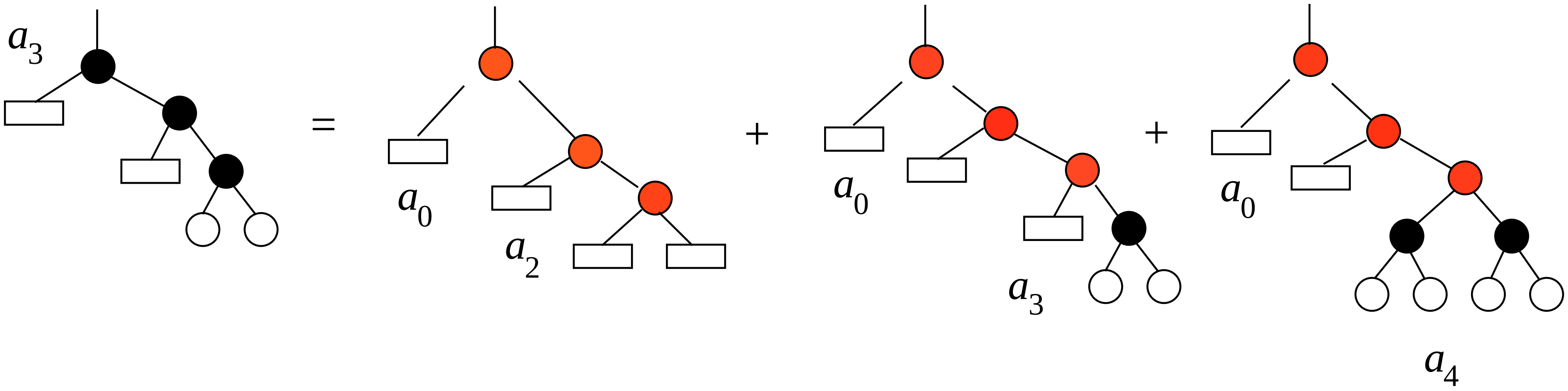


$$\boxed{A_j(x) = \sum_{n,k} a_{j;n} \frac{x^n}{n!}}$$

$a_{j;n}$ ... number of trees of size n in class j

Patterns in Trees

Recurrences $A_3 = xuA_0A_2 + xuA_0A_3 + xuA_0A_4$



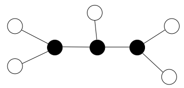
$$A_j(x, u) = \sum_{n,k} a_{j;n,k} \frac{x^n}{n!} u^k$$

$a_{j;n,k}$ ... number of trees of size n in class j with k occurences of $\mathcal{M}$

Patterns in Trees

$$
\begin{aligned}
A_0 &= A_0(x,u) = x + x\sum_{i=0}^{10} A_i + x\sum_{n=3}^{\infty}\frac{1}{n!}\left(\sum_{i=0}^{10} A_i\right)^n,\\
A_1 &= A_1(x,u) = \frac{1}{2}xA_0^2,\\
A_2 &= A_2(x,u) = xA_0A_1,\\
A_3 &= A_3(x,u) = xA_0(A_2+A_3+A_4)u,\\
A_4 &= A_4(x,u) = xA_0(A_5+A_6+A_7+A_8+A_9+A_{10})u^2,\\
A_5 &= A_5(x,u) = \frac{1}{2}xA_1^2u,\\
A_6 &= A_6(x,u) = xA_1(A_2+A_3+A_4)u^2,\\
A_7 &= A_7(x,u) = xA_1(A_5+A_6+A_7+A_8+A_9+A_{10})u^3,\\
A_8 &= A_8(x,u) = \frac{1}{2}x(A_2+A_3+A_4)^2u^3,\\
A_9 &= A_9(x,u) = x(A_2+A_3+A_4)(A_5+A_6+A_7+A_8+A_9+A_{10})u^4,\\
A_{10} &= A_{10}(x,u) = \frac{1}{2}x(A_5+A_6+A_7+A_8+A_9+A_{10})^2u^5.
\end{aligned}
$$

Patterns in Trees

Final Result for $\mathcal{M} =$ 

Central limit theorem with

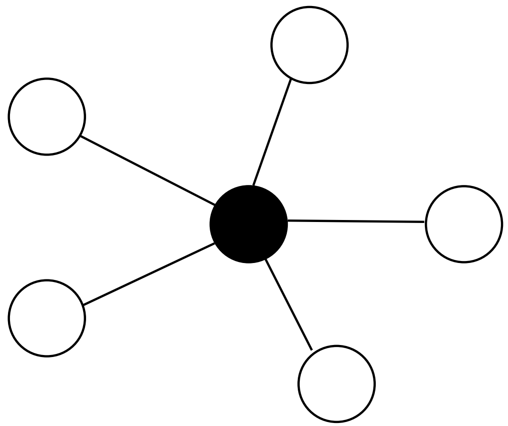
$$\mu = \frac{5}{8e^3} = 0.0311169177\ldots$$

and

$$\sigma^2 = \frac{20e^3 + 72e^2 + 84e - 175}{32e^6} = 0.0764585401\ldots.$$

Degree distribution in random trees

Nodes of given degree



$\ell = 5$

$X_n^{(\ell)}$ = number of nodes of degree ℓ in trees of size n satisfies a **central limit theorem** with

$$\mu^{(\ell)} = \frac{1}{e\,(\ell-1)!} \quad \text{and} \quad (\sigma^{(\ell)})^2 = \frac{1+(\ell-2)^2}{e^2(\ell-1)!^2} + \frac{1}{e\,(\ell-1)!}$$

Degree distribution in random trees

$d_{n,\ell}$ ... probability that a random node in a random labelled tree of size n has degree ℓ:

$$\mathbb{E}\, X_n^{(\ell)} = n\, d_{n,\ell}$$

$$d_\ell := \lim_{n \to \infty} d_{n,\ell} = \frac{1}{e\,(\ell-1)!} = \mu^{(\ell)}$$

Probability generating function of the degree distribution:

$$\boxed{p(w) := \sum_{\ell \geq 1} d_\ell w^\ell = w e^{w-1}}$$

Degree distribution in random planar graphs

- **Outerplanar graph**: no K_4 and no $K_{2,3}$ as a minor.

- **Series-parallel graph**: no K_4 as a minor.

- **Planar graph**: no K_5 and no $K_{3,3}$ as a minor.

