THE SUM OF DIGITS FUNCTION OF POLYNOMIAL SEQUENCES

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Abstract. Let $q \geq 2$ be an integer and $s_q(n)$ denote the sum of the digits in base q of the positive integer n. The goal of this work is to study a problem of Gelfond concerning the the repartition of the sequence $(s_q(P(n)))_{n\in\mathbb{N}}$ in arithmetic progressions when $P \in \mathbb{Z}[X]$ is such that $P(\mathbb{N}) \subset \mathbb{N}$. We answer Gelfond's question and we show the uniform distribution modulo 1 of the sequence $(\alpha s_q(P(n)))_{n\in\mathbb{N}}$ for $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ provided that q is a large enough prime number coprime with the leading coefficient of P.

1. Introduction

Let $s_q(n)$ denote the sum of digits function, defined for any non negative integer n by

$$s_q(n) = \sum_{j\geq 0} \varepsilon_j(n),$$

where, for any non negative integer j, $\varepsilon_j(n) \in \{0, 1, \ldots, q-1\}$ are the digits in the q-ary digital expansion

$$n = \sum_{j\geq 0} \varepsilon_j(n) q^j.$$

For $x \in \mathbb{R}$ we set $e(x) = \exp(2\pi i x)$ and if $\ell = \max\{j : \varepsilon_j(n) \neq 0\}$ we denote by $\mathrm{rep}_q(n) = \varepsilon_\ell(n) \ldots \varepsilon_0(n)$ the q-adic representation of the integer n.

The sum of digits function appears in many different mathematical questions (see [1] and [14] for a survey on this aspect). Mahler introduced in [13] the sequence $\left((-1)^{s_2(n)}\right)_{n\in\mathbb{N}}$ in order to illustrate several results of spectral analysis obtained by Wiener in [26]. In particular, Mahler showed the convergence, for any non negative integer k, of the sequence $(\gamma_k(N))_{N\geq 1}$ defined for any positive integer N by

$$\gamma_k(N) = \frac{1}{N} \sum_{n<N} (-1)^{s_2(n)} (-1)^{s_2(n+k)},$$

and moreover that this limit is non zero for infinitely many integers k.

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Nowadays we know (see [12]) that for any non-negative integer k this limit is equal to the k-th Fourier coefficient of the correlation measure associated to the symbolic dynamical system generated by the sequence $\left((-1)^{s_2(n)}\right)_{n\in\mathbb{N}}$ and that this convergence can be understood as a consequence of the unique ergodicity of this symbolic dynamical system (see [23] or [24]).

Only few results are known concerning the q-adic representation of the sequence $(P(n))_{n\in\mathbb{N}}$ when P is an integer valued polynomial. Davenport and Erdős proved in [5] the normality of the real number whose q-adic representation is

$$0.\operatorname{rep}_q(P(1))\ldots\operatorname{rep}_q(P(n))\ldots$$

when P is an integer valued polynomial. A consequence of their theorem is that in this case we have

$$\sum_{n\le x} s_q(P(n)) \sim \frac{q-1}{2}\, d\, x\, \log_q x \quad (x\to+\infty),$$

where d is the degree of P.

Peter generalized in [21] a result obtained by Delange [6] in the case $P(X)=X$ and proved the following more precise estimate in the case $P(X)=X^d$:

Theorem A. There exist $c\in\mathbb{R}$, $\varepsilon>0$, and $\Phi_{q,d}$ a continuous function on $\mathbb{R}$, 1-periodic and nowhere differentiable such that for all $x\ge 1$,

$$\sum_{n\le x} s_q(n^d) = \frac{q-1}{2}\ d\ x\ \log_q x + cx + x\,\Phi_{q,d}(d\log_q x) + O(x^{1-\varepsilon}).$$

Furthermore Bassily and Katai showed in [2] that there is a central limit theorem for the sum of digits function on polynomial sequences:

Theorem B. Let $P\in\mathbb{Z}[X]$ such that $P(\mathbb{N})\subset\mathbb{N}$ then

$$\frac{1}{x}\operatorname{card}\left\{n\le x,\ s_q(P(n)) \le \frac{q-1}{2}\ d\ x\ \log_q x + y\sqrt{\frac{q^2-1}{12}\ d\ x\ \log_q x}\right\} = \Phi(y)+o(1),$$

where $\Phi(y)$ denotes the normal distribution function.

In 1967 Gelfond studied in [10] the distribution in arithmetic progressions of the sequence $(s_q(P(n)))_{n\in\mathbb{N}}$ when P is an integer valued polynomial of degree 1 and proposed the case of higher degree as an open problem:

Problem 1 (*Gelfond's problem for integer valued polynomials)***.** For any integer valued polynomial P and any fixed integers $a\in\mathbb{Z}$ and $m\ge 1$, give the number of integers $n\le x$ such that $s_q(P(n))\equiv a \bmod m$.

Following the ideas of Piatetski-Shapiro, who studied in [22] the distribution of prime numbers in the sequence $(\lfloor n^c\rfloor)_{n\in\mathbb{N}}$ for $c>1$, a first approach to Gelfond's problem was developed by Mauduit and Rivat in [15, 16] and continued by Morgenbesser in [20] who proved the following results:

Theorem C. If $c \in [1, 7/5)$ and $q \geq 2$ (by [16]) or if $c \in \mathbb{R}^+ \setminus \mathbb{N}$ and $q \geq q_0(c)$ sufficiently large (by [20]) then

- for all $(a, m) \in \mathbb{Z} \times \mathbb{N}^*$, we have
$$\lim_{N \to +\infty} \frac{1}{N} \operatorname{card}\{n < N \ : \ s_q([n^c]) \equiv a \bmod m\} = \frac{1}{m},$$
- the sequence $(\alpha\, s_q([n^c]))_{n \in \mathbb{N}}$ is uniformly distributed modulo 1 if and only if α is an irrational number.

A first answer to Gelfond's original problem for integer valued polynomials was given by Dartyge and Tenenbaum in [3, 4] where they obtained the following general lower bound:

Theorem D. Let q and m be positive integers such that $q \geq 2$ and $\gcd(m, q-1) = 1$ and let $P \in \mathbb{Z}[X]$ be such that $P(\mathbb{N}) \subset \mathbb{N}$. Then there exist two constants $C = C(P, q, m) > 0$ and $N_0 = N_0(P, q, m) \geq 1$ such that for any $a \in \{0, 1, \ldots, m-1\}$ and for any integer $N \geq N_0$, we have
$$\operatorname{card}\{n < N \ : \ s_q(P(n)) \equiv a \bmod m\} \geq C N^{\min(1, 2/d!)},$$
where d is the degree of P.

Recently Mauduit and Rivat gave in [17] a precise answer to Gelfond's problem in the case where the polynomial P is of degree 2 (their paper presents only a proof for the polynomial $P(X) = X^2$ but it could be adapted for any integer valued polynomial P of degree 2 at the price of dealing with a technical discussion concerning the arithmetic properties of the coefficients of P):

Theorem E. For any integers $q \geq 2$ and $m \geq 2$, there exists $\sigma_{q,m} > 0$ such that for any $a \in \mathbb{Z}$,
$$\operatorname{card}\{n \leq x \ : \ s_q(n^2) \equiv a \bmod m\} = \frac{x}{m} Q(a, D) + O_{q,m}(x^{1-\sigma_{q,m}}), \tag{1.1}$$
where $D = \gcd(q-1, m)$ and
$$Q(a, D) = \operatorname{card}\{0 \leq n < D \ : \ n^2 \equiv a \bmod D\}. \tag{1.2}$$

2. Results

The main purpose of this paper is to analyze the distribution of the sum of digits function $s_q(P(n))$ for polynomials $P \in \mathbb{Z}[X]$ such that $P(\mathbb{N}) \subset \mathbb{N}$ when the degree d of the polynomial P is greater or equal to 3.