Remark.

$$\text{outerplanar} \subseteq \text{series-parallel} \subseteq \text{planar}$$

Degree distribution in random planar graphs

Outerplanar Graphs

g_n ... number of **labelled outer-planar** graphs with n vertices:

$$G(x) = \sum_{n \ge 0} g_n \frac{x^n}{n!}$$

$$\begin{aligned} G(x) &= e^{C(x)}, \\ C'(x) &= e^{B'(xC'(x))}, \\ B'(x) &= x + \frac{1}{2} x\, A(x), \\ A(x) &= x(1+A(x))^2 + x(1+A(x))A(x) \\ &= \frac{1 - 3x - \sqrt{1 - 6x + x^2}}{4x}. \end{aligned}$$

Degree distribution in random planar graphs

Series-Paralles Graphs

$g_{n,m}$ ... number of **labelled series-parallel** graphs with n vertices and m edges:

$$G(x,y) = \sum_{n\ge 0} g_{n,m} \frac{x^n}{n!} y^m$$

$$\begin{aligned}
G(x,y) &= e^{C(x,y)} \\
\frac{\partial C(x,y)}{\partial x} &= \exp\left(\frac{\partial B}{\partial x}\left(x\frac{\partial C(x,y)}{\partial x}, y\right)\right), \\
\frac{\partial B(x,y)}{\partial y} &= \frac{x^2}{2}\,\frac{1+D(x,y)}{1+y}, \\
D(x,y) &= (1+y)e^{S(x,y)} - 1, \\
S(x,y) &= (D(x,y) - S(x,y))xD(x,y).
\end{aligned}$$

Degree distribution in random planar graphs

Planar Graphs

$g_{n,m}$ ... number of **labelled planar** graphs with n vertices and m edges:

$$G(x,y) = \sum_{n\geq 0} g_{n,m} \frac{x^n}{n!} y^m$$

$$\begin{aligned}
G(x,y) &= \exp\left(C(x,y)\right), \\
\frac{\partial C(x,y)}{\partial x} &= \exp\left(\frac{\partial B}{\partial x}\left(x\frac{\partial C(x,y)}{\partial x}, y\right)\right), \\
\frac{\partial B(x,y)}{\partial y} &= \frac{x^2}{2}\frac{1+D(x,y)}{1+y}, \\
\frac{M(x,D)}{2x^2D} &= \log\left(\frac{1+D}{1+y}\right) - \frac{xD^2}{1+xD}, \\
M(x,y) &= x^2y^2\left(\frac{1}{1+xy} + \frac{1}{1+y} - 1 - \frac{(1+U)^2(1+V)^2}{(1+U+V)^3}\right), \\
U &= xy(1+V)^2, \\
V &= y(1+U)^2.
\end{aligned}$$

Degree distribution in random planar graphs

[Gimenez+Noy (2005)]

g_n .... number of all labelled planar graphs

$$\boxed{\boxed{g_n \sim c \cdot n^{-\frac{7}{2}} \gamma^n n!}}, \quad \gamma = 27.22...$$

Degree distribution in random planar graphs

Outerplanar graphs

Theorem

$X_n^{(\ell)}$ ... number of vertices of degree ℓ in random 2-connected, connected or unrestricted **labelled outerplanar** graphs with n vertices.

$\Longrightarrow$ $X_n^{(\ell)}$ satisfies a **central limit theorem** with

$$\mathbb{E}\, X_n^{(\ell)} \sim \mu^{(\ell)} n \quad \text{and} \quad \mathbb{V}\, X_n^{(\ell)} \sim (\sigma^{(\ell)})^2 n.$$

Degree distribution in random planar graphs

Outerplanar graphs $d_\ell = \mu^{(\ell)}$, $p(w) = \sum_{\ell \ge 1} d_\ell w^\ell$

- **2-connected**

$$p(w) = \frac{2(3-2\sqrt{2})w^2}{(1-(\sqrt{2}-1)w)^2}$$

- **connected** or **unrestricted**:

$$p(w) = \frac{c_1 w^2}{(1-c_2 w)^2} \exp\left(c_3 w + \frac{c_4 w^2}{(1-c_2 w)}\right)$$

(with certain constants $c_1, c_2, c_3, c_4 > 0$).

Degree distribution in random planar graphs

Series-parallel graphs

Theorem

$X_n^{(\ell)}$ ... number of vertices of degree ℓ in random 2-connected, connected or unrestricted **labelled series-parallel** graphs with n vertices.

$\Longrightarrow$ $X_n^{(\ell)}$ satisfies a **central limit theorem** with

$$\mathbb{E}\, X_n^{(\ell)} \sim \mu^{(\ell)} n \quad \text{and} \quad \mathbb{V}\, X_n^{(\ell)} \sim (\sigma^{(\ell)})^2 n.$$

Degree distribution in random planar graphs

2-connected series-parallel graphs $d_\ell = \mu^{(\ell)}$, $p(w) = \sum_{\ell \ge 1} d_\ell w^\ell$:

$$\boxed{p(w) = \frac{B_1(1,w)}{B_1(1,1)}},$$

where $B_1(y,w)$ is given by the following procedure ...

Degree distribution in random planar graphs

$$\frac{E_0(y)^3}{E_0(y)-1} = \left(\log\frac{1+E_0(y)}{1+R(y)} - E_0(y)\right)^2,$$

$$R(y) = \frac{\sqrt{1-1/E_0(y)}-1}{E_0(y)},$$

$$E_1(y) = -\left(\frac{2R(y)E_0(y)^2(1+R(y)E_0(y))^2}{(2R(y)E_0(y)+R(y)^2E_0(y)^2)^2+2R(y)(1+R(y)E_0(y))}\right)^{\frac{1}{2}},$$

$$D_0(y,w) = (1+yw)e^{\frac{R(y)E_0(y)}{1+R(y)E_0(y)}D_0(y,w)} - 1,$$

$$D_1(y,w) = \frac{(1+D_0(y,w))^{\frac{R(y)E_1(y)D_0(y,w)}{1+R(y)E_0(y)}}}{1-(1+D_0(y,w))^{\frac{R(y)E_0(y)D_0(y,w)}{1+R(y)E_0(y)}}},$$

$$B_0(y,w) = \frac{R(y)D_0(y,w)}{1+R(y)E_0(y)} - \frac{R(y)^2E_0(y)D_0(y,w)^2}{2(1+R(y)E_0(y))},$$

$$B_1(y,w) = \frac{R(y)D_1(y,w)}{1+R(y)E_0(y)} - \frac{R(y)^2E_0(y)D_0(y,w)D_1(y,w)}{1+R(y)E_0(y)} - \frac{R(y)^2E_1(y)D_0(y,w)(1+D_0(y,w)/2)}{(1+R(y)E_0(y))^2}.$$

Degree distribution in random planar graphs

Theorem

$X_n^{(\ell)}$ ... number of vertices of degree ℓ in random 3-connected, 2-connected, connected or unrestricted **labelled planar** graphs with n vertices.

$\Longrightarrow \ \mathbb{E}\, X_n^{(\ell)} \sim \mu^{(\ell)} n$

For $\ell \le 2$, $X_n^{(\ell)}$ satisfies also a central limit theorem.

Degree distribution in random planar graphs

unrestricted planar graphs $d_\ell = \mu^{(\ell)}$, $p(w) = \sum_{\ell \ge 1} d_\ell w^\ell$:

d_1	d_2	d_3	d_4	d_5	d_6
0.0367284	0.1625794	0.2354360	0.1867737	0.1295023	0.0861805

$$p(w) = -e^{B_0(1,w)-B_0(1,1)} B_2(1,w) + e^{B_0(1,w)-B_0(1,1)} \frac{1+B_2(1,1)}{B_3(1,1)} B_3(1,w)$$

where $B_j(y,w)$ are given by the following procedure ...

Degree Distribution

- Implicit equation for $D_0(y,w)$:

$$1 + D_0 = (1 + y\boxed{w}) \exp\left(\frac{\sqrt{S}(D_0(t-1)+t)}{4(3t+1)(D_0+1)} - \right.$$
$$\left. - \frac{D_0^2(t^4 - 12t^2 + 20t - 9) + D_0(2t^4 + 6t^3 - 6t^2 + 10t - 12) + t^4 + 6t^3 + 9t^2}{4(t+3)(D_0+1)(3t+1)}\right),$$

where $t = t(y)$ satisfies $y + 1 = \frac{1+2t}{(1+3t)(1-t)} \exp\left(-\frac{1}{2}\frac{t^2(1-t)(18+36t+5t^2)}{(3+t)(1+2t)(1+3t)^2}\right)$.
and $S = (D_0(t-1)+t)(D_0(t-1)^3 + t(t+3)^2)$.

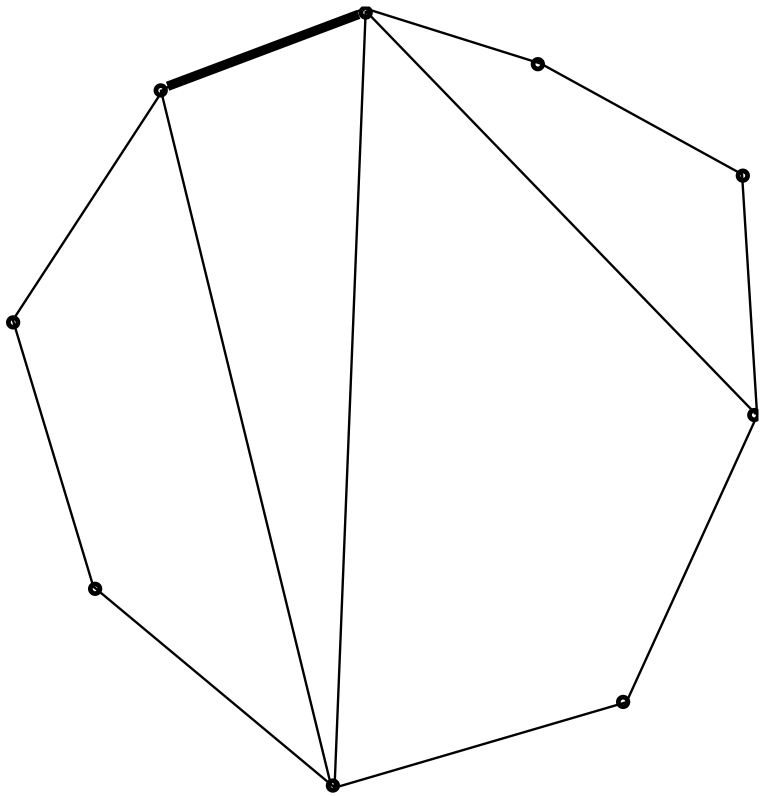
- Explicit expressions in terms of $D_0(y,w)$:

$$D_2(y,w),\ D_3(y,w),\ B_0(y,w),\ B_2(y,w),\ B_3(y,w)$$

- Explict expression for $p(w)$:

$$\boxed{p(w) = -e^{B_0(1,w)-B_0(1,1)} B_2(1,w) + e^{B_0(1,w)-B_0(1,1)} \frac{1 + B_2(1,1)}{B_3(1,1)} B_3(1,w)}$$

Dissections

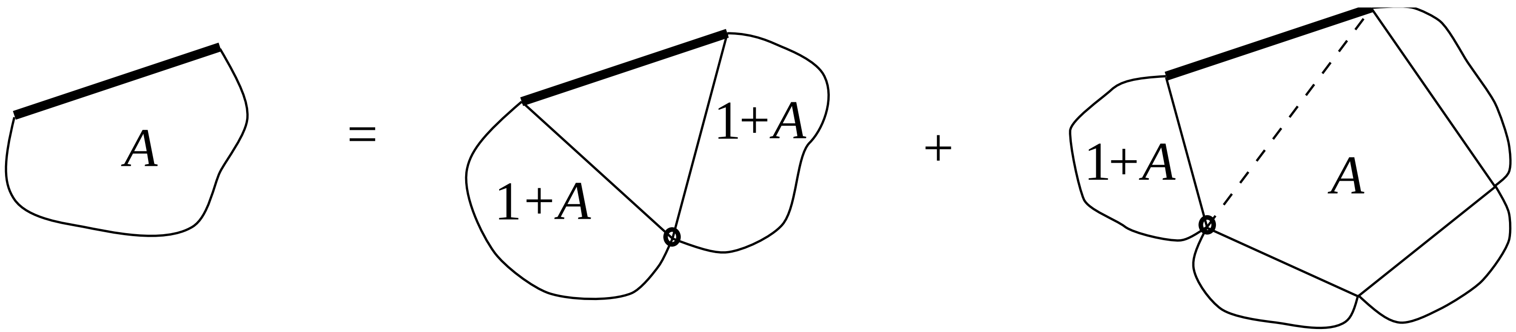


$\mathcal{A}$ ... set of dissections
(unlabelled planar graphs, where all nodes are on the outer face, one edge is marked, and there are at least 3 edges)

Dissections

a_n ... number of dissections with $n+2$ nodes, $n \geq 1$,
(the nodes of the marked edge are not counted)

$A(x) = \sum_{n \geq 1} a_n x^n$ ... generating function of dissections



$$A(x) = x(1 + A(x))^2 + x(1 + A(x))A(x)$$

Dissections

Quadratic equation:

$$A^2 + \frac{3x-1}{2x} A + \frac{1}{2} = 0$$

Solution:

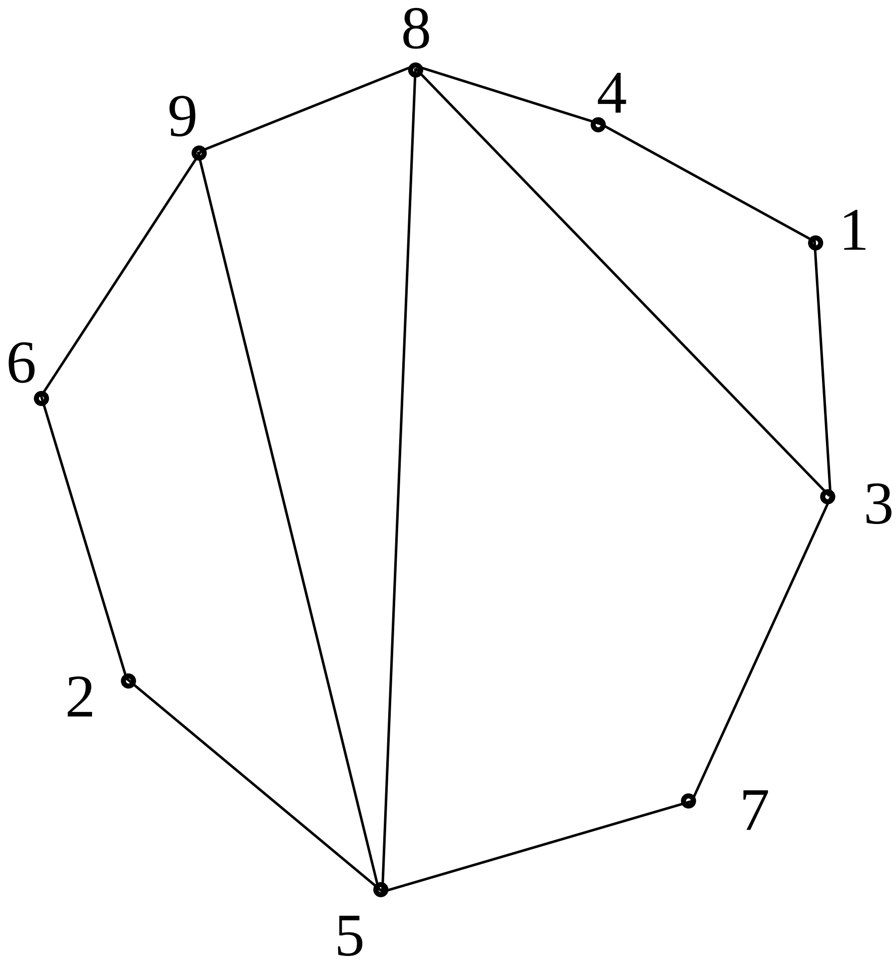
$$\boxed{A(x) = \frac{1 - 3x - \sqrt{1 - 6x + x^2}}{4x}}$$

Radius of convergence: $\rho_1 = 3 - 2\sqrt{2}$.

Lagrange inversion formula:

$$a_n = \frac{1}{n} \sum_{\ell=0}^{n-1} \binom{n}{\ell} \binom{n}{\ell+1} 2^\ell.$$

2-Connected Outer Planar Graphs



b_n ... number of 2-connected vertex labelled outer planar graphs

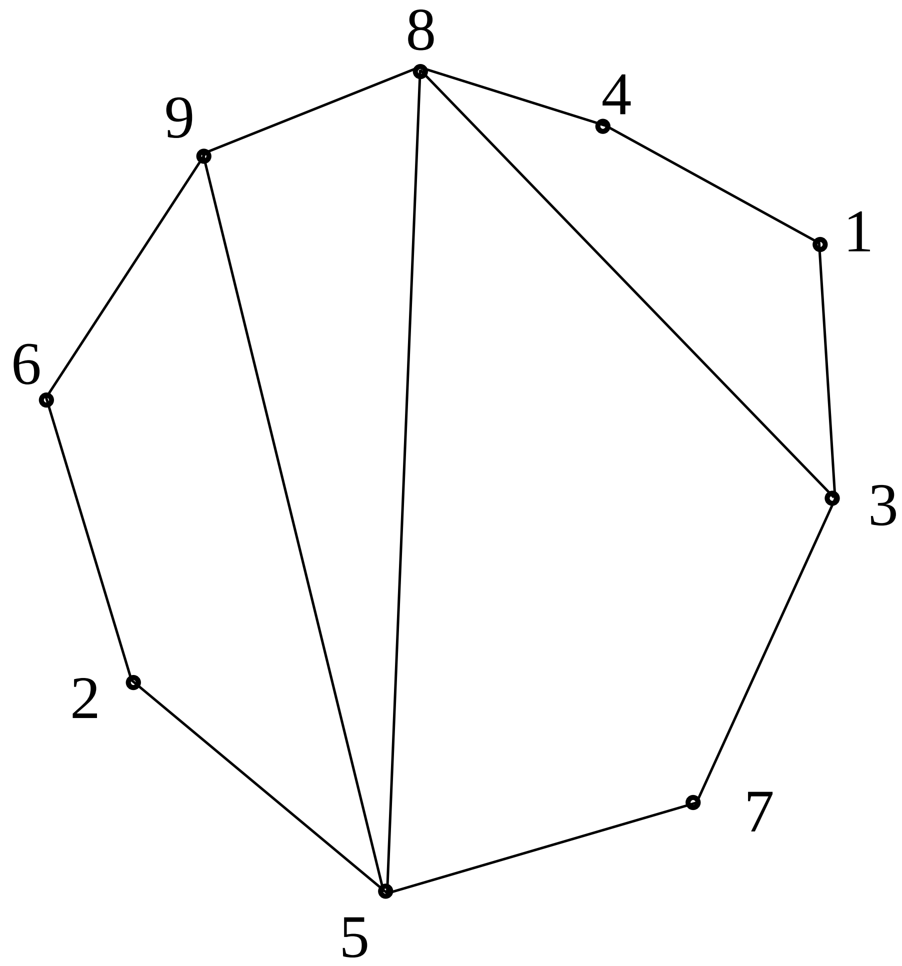
2-Connected Outer Planar Graphs

$B(x) = \sum_{n \geq 1} b_n \frac{x^n}{n!}$ ... exponential generating function of 2-connected labelled outer planar graphs:

$$\boxed{B'(x) = x + \frac{1}{2} x A(x)}$$

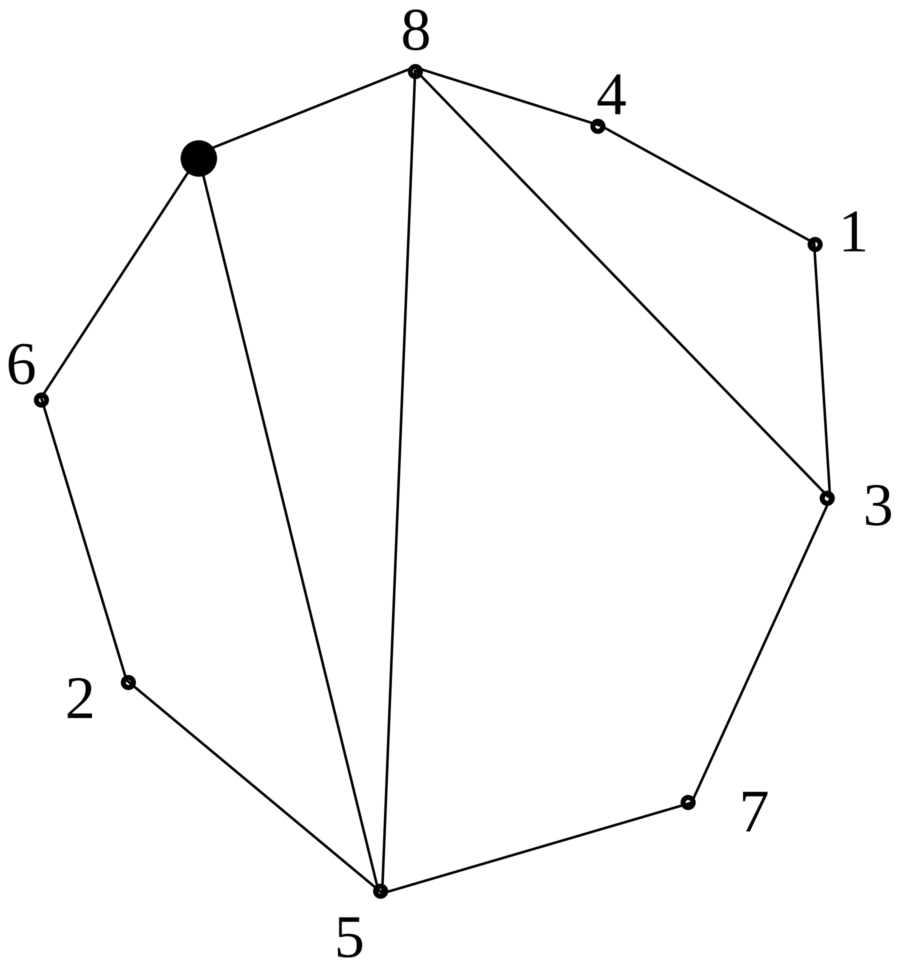
The derivative $B'(x)$ can be also interpreted as the exponential generating function $B^\bullet(x)$ of 2-connected labelled outer planar graphs, where one node is marked (and is not counted).