For $d = 2$ the method introduced by Mauduit and Rivat in order to establish Theorem E lies on a carry lemma that allows them to concentrate the Fourier analysis on a very short window of digits. Then the remaining exponential sums can be handle efficiently by estimates on incomplete quadratic Gaussian sums. Two new difficulties arise when $d \geq 3$. First the estimates for the incomplete exponential sums are not as good as for $d = 2$. Secondly the carry lemma permits only to remove a smaller proportion of digits (see remark 4). This leads to several difficulties in the control of the Fourier transforms.

Using Vinogradov estimates on incomplete exponential sums and a more precise control of the Fourier transforms, we will be able to give a partial answer to Gelfond's problem valid for integer polynomials of any degree.

The main result of this paper is the following one.

Theorem 1. *Let $d \geq 2$ be an integer, $q \geq q_0(d)$ a sufficiently large prime number, and $P \in \mathbb{Z}[X]$ of degree d such that $P(\mathbb{N}) \subset \mathbb{N}$ for which the leading coefficient a_d is co-prime to q. If $(q-1)\alpha \in \mathbb{R} \setminus \mathbb{Z}$ then there exists $\sigma > 0$ with*

$$\sum_{n<x} e(\alpha s_q(P(n))) \ll x^{1-\sigma} \tag{2.1}$$

where the implied constant depends on q, d and α.

Remark 1. It follows from the proof of Theorem 1 that we can choose $\sigma = c\,\|(q-1)\alpha\|^2$ for some constant $c > 0$ depending only on q and d. Furthermore we will show that

$$q_0(d) \leq e^{67d^3(\log d)^2}. \tag{2.2}$$

Remark 2. The assumptions that q is prime and that a_d is co-prime to q are not really necessary. The method we introduce to prove theorem 1 holds for general $q \geq q_0(d)$ and $a_d > 0$. However, the proof would be even much more technical. Therefore we decided to restrict ourselves to this simplified case, since the main *incompleteness* of the theorem, namely that we cannot say anything for small $q < q_0(d)$, remains being an open problem and it is questionable whether the methods we use are sufficient to cover the cases of small q.

The following theorems can be easily deduced from Theorem 1.

Theorem 2. *Let $d \geq 2$ be an integer, $q \geq q_0(d)$ a sufficiently large prime number, $P \in \mathbb{Z}[X]$ of degree d such that $P(\mathbb{N}) \subset \mathbb{N}$ for which the leading coefficient a_d is co-prime to q, and m an integer, $m \geq 1$. Then there exists $\sigma_{q,m} > 0$ such that for all integers a*

$$\mathrm{card}\{n \leq x : s_q(P(n)) \equiv a \bmod m\} = \frac{x}{m} Q(a,D) + O(x^{1-\sigma_{q,m}}),$$

where $D = (q-1,m)$ and

$$Q(a,D) = \mathrm{card}\{0 \leq n < D : P(n) \equiv a \bmod D\}.$$

Remark 3. There is no simple formula to express $Q(a,D)$ in the general case, but for any a and D fixed, we have

$$Q(a,D) = \prod_{p\,|\,D} Q(a, p^{v_p(D)})$$

(see [25, chapitre 5.9]). In the special case where $D = 1$ we have $Q(a,D) = 1$.

Theorem 3. *Let $d \geq 2$ be an integer, $q \geq q_0(d)$ a sufficiently large prime number, and $P \in \mathbb{Z}[X]$ of degree d such that $P(\mathbb{N}) \subset \mathbb{N}$ for which the leading coefficient a_d is co-prime to q. Then the sequence $(\alpha s_q(P(n)))_{n\in\mathbb{N}}$ is uniformly distributed modulo 1 if and only if α is an irrational number.*

Let us consider the following question:

Problem 2. For any integer valued polynomial P of degree d and for any integer k is close to $\frac{q-1}{2}dx\log_q x$, give the number of integers $n \leq x$ such that $s_q(P(n)) = k\}$.

For $P(X) = X^2$ the estimates obtained in [17] are uniform in α so that the methods we used in [7] permit to answer Problem 2 when $d = 2$.

But, as the estimate (2.1) is not uniform in α Problem 2 w remains open for $d \geq 3$.

The structure of the paper is the following one: in section 3 we present some auxiliary results concerning combinatorial lemmas and Fourier transforms estimates, in Section 4 we prove Theorem 1 and then Theorems 2 and 3 are derived in Section 5.

3. Auxiliary Results

3.1. **Van der Corput's inequality.** We recall van der Corput's inequality:

Lemma 1. *For all complex numbers $z_1, z_2, \ldots, z_N$ and integer $R \geq 1$ we have*

$$\left|\sum_{n=1}^{N} z_n\right| \leq \left(\frac{N-1+R}{R}\sum_{|r|<R}\left(1-\frac{|r|}{R}\right)\sum_{1\leq n,n+r\leq N} z_{n+r}\overline{z_n}\right)^{1/2}.$$

Proof. See for example Lemme 4 of [18] . □

We will need also the following variant of van der Corput's inequality, which gives some flexibility in the indexes:

Lemma 2. *For all integers $1 \leq A \leq B \leq N$, all integers $R \geq 1$ and all complex numbers $z_1, z_2, \ldots, z_N$ of modulus ≤ 1 we have*

$$\left|\sum_{n=A}^{B} z_n\right| \leq \left(\frac{B-A+1}{R}\sum_{|r|<R}\left(1-\frac{|r|}{R}\right)\sum_{1\leq n,n+r\leq N} z_{n+r}\overline{z_n}\right)^{1/2} + \frac{R}{2}.$$

Proof. This is Lemme 15 of [17, p. 123] . □

3.2. **A Carry-Lemma.** Let $s_q^{[<\lambda]}$ denote the truncated sum-of-digits function

$$s_q^{[<\lambda]} = \sum_{j<\lambda} \varepsilon_j(n).$$

The truncated sum-of-digits function was introduced in [8] and the following property is a generalization of [17, Lemme 16], where the polynomial $P(X) = X^2$ is considered.

Lemma 3. *Suppose that $P \in \mathbb{Z}[X]$ of degree $d \geq 2$ is such yhat $P(\mathbb{N}) \subset \mathbb{N}$ and that ν and ρ are integers with $\nu \geq 2$ and $1 \leq \rho \leq \nu/d$. For every integer r with $|r| < q^\rho$ let $E(r,\nu,\rho)$ denote the number of integers n with $q^{\nu-1} < n \leq q^\nu$ and*

$$s_q(P(n+r)) - s_q(P(n)) \neq s_q^{[<(d-1)\nu+2\rho]}(P(n+r)) - s_q^{[<\nu+2\rho]}(P(n)). \tag{3.1}$$

Then we have

$$E(r,\nu,\rho) \le C\, q^{\nu-\rho}, \tag{3.2}$$

where the constant $C>0$ depends on the polynomial P.

Proof. First observe that $|P(n+r)-P(n)| \le c_1 q^{(d-1)\nu+\rho} \le q^{(d-1)\nu+\rho+C_1}$ for some constants $c_1>0$ and $C_1>0$. If $\rho \le C_1$ then (3.2) is certainly true (for a proper constant $C>0$). Thus we may assume that $\rho > C_1$.