2-Connected Outer Planar Graphs



$$\boxed{b_n = \frac{1}{2} a_{n-2} \cdot (n-1)!} \qquad (n \geq 3)$$

2-Connected Outer Planar Graphs



$$\boxed{b_n = \frac{1}{2} a_{n-2} \cdot (n-1)!} \qquad (n \geq 3)$$

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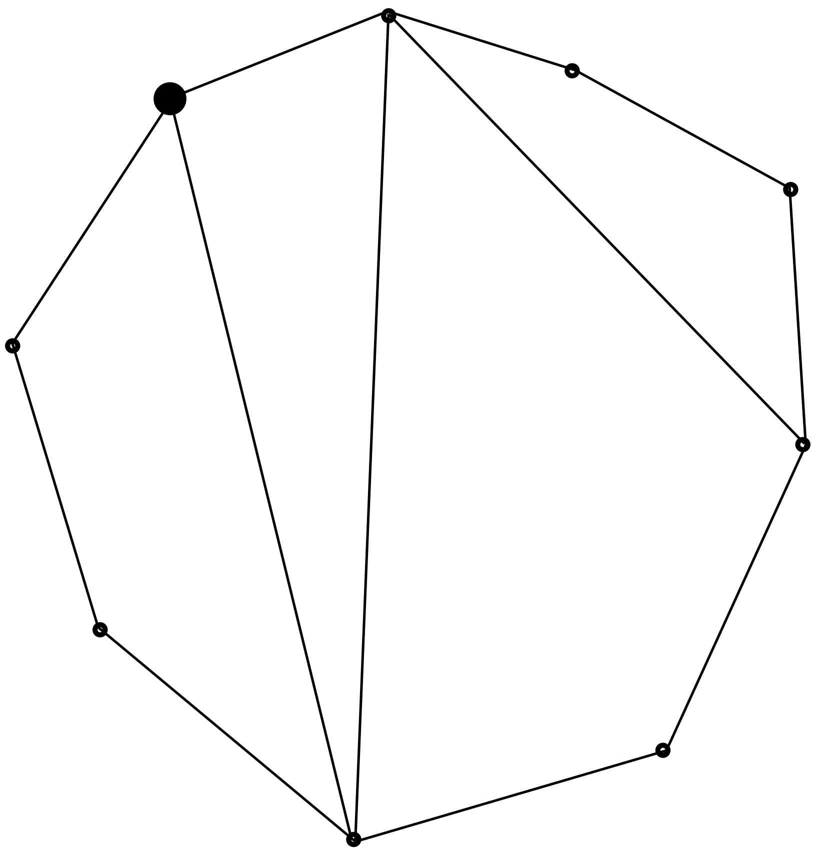

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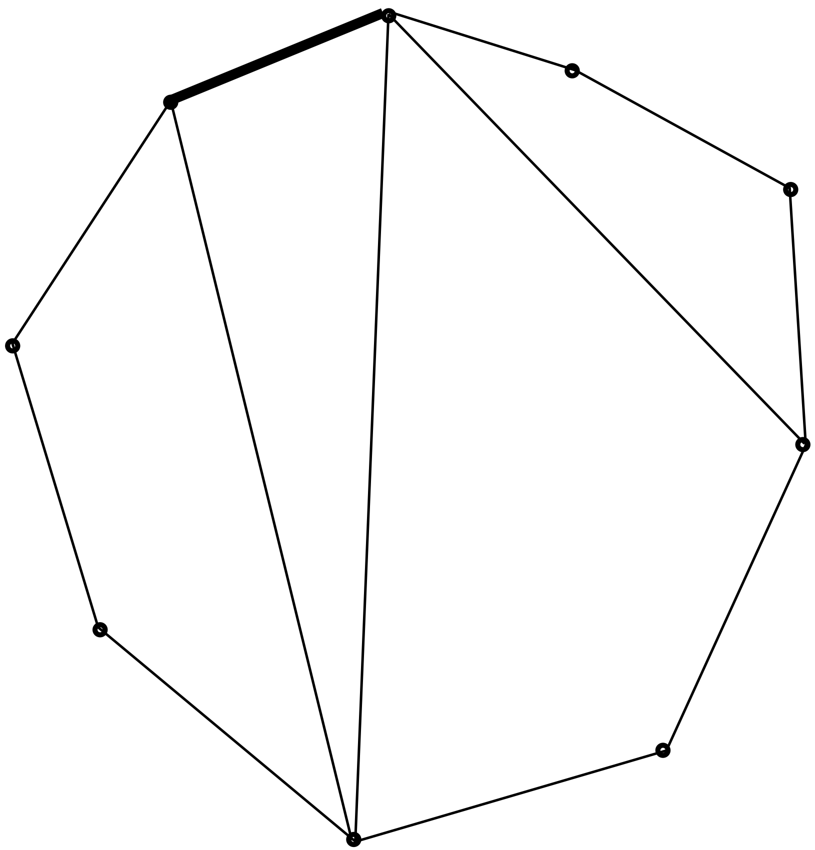