Assume that $P(n+r)-P(n)>0$. This means that if we add $P(n+r)-P(n)$ to $P(n)$ then this will affect certainly the first $(d-1)\nu+\rho+C_1$ digits. Furthermore, if n satisfies (3.1) then the digits of $a_j = \varepsilon_j(P(n))$ have to satisfy $a_j = q-1$ for $(d-1)\nu+\rho+C_1 \le j < (d-1)\nu+2\rho$. Hence, it is sufficient to estimate the number of n with this property. It is clear that this property is equivalent to the statement that there exists a positive integer $m \le q^{\nu-2\rho}$ with $\lfloor P(n)/q^{(d-1)\nu+\rho+C_1} \rfloor = q^{\rho-C_1}m-1$. Equivalently this means that

$$q^{\rho-C_1}m - 1 \le \frac{P(n)}{q^{(d-1)\nu+\rho+C_1}} < q^{\rho-C_1}m. \tag{3.3}$$

Hence, for given $m \le q^{\nu-2\rho}$ the number of n (with $q^{\nu-1} < n \le q^\nu$) that satisfy (3.3) is bounded by

$$1 + c_2 q^{\frac{(d-1)\nu+2\rho}{d}-\rho} m^{\frac{1}{d}-1} + c_3 q^{\frac{(d-1)\nu+2\rho}{d}-\nu} m^{\frac{1}{d}}$$

for certain constants $c_2, c_3 > 0$. Consequently the total number of n with these restrictions if bounded by

$$\begin{aligned} &\ll q^{\nu-2\rho} + q^{\frac{(d-1)\nu+2\rho}{d}-\rho} q^{\frac{\nu-2\rho}{d}} + q^{\frac{(d-1)\nu+2\rho}{d}-\nu} q^{\frac{\nu-2\rho}{d}+\nu-2\rho} \\ &\ll q^{\nu-\rho}. \end{aligned}$$

A similar estimate holds for those n with $P(n+r)-P(n) \le 0$. This proves the lemma. □

Remark 4. Heuristically this lemma allows us, for most integers n, to get rid of the digits of index between $(d-1)\nu$ and $d\nu$. In the case $d=2$ we can remove in this way almost half of the digits and this was a crucial argument in the proof of Theorem E. When $d \ge 3$, we remove only a smaller proportion $(1/d)$ of digits and this leads to a more difficult situation.

3.3. **Exponential Sum Estimates.** In what follows we will use several estimates of exponential sums. The first one is the following version of Vinogradov's estimate that is due to Montgomery [19].

Lemma 4. *Suppose that P is a polynomial of degree $d \ge 2$ with real coefficients whose leading coefficient α_d satisfies*

$$\left|\alpha_d - \frac{a}{q}\right| \le \frac{1}{q^2}$$

with $(a,q)=1$ and $N \le q \le N^{d-1}$. Then

$$\sum_{n=1}^{N} e(P(n)) \ll N^{1-\frac{1}{11d^2\log d}}, \tag{3.4}$$

where the constant implied by $\ll$ depends on the degree d.

Note that the condition $N \le q \le N^{d-1}$ can be weakened but then the exponential saving gets worse. For example, if $q = N^{d-\tau}$ for some $\tau \in [0,1]$ then we have

$$\sum_{n=1}^{N} e(P(n)) \ll N^{1-\frac{\tau}{11d^2\log d}}. \tag{3.5}$$

For example, in the proof of Theorem 1 we will need estimates for exponential sums of the form

$$S = \sum_{n<q^\nu} e(P(n)),$$

where P is of the form $P(x) = \frac{a}{q^\lambda}x^d + \cdots$, $\lambda = (d-1)\nu + 2\rho > (d-1)\nu$, and $(a, q^\lambda) = 1$. By splitting up the sum according to $n = q^{2\mu}n' + \ell$ with $0 \le \ell < q^{2\rho}$ and $0 \le n' < q^{\nu-2\rho}$ we obtain (since $\lambda - 2\rho d = (d-1)(\nu - 2\rho)$)

$$\begin{aligned} S &= \sum_{0\le \ell < q^{2\rho}} \sum_{n'<q^{\nu-2\rho}} e\left(\frac{a}{q^{(d-1)(\nu-2\rho)}}(n')^d + \cdots\right) \\ &\ll q^{2\rho} q^{(\nu-2\rho)\left(1-\frac{1}{11d^2\log d}\right)} \\ &= q^{\nu\left(1-\left(1-\frac{\nu}{2\rho}\right)\frac{1}{11d^2\log d}\right)}. \end{aligned}$$

This is in accordance with (3.5).

Finally we formulate a lemma that applies also in the range that is not covered by Lemma 4, see [11, Proposition 8.2].

Lemma 5. *Suppose that $d \ge 2$ and $P(x) = \alpha_d x^d + \cdots + \alpha_0$ is a polynomial with rational leading coefficient $\alpha_d \ne 0$. Then*

$$\sum_{n<N} e(P(n)) \ll N^{1-2^{1-d}} + N^{1-d\,2^{1-d}} \left(\sum_{1\le |s_1|,\ldots|s_{d-1}|<N} \min\left(N, \frac{1}{|\sin(\pi\alpha_d d! s_1\cdots s_{d-1})|}\right)\right)^{2^{1-d}},$$

where the implied constant depends on d.

Proof. For the reader's convenience we present a short proof.

For $d = 2$ one just applies van der Corput's inequality (Lemma 1) with $a = 0$, $B = N-1$, and $R = N$ and obtains

$$\left|\sum_{n<N} e\left(\alpha_2 n^2 + \alpha_1 n\right)\right| \ll N^{1/2} + \left(\sum_{1\le|s|<N} \left|\sum_{0\le n, n+s<N} e\left(2\alpha_2 s n\right)\right|\right)^{1/2}.$$

Now one proceeds by induction and shows (by applying van der Corput's inequality and Hölder's inequality)

$$\sum_{n<N} e(P(n)) \ll N^{1-2^{1-d}}$$

$$+ N^{1-d\,2^{1-d}} \left(\sum_{1\le |s_1|,\ldots |s_{d-1}|<N} \left| \sum_{0\le n, n+s_1,\ldots,n+s_{d-1}<N} e\left(\alpha_d d! s_1 \cdots s_{d-1} n\right) \right| \right)^{2^{1-d}}.$$

Finally, since we have for any interval I (of length $|I|\ge 1$)

$$\sum_{n\in I} e\left(\alpha n\right) \ll \min\left(|I|, \frac{1}{|\sin\left(\pi\alpha\right)|}\right),$$

the lemma follows directly. □

3.4. **Fourier-Analytic Tools.** A major ingredient of the proof of Theorem 1 is the discrete Fourier analysis of the function

$$n \mapsto e(f_\lambda(n)),$$

where $f_\lambda(n)$ denotes the function

$$f_\lambda(n) = \alpha \sum_{j<\lambda} \varepsilon_j(n) = \alpha\, s_q^{[<\lambda]}(n). \tag{3.6}$$

Observe that f_λ is periodic with period q^λ.

We set

$$F_\lambda(h,\alpha) = q^{-\lambda} \sum_{0\le u<q^\lambda} e\left(f_\lambda(u) - huq^{-\lambda}\right). \tag{3.7}$$

Furthermore set

$$\varphi_q(t) = \frac{|\sin(\pi q t)|}{|\sin(\pi t)|},$$

$$\psi_q(t) = \frac{1}{q}\sum_{0\le r<q} \varphi_q\left(t+\frac{r}{q}\right),$$

$$\eta_q = \frac{\log\psi_q\left(\frac{1}{2q}\right)}{\log q}.$$

Then the following properties hold.