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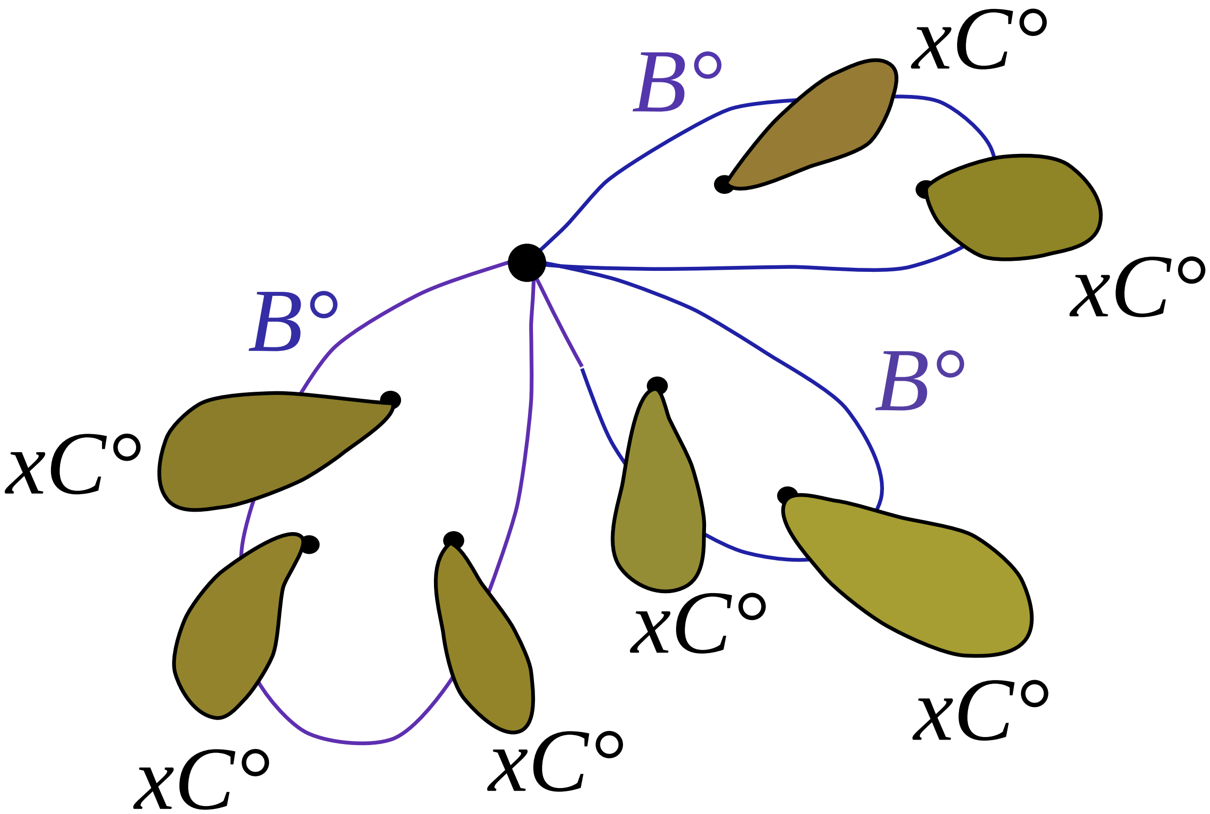
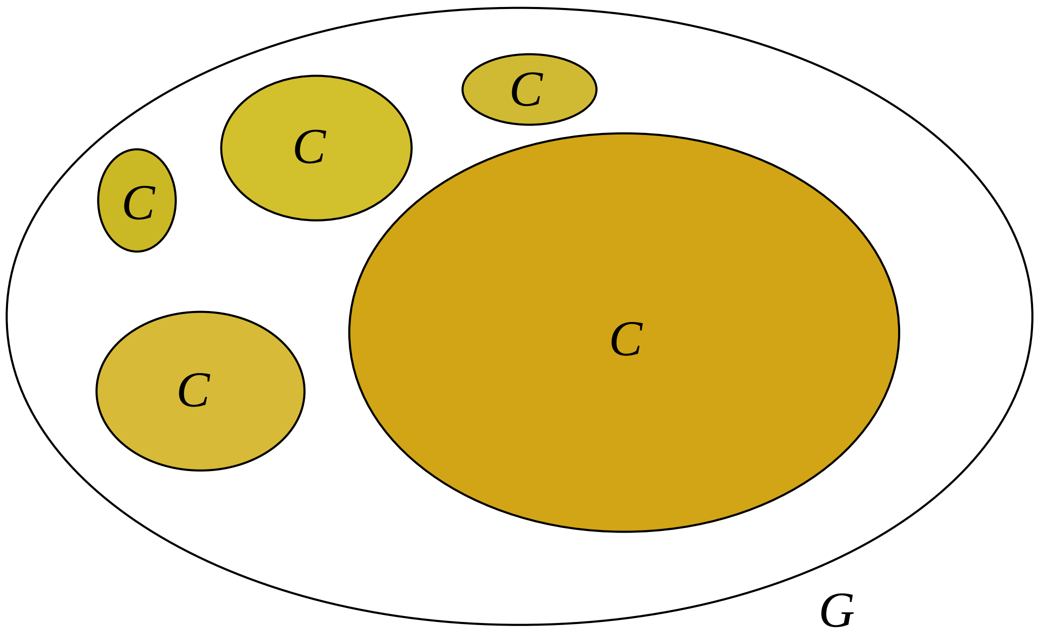
Connected Outer Planar Graphs