Lemma 6. *Let $q\ge 2$ and $\lambda\ge 1$ be integers and $F_\lambda(h,\alpha)$ and η_q be defined as above.*

(1) *Set $c_q = \frac{\pi^2}{12\log q}\left(1-\frac{2}{q+1}\right)$. Then we have uniformly for all real α*

$$|F_\lambda(h,\alpha)| \le e^{\pi^2/48} q^{-c_q \|(q-1)\alpha\|^2 \lambda}.$$

(2) *Suppose that* $0 \le \delta \le \lambda$. *Then for all integers* a

$$\sum_{\substack{0 \le h < q^\lambda \\ h \equiv a \bmod q^\delta}} |F_\lambda(h,\alpha)|^2 = |F_\delta(a,\alpha)|^2.$$

(3) *Suppose that* $0 \le \delta \le \lambda$. *Then for all integers* a

$$\sum_{\substack{0 \le h < q^\lambda \\ h \equiv a \bmod q^\delta}} |F_\lambda(h,\alpha)| \le q^{\eta_q(\lambda-\delta)} |F_\delta(a,\alpha)|.$$

(4) *Suppose that* $0 \le \delta_1 \le \delta \le \lambda$. *Then*

$$\sum_{\substack{0 \le h_1, h_2 < q^\lambda \\ h_1 + h_2 \equiv 0 \bmod q^\delta \\ h_1 \equiv 0 \bmod q^{\delta_1}}} |F_\lambda(h_1,\alpha) F_\lambda(-h_2,\alpha)| \le q^{2\eta_q(\lambda-\delta)} |F_{\delta_1}(0,\alpha)|^2.$$

Proof. These are slight and direct extensions of corresponding estimates from [17, 18]. □

Note that η_q can be estimated by

$$\eta_q \le \frac{\log\left(\frac{2}{q\sin\frac{\pi}{2q}} + \frac{2}{\pi}\log\frac{2q}{\pi}\right)}{\log q}$$

which ensures that $\eta_q \to 0$ as $q \to \infty$. (The upper bound is asymptotically equivalent to $\log\log q/\log q$.) For example, we have $\eta_2 = 0.5$ and $\eta_3 \approx 0.4649$, see [18].

In the proof of Theorem 1 we will need the assumption

$$2(d-1)\eta_q < \frac{1}{11 d^2 \log d}$$

which is implied by (2.2). Note that (2.2) also implies $q > d!$.

The next lemma extends a property of [18] and will be crucial in the proof of Theorem 1.

Lemma 7. *Suppose that* $d \ge 2$, *that* q *is a prime number, and that* $(a,q) = 1$ *Furthermore, let* λ, ν, *and* δ *non-negative integers with* $\lambda \ge (d-2)\nu + \delta$. *Then for every* $\varepsilon > 0$ *we have*

$$\sum_{\substack{1 \le h < q^\lambda \\ (h,q^\lambda) = q^\delta}} |F_\lambda(h,\alpha)|^2 \sum_{1 \le |s_1|,\ldots|s_{d-2}| < q^\nu} \frac{1}{\left|\sin\left(\pi \frac{a h s_1 \cdots s_{d-2}}{q^\lambda}\right)\right|}$$

$$\ll \nu\, q^{\lambda-\delta+\nu\varepsilon}\, q^{-c_q\,(\lambda-(d-2)\nu)\|(q-1)\alpha\|^2}, \tag{3.8}$$

where c_q *is defined in Lemma 6 and the implied constant depends on* d *and on* ε.

Proof. We proceed by induction and start with $\lambda = (d-2)\nu + \delta$. Note that if $1 \le |s_1|, \ldots |s_{d-2}| < q^\nu$ and then we certainly have

$$1 \le |s_1 \cdots s_{d-2}| \le q^{(d-2)\nu} = q^{\lambda-\delta}.$$

Furthermore, the divisor functions $\tau(n) = \mathrm{card}\{d \le n : d|n\}$ satisfies $\tau(n) \ll n^\varepsilon$ for every $\varepsilon > 0$. Hence, it follows that for every $\varepsilon > 0$ we have uniformly for all residue classes $1 \le \ell < q^{\lambda-\delta}$

$$\mathrm{card}\{(s_1,\ldots,s_{d-2}) : 1 \le |s_1|,\ldots|s_{d-2}| < q^\nu,\ s_1\cdots s_{d-2} \equiv \ell \bmod q^{\lambda-\delta}\} \ll q^{\nu\varepsilon}.$$

We recall that $(a,q)=1$ and that $(h,q^\lambda)=q^\delta$. Hence, if we write $H = hq^{-\delta}$ it also follows that for every residue classes $1 \le \ell < q^{\lambda-\delta}$

$$\mathrm{card}\{(s_1,\ldots,s_{d-2}) : 1 \le |s_1|,\ldots|s_{d-2}| < q^\nu,\ aHs_1\cdots s_{d-2} \equiv \ell \bmod q^{\lambda-\delta}\} \ll q^{\nu\varepsilon}. \tag{3.9}$$

Hence, (3.9) implies

$$\begin{aligned}\sum_{1\le|s_1|,\ldots|s_{d-2}|<q^\nu} \frac{1}{\left|\sin\left(\pi\frac{aHs_1\cdots s_{d-2}}{q^{\lambda-\delta}}\right)\right|} &\ll q^{\lambda-\delta+\nu\varepsilon}\log(q^{\lambda-\delta}) \\ &\ll \nu\, q^{\lambda-\delta+\nu\varepsilon}.\end{aligned}$$

Furthermore, by Lemma 6

$$\begin{aligned}\sum_{\substack{1\le h<q^\lambda\\(h,q^\lambda)=q^\delta}} |F_\lambda(h,\alpha)|^2 &\le |F_\delta(0,\alpha)|^2 \\ &\ll q^{-2c_q\|(q-1)\alpha\|^2\delta}\end{aligned}$$

Consequently we obtain (3.8) for $\lambda = (d-2)\nu+\delta$.

Similarly we can check (3.8) for $\lambda = (d-2)\nu+\delta+1$.

Finally we show inductively that if (3.8) is valid for λ then (3.8) is still valid when λ is replaced by $\lambda+2$. For this purpose we consider the property that

$$\sum_{\substack{0\le h<q^{\lambda+2}\\(h,q^{\lambda+2})=q^\delta}} \frac{|F_{\lambda+2}(h,\alpha)|^2}{\left|\sin\left(\pi\frac{Ah}{q^{\lambda+2}}\right)\right|} \le q^{2-2c_q\|(q-1)\alpha\|^2} \sum_{\substack{0\le h<q^{\lambda}\\(h,q^{\lambda})=q^\delta}} \frac{|F_{\lambda}(h,\alpha)|^2}{\left|\sin\left(\pi\frac{Ah}{q^{\lambda}}\right)\right|} \tag{3.10}$$

holds for all integers $0 \le \delta \le \lambda$ and for all A with $(A,q^\lambda) < q^{\lambda-\delta}$.

It is clear that (3.10) implies the induction step. One only has to replace A by $as_1\cdots s_{d-2}$ and take the sum over all $s_1,\cdots s_{d-2}$.

Hence, it remains to check (3.10). Set

$$\Phi_1(A,x) = \frac{1}{q^2}\sum_{0\le j<q} \varphi_q^2\left(\alpha - \frac{x+j}{q}\right)\varphi_q\left(\frac{A(x+j)}{q}\right)$$

and

$$\Phi_2(A,x) = \frac{1}{q^2}\sum_{0\le j<q} \varphi_q^2\left(\alpha - \frac{x+j}{q}\right)\varphi_q\left(\frac{A(x+j)}{q}\right)\Phi_1\left(\frac{x+j}{q}\right).$$

First suppose that $(A,q)=1$. Then it follows as in [18, Lemme 21] that

$$\sum_{\substack{0\le h<q^{\lambda+2}\\ (h,q^{\lambda+2})=q^\delta}} \frac{|F_{\lambda+2}(h,\alpha)|^2}{\left|\sin\left(\pi\frac{Ah}{q^{\lambda+2}}\right)\right|} = \sum_{\substack{0\le h<q^{\lambda}\\ (h,q^{\lambda})=q^\delta}} \frac{|F_{\lambda}(h,\alpha)|^2}{\left|\sin\left(\pi\frac{Ah}{q^{\lambda}}\right)\right|}\Phi_2\left(A,\frac{h}{q^\lambda}\right). \tag{3.11}$$

Since

$$\left|F_{\lambda+1}(h'+\ell q^\lambda,\alpha)\right| = |F_\lambda(h',\alpha)|\,\frac{1}{q}\varphi_q\left(\alpha-\frac{h'}{q^{\lambda+1}}-\frac{\ell}{q}\right)$$

and

$$\frac{1}{\left|\sin\left(\pi\frac{A(h'+\ell q^\lambda)}{q^{\lambda+1}}\right)\right|} = \frac{1}{\left|\sin\left(\pi\frac{Ah'}{q^{\lambda}}\right)\right|}\,\varphi_q\left(\frac{A(h'+\ell q^\lambda)}{q^{\lambda+1}}\right)$$

it follows that

$$\begin{aligned}
\sum_{\substack{0\le h<q^{\lambda+1}\\ (h,q^{\lambda+1})=q^\delta}} \frac{|F_{\lambda+1}(h,\alpha)|^2}{\left|\sin\left(\pi\frac{Ah}{q^{\lambda+1}}\right)\right|} &= \sum_{\substack{0\le h'<q^{\lambda}\\ (h',q^{\lambda})=q^\delta}} \sum_{0\le\ell<q} \frac{\left|F_{\lambda+1}(h'+\ell q^\lambda,\alpha)\right|^2}{\left|\sin\left(\pi\frac{A(h'+\ell q^\lambda)}{q^{\lambda+1}}\right)\right|} \\
&= \sum_{\substack{0\le h'<q^{\lambda}\\ (h',q^{\lambda})=q^\delta}} \frac{|F_{\lambda}(h',\alpha)|^2}{\left|\sin\left(\pi\frac{Ah'}{q^{\lambda}}\right)\right|} \sum_{0\le\ell<q} \frac{1}{q^2}\varphi_q^2\left(\alpha-\frac{h'}{q^{\lambda+1}}-\frac{\ell}{q}\right)\varphi_q\left(\frac{A(h'+\ell q^\lambda)}{q^{\lambda+1}}\right) \\
&= \sum_{\substack{0\le h'<q^{\lambda}\\ (h',q^{\lambda})=q^\delta}} \frac{|F_{\lambda}(h',\alpha)|^2}{\left|\sin\left(\pi\frac{Ah'}{q^{\lambda}}\right)\right|}\,\Phi_1\left(a,\frac{h'}{q^\lambda}\right).
\end{aligned}$$

In completely the same way one obtains (3.11).

Moreover it follows from the proof of Lemme 21 of [18] that one has uniformly in x and for all A with $(A,q)=1$

$$\Phi_2(a,x)\le q^{2\gamma_q(\alpha)}$$

with $\gamma_q(\alpha)$ defined by

$$q^{\gamma_q(\alpha)} = \max_{t\in\mathbb{R}} \sqrt{\phi_q(\alpha+t)\phi_q(\alpha+qt)}.$$

It follows from Lemme 7 of [17] that

$$\gamma_q(\alpha)\le 1-c_q\|(q-1)\alpha\|^2$$

where c_q is defined in Lemma 6, so that we have

$$\Phi_2(a,x)\le q^{2-2c_q\|(q-1)\alpha\|^2}.$$

Of course, this proves (3.10) in this case.

Now suppose that $(A,q^\lambda)=q^\mu$ with $\lambda-\mu>\delta$. We also set $A_1=Aq^{-\mu}$. Then it follows from Lemma 6 that

$$\begin{aligned}
\sum_{\substack{0\le h<q^\lambda\\ (h,q^\lambda)=q^\delta}} \frac{|F_\lambda(h,\alpha)|^2}{\left|\sin\left(\pi\frac{Ah}{q^\lambda}\right)\right|} &= \sum_{\substack{0\le h<q^\lambda\\ (h,q^\lambda)=q^\delta}} \frac{|F_\lambda(h,\alpha)|^2}{\left|\sin\left(\pi\frac{A_1h}{q^{\lambda-\mu}}\right)\right|}\\
&= \sum_{\substack{0\le h'<q^{\lambda-\mu}\\ (h',q^{\lambda-\mu})=q^\delta}} \frac{1}{\left|\sin\left(\pi\frac{A_1h'}{q^{\lambda-\mu}}\right)\right|} \sum_{\substack{0\le h<q^\lambda\\ h\equiv h' \bmod q^{\lambda-\mu}}} |F_\lambda(h,\alpha)|^2\\
&= \sum_{\substack{0\le h'<q^{\lambda-\mu}\\ (h',q^{\lambda-\mu})=q^\delta}} \frac{|F_{\lambda-\mu}(h',\alpha)|^2}{\left|\sin\left(\pi\frac{A_1h'}{q^{\lambda-\mu}}\right)\right|}
\end{aligned}$$

This means that we can reduce the general case $\mu>1$ to the case $\mu=0$ and, thus, (3.10) holds in all cases.

This completes the proof of the lemma. □

4. Proof of Theorem 1

4.1. **Reduction of the Problem.** In order to simplify notation we set $f(n)=\alpha s_q(n)$. The major aim is to estimate the exponential sum

$$S=\sum_{n\le x} e(f(P(n))).$$

We also make the general assumption $d\ge 3$ since the case of quadratic polynomials is completely covered in the analysis of [17].

As usual we will only consider sums of the following form.

Proposition 1. *Let $d\ge 3$ be an integer, $q\ge q_0(d)$ be a prime number, and $P\in\mathbb{Z}[X]$ of degree d such that $P(\mathbb{N})\subset\mathbb{N}$ for which the leading coefficient a_d is co-prime to q. Then*

$$S_1=\sum_{q^{\nu-1}<n\le x} e(f(P(n))) \ll q^{\nu-c\|(q-1)\alpha\|^2\nu}$$

uniformly for $q^{\nu-1}<x\le q^\nu$, where $\nu\ge\nu_1=\nu_1(q,\alpha)$ is sufficiently large, $c>0$ depends on q and d and the implied constant depends on q, d and α.

It is an easy task to derive Theorem 1 from Proposition 1. From the obvious decomposition

$$\begin{aligned}
\sum_{1\le n\le x} e(f(P(n))) = e(f(P(1)) &+ \sum_{1\le i\le \nu_1-1}\ \sum_{q^{i-1}<n\le q^i} e(f(P(n)))\\
&+ \sum_{\nu_1\le i\le\nu-1}\ \sum_{q^{i-1}<n\le q^i} e(f(P(n))) + \sum_{q^{\nu-1}<n\le x} e(f(P(n)))
\end{aligned}$$

we obtain immediately

$$\begin{aligned}\left|\sum_{1\le n\le x} e(f(P(n)))\right| &\ll q^{\nu_1-1} + \sum_{\nu_1\le i\le \nu} q^{i-c\|(q-1)\alpha\|^2 i}\\ &\ll q^{\nu-c\|(q-1)\alpha\|^2\nu}\\ &\ll x^{1-c\|(q-1)\alpha\|^2}\end{aligned}$$

which is precisely the statement of Theorem 1.