$$C^{\bullet}(x) = e^{B^{\bullet}(xC^{\bullet}(x))}$$

Connected Outer Planar Graphs

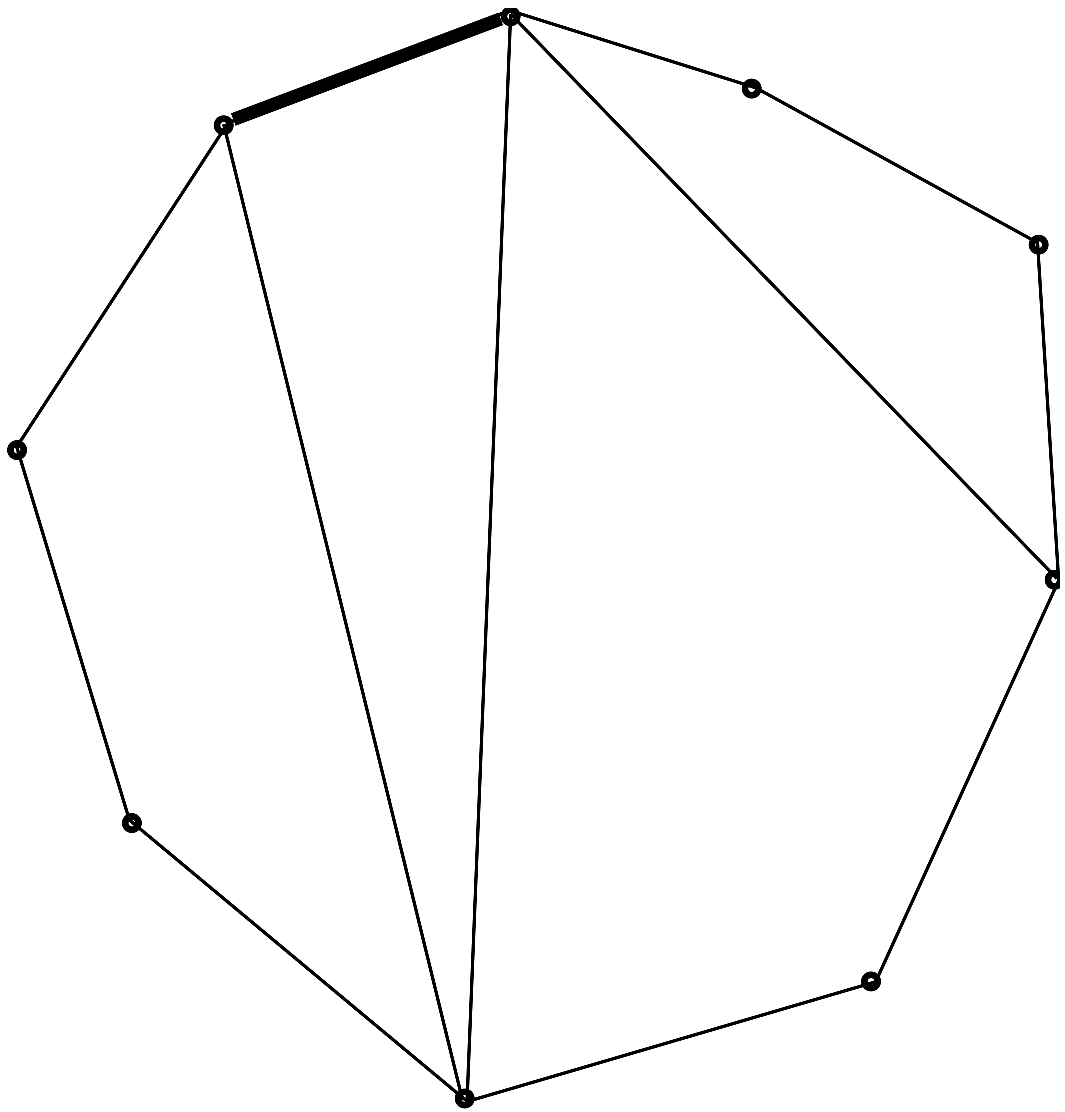
Generating Functions



$$G(x) = \exp\left(C(x)\right)$$

Nodes of Given Degree

Dissections:



Nodes of Given Degree

- v_2 counts the number of nodes with degree 2,

- v_3 counts the number of nodes with degree 3,

- v counts the number of nodes with degree > 3, and

- in all cases the two **nodes of the rooted edge are are not taken into account.**

Nodes of Given Degree

- $A_{22}(v_2, v_3, v)$ ... generating function of dissections with the properties that both nodes of the rooted edge have degree 2,

- $A_{23}(v_2, v_3, v)$ ... generating function of dissections with the properties that the left node of the rooted edge has degree 2 and right one has degree 3,

- $A_{33}(v_2, v_3, v)$ ... generating function of dissections with the properties that both nodes of the rooted edge have degree 3,

Nodes of Given Degree

- $A_{2>}(v_2, v_3, v)$ ... generating function of dissections with the properties that the left node of the rooted edge has degree 2 and the right has degree > 3,

- $A_{3>}(v_2, v_3, v)$ ... generating function of dissections with the properties that the left node of the rooted edge has degree 3 and the right one has degree > 3, and

- $A_{>>}(v_2, v_3, v)$ ... generating function of dissections with the properties that both nodes of the rooted edge have degree > 3.

Nodes of Given Degree

The sum

$$A(v_2, v_3, v) = A_{22} + 2A_{23} + A_{33} + 2A_{2>} + 2A_{3>} + A_{>>}$$

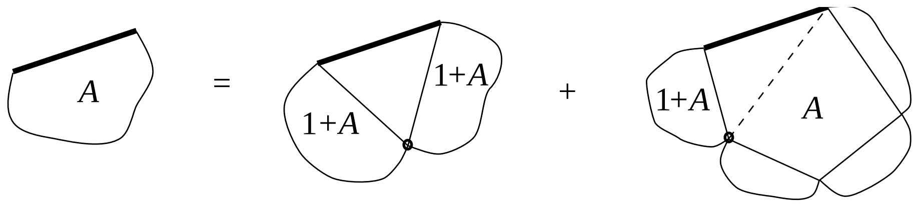
is the generating function of all dissections (with the corresponding counting in v_2, v_3, v).

In particular,

$$A(x) = A(x, x, x) = \frac{1 - 3x - \sqrt{1 - 6x + x^2}}{4x}.$$

Nodes of Given Degree

Lemma 3



$$
\begin{aligned}
A_{22} &= v_2 \\
&+ v_2 A_{22} + v_3 A_{23} + v A_{2>}, \\
A_{23} &= v_3 A_{22} + v(A_{23} + A_{2>}) \\
&= v_2 A_{23} + v_3 A_{33} + v A_{3>}, \\
A_{33} &= v(A_{22} + A_{23} + A_{2>})^2 \\
&+ v(A_{22} + A_{23} + A_{2>})(A_{23} + A_{33} + A_{3>}), \\
A_{2>} &= v_3 A_{23} + v(A_{33} + A_{3>}) + v(A_{2>} + A_{3>} + A_{>>}) \\
&+ v_2 A_{2>} + v_3 A_{3>} + v A_{>>}, \\
A_{3>} &= v(A_{23} + A_{33} + A_{3>})(A_{2>} + A_{3>} + A_{>>}) \\
&+ v(A_{22} + A_{23} + A_{2>})(A_{2>} + A_{3>} + A_{>>}), \\
A_{>>} &= v(A_{23} + A_{33} + A_{3>} + A_{2>} + A_{3>} + A_{>>})^2 \\
&+ v(A_{23} + A_{33} + A_{3>} + A_{2>} + A_{3>} + A_{>>})(A_{2>} + A_{3>} + A_{>>}).
\end{aligned}
$$

Nodes of Given Degree

- $B_1^\bullet(v_1, v_2, v_3, v)$ ... exponential genenerating functions of 2-connected labelled and marked outer planar graphs, where the root node (that is not counted) has degree 1.

- $B_2^\bullet(v_1, v_2, v_3, v)$ ... exponential genenerating functions of 2-connected labelled and marked outer planar graphs, where the root node (that is not counted) has degree 2.

- $B_3^\bullet(v_1, v_2, v_3, v)$ ... exponential genenerating functions of 2-connected labelled and marked outer planar graphs, where the root node (that is not counted) has degree 3.

- $B_>^\bullet(v_1, v_2, v_3, v)$ ... exponential genenerating functions of 2-connected labelled and marked outer planar graphs, where the root node (that is not counted) has degree > 3.

Nodes of Given Degree

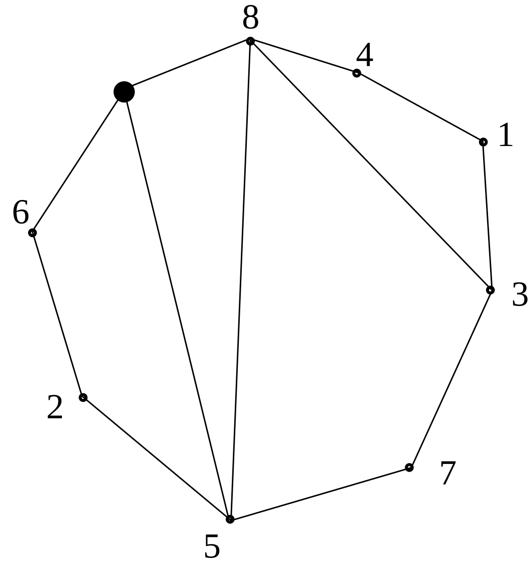
Lemma 4

$$B_1^\bullet(v_1,v_2,v_3,v) = v_1,$$

$$B_2^\bullet(v_1,v_2,v_3,v) = \frac{1}{2}\left(v_2A_{22} + v_3A_{23} + vA_{2>}\right),$$

$$B_3^\bullet(v_1,v_2,v_3,v) = \frac{1}{2}\left(v_2A_{23} + v_3A_{33} + vA_{3>}\right),$$

$$B_>^\bullet(v_1,v_2,v_3,v) = \frac{1}{2}\left(v_2A_{2>} + v_3A_{3>} + vA_{>>}\right).$$



Nodes of Given Degree

- $C_0^\bullet(v_1, v_2, v_3, v)$ ... exponential generating functions of connected labelled and marked outer planar graphs, where the root node (that is not counted) has degree 0.