The first step is to use van der Corput's inequality (Lemma 2). With $A=1$, $B=\lfloor x\rfloor - q^{\nu-1}$, $N=q^\nu-q^{\nu-1}$, $z_n = e(f(P(q^{\nu-1}+n))$ and $R=q^\rho$ we obtain

$$|S_1| \ll q^{(\nu-\rho)/2}\left|\sum_{|r|<q^\rho}\left(1-\frac{|r|}{q^\rho}\right)\left(\sum_{q^{\nu-1}<n,n+r\le q^\nu} e(f(P(n+r))-f(P(n)))\right)\right|^{1/2} + \frac{q^\rho}{2}.$$

By separating the case $r=0$ and by suppressing the condition $q^{\nu-1}<n+r\le q^\nu$ (by adding proper error terms) we get the upper bound

$$|S_1| \ll q^{\nu-\frac{\rho}{2}} + q^{\frac{\nu+\rho}{2}} + q^\rho + q^{\frac{\nu}{2}} \max_{1\le|r|<q^\rho}\left|\sum_{q^{\nu-1}<n\le q^\nu} e(f(P(n+r))-f(P(n)))\right|^{1/2}.$$

In order to simplify our estimates we will assume (without loss of generality) that $\nu\ge 10$ and

$$1\le \rho\le \frac{\nu}{10} \tag{4.1}$$

which ensures that

$$q^{\nu-\frac{\rho}{2}} + q^{\frac{\nu+\rho}{2}} + q^\rho \ll q^{\nu-\frac{\rho}{2}}.$$

The next step is to replace the difference $f(P(n+r))-f(P(n))$ by $f_{(d-1)\nu+2\rho}(P(n+r))-f_{(d-1)\nu+2\rho}(P(n))$ where $f_{(d-1)\nu+2\rho}$ is defined by (3.6). By setting

$$S_2(r,\nu,\rho) = \sum_{q^{\nu-1}<n\le q^\nu} e(f_{(d-1)\nu+2\rho}(P(n+r)) - f_{(d-1)\nu+2\rho}(P(n)))$$

we obtain (with the help of Lemma 3)

$$\begin{aligned}|S_1| &\ll q^{\nu-\frac{\rho}{2}} + q^{\frac{\nu}{2}}\max_{1\le|r|<q^\rho}\left(|S_2(r,\nu,\rho)| + E(r,\nu,\rho)\right)^{1/2}\\ &\ll q^{\nu-\frac{\rho}{2}} + q^{\frac{\nu}{2}}\max_{1\le|r|<q^\rho}|S_2(r,\nu,\rho)|^{1/2}.\end{aligned}$$

Therefore we only have to discuss the sums $S_2(r,\nu,\rho)$.

4.2. **Fourier Analysis of $S_2(r,\nu,\rho)$.** By using the orthogonality relation for of the exponential function it follows with

$$\lambda = (d-1)\nu + 2\rho$$

that

$$\begin{aligned} S_2(r,\nu,\rho) &= \sum_{q^{\nu-1}<n\le q^\nu} e\left(f_\lambda(P(n+r)) - f_\lambda(P(n))\right) \\ &= \frac{1}{q^{2\lambda}} \sum_{0\le u_1<q^\lambda} \sum_{0\le u_2<q^\lambda} e(f_\lambda(u_1) - f_\lambda(u_2)) \\ &\quad \times \sum_{q^{\mu-1}\le n<q^\nu} \sum_{0\le h_1<q^\lambda} e\left(\frac{h_1(P(n+r)-u_1)}{q^\lambda}\right) \sum_{0\le h_2<q^\lambda} e\left(\frac{h_2(P(n)-u_2)}{q^\lambda}\right) \\ &= \sum_{0\le h_1<q^\lambda} \sum_{0\le h_2<q^\lambda} F_\lambda(h_1,\alpha)\,\overline{F_\lambda(-h_2,\alpha)} \sum_{q^{\nu-1}\le n<q^\nu} e\left(\frac{h_1P(n+r)+h_2P(n)}{q^\lambda}\right), \end{aligned}$$

where F_λ is defined by (3.7). In order to estimate $S_2(r,\nu,\rho)$ we will have a close look to the exponential sum

$$S_3(r,\nu,\rho,h_1,h_2) = \sum_{q^{\nu-1}\le n<q^\nu} e\left(\frac{h_1P(n+r)+h_2P(n)}{q^\lambda}\right).$$

Suppose that a_j, $0\le j\le d$, are the coefficients of P. Then we have

$$h_1P(x+r) + h_2P(x) = (h_1+h_2)a_dx^d + \left(h_1da_dr + (h_1+h_2)a_{d-1}\right)x^{d-1} + \cdots$$

and consequently

$$\frac{h_1P(x+r)+h_2P(x)}{q^\lambda} = \frac{a_d(h_1+h_2)}{q^\lambda}x^d + \frac{h_1da_dr+(h_1+h_2)a_{d-1}}{q^\lambda}x^{d-1} + \cdots.$$

We now use the assumption that q is prime and that a_d is co-prime to q. In order to apply Lemma 4 we have to assume that the leading coefficient of the polynomial is close or equal to a rational number with co-prime numerator and denominator. This means that we have to distinguish between the cases $(h_1+h_2,q^\lambda) = q^\delta$, where $0\le\delta\le\lambda$. In particular we have to cut this range into three pieces. For this purpose we introduce an additional parameter μ that satisfies $(d-2)\nu+2\rho<\mu\le\lambda$ (and in fact it will be chosen very close to λ, see Section 4.6) and we consider the three following cases:

(1) $0\le\delta\le(d-2)\nu+2\rho$. In this case we will apply Lemma 4.
(2) $(d-2)\nu+2\rho<\delta\le\mu$. In this case we will also work directly with Lemma 4 but in a slightly different way.
(3) $\mu<\delta\le\lambda$. This is the most difficult case. Here we will apply Lemma 5 and proper estimates for the Fourier terms $F_\lambda(h,\alpha)$.

The next three sections deal with these cases separately.

4.3. **Small δ.** Set

$$S_4(r,\nu,\rho) = \sum_{0\le\delta\le(d-2)\nu+2\rho} \sum_{\substack{0\le h_1,h_2<q^\lambda\\ (h_1+h_2,q^\lambda)=q^\delta}} F_\lambda(h_1,\alpha)\overline{F_\lambda(-h_2,\lambda)} \times \sum_{q^{\nu-1}\le n<q^\nu} e\left(\frac{h_1P(x+r)+h_2P(x)}{q^\lambda}\right).$$

If $(h_1+h_2,q^\lambda)=q^\delta$ we have

$$\frac{h_1P(x+r)+h_2P(x)}{q^\lambda} = \frac{a_dH}{q^{\lambda-\delta}}x^d+\cdots$$

for some integer H with $(H,q)=1$. Note also that $\delta\le(d-2)\nu+2\rho$ implies $\lambda-\delta\ge\nu$. Hence, by Lemma 4 (and its extension (3.5)) we have

$$\sum_{q^{\nu-1}\le n<\nu} e\left(\frac{h_1P(x+r)+h_2P(x)}{q^\lambda}\right) \ll q^{\nu(1-(1-\frac{2\rho}{\nu})C_d)},$$

where C_d abbreviates

$$C_d = \frac{1}{11d^2\log d}.$$

Furthermore, by Lemma 6 we have

$$\sum_{\substack{0\le h_1,h_2<q^\lambda\\ (h_1+h_2,q^\lambda)=q^\delta}} F_\lambda(h_1,\alpha)\overline{F_\lambda(-h_2,\lambda)} \ll q^{2\eta_q(\lambda-\delta)}$$

Consequently

$$\begin{aligned} S_4(r,\nu,\rho) &\ll \sum_{0\le\delta\le(d-2)\nu+2\rho} q^{2\eta_q(\lambda-\delta)}q^{\nu(1-(1-\frac{2\rho}{\nu})C_d)} \\ &\ll q^{\nu+2\rho(2\eta_q+C_d)-\nu(C_d-2(d-1)\eta_q)}. \end{aligned}$$

If $q\ge q_0(d)$ is sufficiently large then $2(d-1)\eta_q<C_d$. Furthermore if we suppose that

$$0<\rho\le\frac{C_d-2(d-1)\eta_q}{4(2\eta_q+C_d)}\nu \tag{4.2}$$

then

$$S_4(r,\nu,\rho)\ll q^{\nu(1-\kappa)} \tag{4.3}$$

with $\kappa=\frac{1}{2}\left(C_d-2(d-1)\eta_q\right)>0$ that is independent from r and α (provided that $q\ge q_0(d)$).

4.4. **Medium δ.** Next set

$$S_5(r,\nu,\rho,\mu) = \sum_{(d-2)\nu+2\rho<\delta\le\mu} \sum_{\substack{0\le h_1,h_2<q^\lambda \\ (h_1+h_2,q^\lambda)=q^\delta}} F_\lambda(h_1,\alpha)\overline{F_\lambda(-h_2,\lambda)} \times \sum_{q^{\nu-1}\le n<q^\nu} e\left(\frac{h_1P(x+r)+h_2P(x)}{q^\lambda}\right),$$

where $(d-2)\nu+2\rho<\mu\le\lambda$. Again if $(h_1+h_2,q^\lambda)=q^\delta$ we have

$$\frac{h_1P(x+r)+h_2P(x)}{q^\lambda} = \frac{a_dH}{q^{\lambda-\delta}}x^d+\cdots$$

for some integer H with $(H,q)=1$. However, if $\delta>(d-2)\nu+2\rho$ then we have $\lambda-\delta<\nu$. Thus we subdivide the interval $[q^{\nu-1},q^\nu)$ into $q^{\nu-\lambda+\delta-1}$ sub-intervals of length $q^{\lambda-\delta}$ and apply then Lemma 6. Hence we have

$$\sum_{q^{\nu-1}\le n<\nu} e\left(\frac{h_1P(x+r)+h_2P(x)}{q^\lambda}\right) \ll q^{\nu-\lambda+\delta}q^{(\lambda-\delta)(1-C_d)} = q^{\nu-(\lambda-\delta)C_d}.$$

Consequently we obtain

$$\begin{aligned} S_5(r,\nu,\rho,\mu) &\ll \sum_{(d-2)\nu+2\rho<\delta\le\mu} q^{2\eta_q(\lambda-\delta)}q^{\nu-(\lambda-\delta)C_d} \\ &\ll q^{\nu-(C_d-2\eta_q)(\lambda-\mu)}. \end{aligned} \tag{4.4}$$

4.5. **Large δ.** Set

$$S_6(r,\nu,\rho,\mu) = \sum_{\mu<\delta\le\lambda} \sum_{\substack{0\le h_1,h_2<q^\lambda \\ (h_1+h_2,q^\lambda)=q^\delta}} F_\lambda(h_1,\alpha)\overline{F_\lambda(-h_2,\lambda)} \times \sum_{q^{\nu-1}\le n<\nu} e\left(\frac{h_1P(x+r)+h_2P(x)}{q^\lambda}\right).$$

This case of large δ is the most difficult one. The reason is that the denominator $q^{\lambda-\delta}$ gets too small so that Lemma 4 gives no proper error term. In fact by considering proper residue classes we will omit the leading term $a_d(h_1+h_2)q^{-\lambda}n^d$ completely.

Set $\tau=\lceil\frac{\lambda-\delta}{d-1}\rceil$ and write $n=q^\tau n'+\ell$ with $0\le\ell<q^\tau$. Then, with $H=(h_1+h_2)q^{-\delta}$ we have

$$\begin{aligned} \frac{h_1P(n+r)+h_2P(n)}{q^\lambda} &= Ha_dq^{d\tau-(\lambda-\delta)}(n')^d + dH\ell a_dq^{(d-1)\tau-(\lambda-\delta)}(n')^{d-1} \\ &\quad + Ha_{d-1}q^{(d-1)\tau-(\lambda-\delta)}(n')^{d-1} + \frac{da_dh_1r}{q^{\lambda-(d-1)\tau}}(n')^{d-1}+\cdots \end{aligned}$$

and consequently

$$e\left(\frac{h_1P(x+r)+h_2P(x)}{q^\lambda}\right) = e\left(\frac{da_dh_1r}{q^{\lambda-(d-1)\tau}}(n')^{d-1}+\cdots\right).$$

This means that the polynomial $f(x) = (h_1P(x+r)+h_2P(x))q^{-\lambda}$ of degree d is replaced by a polynomial of degree $d-1$.

Suppose that $(r,q^\lambda) = q^{\rho_1}$ for some $0 \le \rho_1 \le \rho$ and $(h_1,q^\lambda) = q^{\delta_1}$ for some $0 \le \delta_1 < \lambda$. We will distinguish again between several ranges of δ_1. Set

$$S_6'(r,\nu,\rho,\mu) = \sum_{\mu<\delta\le\lambda}\ \sum_{0\le\delta_1\le\nu+\tau-\rho_1} S_7(r,\nu,\rho,\mu,\delta,\delta_1),$$

$$S_6''(r,\nu,\rho,\mu) = \sum_{\mu<\delta\le\lambda}\ \sum_{\nu+\tau-\rho_1<\delta_1\le\mu-2\rho} S_7(r,\nu,\rho,\mu,\delta,\delta_1),$$

$$S_6'''(r,\nu,\rho,\mu) = \sum_{\mu<\delta\le\lambda}\ \sum_{\mu-2\rho<\delta_1\le\lambda} S_7(r,\nu,\rho,\mu,\delta,\delta_1),$$

where

$$S_7(r,\nu,\rho,\mu,\delta,\delta_1) = \sum_{\substack{0\le h_1,h_2<q^\lambda\\(h_1+h_2,q^\lambda)=q^\delta\\(h_1,q^\lambda)=q^{\delta_1}}} F_\lambda(h_1,\alpha)\overline{F_\lambda(-h_2,\lambda)}$$

$$\times\, q^\tau \sum_{q^{\nu-\tau-1}\le n'<q^{\nu-\tau}} e\left(\frac{da_dh_1r}{q^{\lambda-(d-1)\tau}}(n')^{d-1}+\cdots\right).$$

4.5.1. *Large δ_1.* First let us consider the sum $S_6'''(r,\nu,\rho,\mu)$. In this case we have $\delta_1 > \mu-2\rho$ which assures that the case where $\rho_1+\delta_1$ larger than $\lambda-(d-1)\tau$ or almost as large as $\lambda-(d-1)\tau$. In particular we have $\lambda-\delta_1 \le \lambda-\mu+2\rho$. Consequently the number of pairs (h_1,h_2) with the properties $0\le h_1,h_2<q^\lambda$, $(h_1,q^\lambda)=q_1^\delta$, $(h_1+h_2,q^\lambda)=q^\delta$ is bounded by $q^{2(\lambda-\mu+\rho)}$. Furthermore, we have by Lemma 6 $|F_\lambda(h,\alpha)| \ll q^{-c_q\|(q-1)\alpha\|^2\lambda}$.

Hence we have

$$S_7(r,\nu,\rho,\mu,\delta,\delta_1) \le q^{\nu+2(\lambda-\mu+\rho)}q^{-2c_q\|(q-1)\alpha\|^2\lambda}$$

and consequently

$$S_6'''(r,\nu,\rho,\mu) \le \lambda^2\, q^{\nu+2(\lambda-\mu+\rho)-2c_q\|(q-1)\alpha\|^2\lambda}. \tag{4.5}$$

In what follows we will choose ρ and μ appropriately so that the term $2c_q\|(q-1)\alpha\|^2\lambda$ dominates $2(\lambda-\mu+\rho)$ and $S_6'''(r,\nu,\rho,\mu)$ is small enough.

Next let us consider the sum $S_6''(r,\nu,\rho,\mu)$. Here we will use Lemma 4 to estimate the exponential sum

$$S_8 = \sum_{q^{\nu-\tau-1}\le n'<q^{\nu-\tau}} e\left(\frac{da_dH_1r_1}{q^{\lambda-(d-1)\tau-\delta_1-\rho_1}}(n')^{d-1}+\cdots\right),$$

where $H_1 = h_1q^{-\delta_1}$ and $r_1 = rq^{-\rho_1}$. Note that $(da_dH_1r_1,q)=1$. Note also that in this case $\delta_1<\delta$.