- $C_1^\bullet(v_1, v_2, v_3, v)$ ... exponential generating functions of connected labelled and marked outer planar graphs, where the root node (that is not counted) has degree 1.

- $C_2^\bullet(v_1, v_2, v_3, v)$ ... exponential generating functions of connected labelled and marked outer planar graphs, where the root node (that is not counted) has degree 2.

Nodes of Given Degree

- $C_3^\bullet(v_1, v_2, v_3, v)$ … exponential generating functions of connected labelled and marked outer planar graphs, where the root node (that is not counted) has degree 3.

- $C_>^\bullet(v_1, v_2, v_3, v)$ … exponential generating functions of connected labelled and marked outer planar graphs, where the root node (that is not counted) has degree > 3.

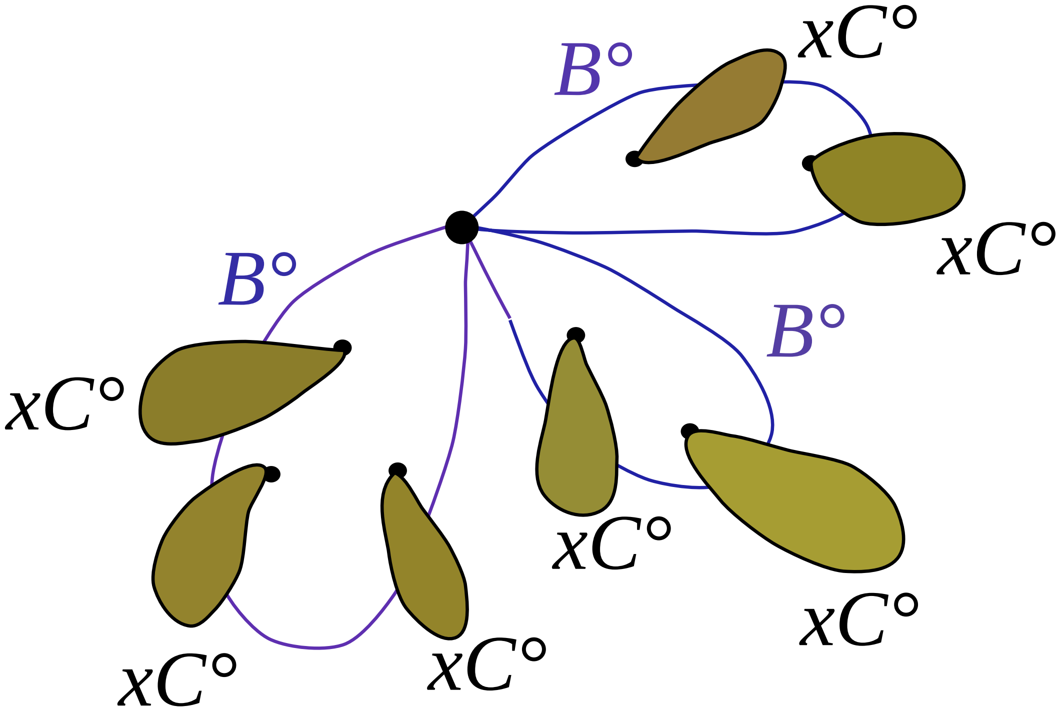
Nodes of Given Degree

Lemma 5

$$
\begin{aligned}
C_0^\bullet(v_1,v_2,v_3,v) &= 1,\\
C_1^\bullet(v_1,v_2,v_3,v) &= B_1^\bullet(W_1,W_2,W_3,W),\\
C_2^\bullet(v_1,v_2,v_3,v) &= \frac{1}{2!}(B_1^\bullet(W_1,W_2,W_3,W))^2 + B_2^\bullet(W_1,W_2,W_3,W),\\
C_3^\bullet(v_1,v_2,v_3,v) &= \frac{1}{3!}(B_1^\bullet(W_1,W_2,W_3,W))^3\\
&\quad + \frac{1}{1!1!}B_1^\bullet(W_1,W_2,W_3,W)B_2^\bullet(W_1,W_2,W_3,W)\\
&\quad + B_3^\bullet(W_1,W_2,W_3,W),\\
C_>^\bullet(v_1,v_2,v_3,v) &= e^{B_1^\bullet(W_1,W_2,W_3,W)+B_2^\bullet(\ldots)+B_3^\bullet(\ldots)+B_>^\bullet(W_1,W_2,W_3,W)}\\
&\quad - 1 - B_1^\bullet(W_1,W_2,W_3,W) - B_2^\bullet(\ldots) - B_3^\bullet(\ldots)\\
&\quad - \frac{1}{1!}(B_1^\bullet(W_1,W_2,W_3,W))^2 - \frac{1}{3!}(B_1^\bullet(W_1,W_2,W_3,W))^3\\
&\quad - \frac{1}{1!1!}B_1^\bullet(W_1,W_2,W_3,W)B_2^\bullet(W_1,W_2,W_3,W),
\end{aligned}
$$

where on the right hand side

$$W_1 = v_1 C_0^\bullet + v_2 C_1^\bullet + v_3 C_2^\bullet + v(C_3^\bullet + C_>^\bullet),$$
$$W_2 = v_2 C_0^\bullet + v_3 C_1^\bullet + v(C_2^\bullet + C_3^\bullet + C_>^\bullet),$$
$$W_3 = v_3 C_0^\bullet + v(C_1^\bullet + C_2^\bullet + C_3^\bullet + C_>^\bullet),$$
$$W = v(C_0^\bullet + C_1^\bullet + C_2^\bullet + C_3^\bullet + C_>^\bullet).$$



Nodes of Given Degree

Counting nodes of degree 3:

$C(v_1, v_2, v_3, v)$ ... exponential generating function of all connected labelled outer planar graphs

$C_{d=3}(x,u)$ ... exponential generating function that counts the number of nodes with x and the number of nodes of degree $d = 3$ with u:

$$C_{d=3}(x,u) = C(x,x,xu,x).$$

Also:

$$\frac{\partial C_{d=3}(x,u)}{\partial x} = C_1^\bullet + C_2^\bullet + uC_3^\bullet + C_>^\bullet \quad \text{and} \quad \frac{\partial C_{d=3}(x,u)}{\partial u} = xC_3^\bullet$$

Thanks for your attention!