Suppose first that

$$\nu - \tau - \rho_1 \le \delta_1 \le (d-2)\nu + 2\rho - \rho_1 - (d-1)\tau, \tag{4.6}$$

which is equivalent to

$$q^{\nu-\tau} \le q^{\lambda-(d-1)\tau-\delta_1-\rho_1} \le q^{(d-2)(\nu-\tau)+2\rho}.$$

Hence we can apply Lemma 4 (and its extension (3.5)) to obtain the bound

$$\begin{aligned} S_8 &\ll q^{(\nu-\tau)\left(1-\left(1-\frac{2\rho}{\nu-\tau}\right)C_{d-1}\right)} \\ &\ll q^{-\tau} q^{\nu\left(1-\frac{1}{2}C_{d-1}\right)}, \end{aligned}$$

provided that

$$(\nu-\tau)\left(1-\frac{2\rho}{\nu-\tau}\right) \ge \frac{\nu}{2}. \tag{4.7}$$

It will be an easy task to choose the constants ρ and μ such that (4.7) is satisfied. Furthermore we have from Lemma 6

$$\sum_{\substack{0\le h_1,h_2<q^\lambda \\ h_1+h_2\equiv 0 \bmod q^\delta \\ h_1 \equiv 0 \bmod q^{\delta_1}}} |F_\lambda(h_1,\alpha)F_\lambda(-h_2,\alpha)| \le q^{2\eta_q(\lambda-\delta)}. \tag{4.8}$$

This leads to the estimate

$$\begin{aligned} &\sum_{\mu<\delta\le\lambda} \sum_{\nu+\tau-\rho_1<\delta_1\le(d-2)\nu+2\rho-\rho_1-(d-1)\tau} S_7(r,\nu,\rho,\mu,\delta,\delta_1) \\ &\qquad \ll \sum_{\mu<\delta\le\lambda} \sum_{\nu+\tau-\rho_1<\delta_1\le(d-2)\nu+2\rho-\rho_1-(d-1)\tau} q^{2\eta_q(\lambda-\delta)} q^{\nu\left(1-\frac{1}{2}C_{d-1}\right)} \\ &\qquad \ll \lambda\, q^{2\eta_q(\lambda-\mu)} q^{\nu\left(1-\frac{1}{2}C_{d-1}\right)} \\ &\qquad \ll \lambda q^{\nu\left(1-\frac{1}{4}C_{d-1}\right)} \end{aligned}$$

provided that

$$2\eta_q(\lambda-\mu) \le \frac{\nu}{4}C_{d-1}. \tag{4.9}$$

Again it will be easy to choose μ sufficiently close to λ such that (4.9) holds.

4.5.2. *Medium δ_1.* Next suppose that

$$(d-2)\nu + 2\rho - \rho_1 - (d-1)\tau < \delta_1 \le \mu - 2\rho \tag{4.10}$$

which is equivalent to

$$q^{\lambda-\mu-(d-1)\tau+2\rho-\rho_1} \le q^{\lambda-(d-1)\tau-\delta_1-\rho_1} < q^{\nu-\tau}.$$

Hence, by subdividing the interval $[q^{\nu-\tau-1}, q^{\nu-\tau})$ and applying Lemma 4 it follows that

$$\begin{aligned} S_8 &\ll q^{\nu-\tau-(\lambda-(d-1)\tau-\delta_1-\rho_1)} q^{(\lambda-(d-1)\tau-\delta_1-\rho_1)(1-C_{d-1})} \\ &\ll q^{-\tau} q^{\nu-\rho C_{d-1}}. \end{aligned}$$

Here we have used that

$$\lambda-(d-1)\tau-\delta_1-\rho_1 \ge \lambda-\mu-(\lambda-\delta)-1+2\rho-\rho_1 \ge \rho-1.$$

Furthermore, we can assume that

$$(d-2)\nu+2\rho-\rho_1-(d-1)\tau \ge (d-5/2)\nu. \tag{4.11}$$

Consequently $\delta_1 \ge (d-5/2)\nu$ and we get

$$\begin{aligned} &\sum_{\substack{0\le h_1,h_2<q^\lambda \\ h_1\equiv 0 \bmod q^{\delta_1}}} \sum_{h_1+h_2\equiv 0 \bmod q^\delta} |F_\lambda(h_1,\alpha)F_\lambda(-h_2,\alpha)| \\ &\qquad\qquad \ll q^{2\eta_q(\lambda-\delta)} q^{-2(d-5/2)c_q\|(q-1)\alpha\|^2\nu} \end{aligned}$$

and thus

$$\begin{aligned} &\sum_{\mu<\delta\le\lambda} \sum_{(d-2)\nu+2\rho-\rho_1-(d-1)\tau<\delta_1\le\mu-2\rho} S_7(r,\nu,\rho,\mu,\delta,\delta_1) \\ &\quad \ll \sum_{\mu<\delta\le\lambda} \sum_{(d-2)\nu+2\rho-\rho_1-(d-1)\tau<\delta_1\le\mu-2\rho} q^{2\eta_q(\lambda-\delta)} q^{-2(d-5/2)c_q\|(q-1)\alpha\|^2\nu} q^{\nu-\rho C_{d-1}} \\ &\quad \ll \lambda\, q^{2\eta_q(\lambda-\mu)} q^{\nu-\rho C_{d-1}-(2d-5)c_q\|(q-1)\alpha\|^2\nu}. \end{aligned}$$

Putting these two estimates together we obtain an upper bound for $S_6''(r,\nu,\rho,\mu)$ of the form

$$S_6''(r,\nu,\rho,\mu) \ll \lambda q^{\nu\left(1-\frac14 C_{d-1}\right)} + \lambda\, q^{2\eta_q(\lambda-\mu)} q^{\nu-\rho C_{d-1}-(2d-5)c_q\|(q-1)\alpha\|^2\nu} \tag{4.12}$$

provided that (4.7), (4.9), and (4.11) hold.

4.5.3. *Small* δ_1. Finally we deal with $S_6'(r,\nu,\rho,\mu)$ For notational convenience we set $\lambda' = \lambda-(d-1)\tau$ and $\nu' = \nu-\tau$ (where $\tau = \lceil\frac{\lambda-\delta}{d-1}\rceil$). As above we also use the abbreviations $H_1 = h_1 q^{-\delta_1}$ and $r_1 = r q^{-\rho_1}$. Furthermore we define

$$H(\lambda,\delta,\delta_1) = \{(h_1,h_2)\in\mathbb{Z}^2 : 0\le h_1,h_2<q^\lambda,\ h_1+h_2\equiv 0 \bmod q^\delta,\ h_1\equiv 0 \bmod q^{\delta_1}\}.$$

It now follows from Lemma 5 that

$$\begin{aligned} &\sum_{q^{\nu'-1}\le n'<q^{\nu'}} e\left(\frac{d a_{d-1}H_1 r_1}{q^{\lambda'-\delta_1-\rho_1}}(n')^{d-1}+\cdots\right) \ll q^{\nu'(1-2^{2-d})} \\ &\qquad + q^{\nu'(1-(d-1)\,2^{2-d})}\left(\sum_{1\le|s_1|,\ldots,|s_{d-2}|\le q^{\nu'}} \frac{1}{\left|\sin\left(\pi\frac{d!a_d H_1 r_1}{q^{\lambda'-\delta_1-\rho_1}}s_1\cdots s_{d-2}\right)\right|}\right)^{2^{d-2}}, \end{aligned}$$

provided that

$$\lambda' - \rho_1 \geq (d-2)\nu' + \delta_1,$$

or equivalently if

$$\delta_1 \leq \nu + 2\rho - \tau - \rho_1. \tag{4.13}$$

However, by assumption we have $0 \leq \delta_1 \leq \nu + \tau - \rho_1$. Hence, (4.13) is satisfied if

$$\tau \leq \rho. \tag{4.14}$$

Now we write $S_7(r,\nu,\rho,\mu,\delta,\delta_1) = T_1 + T_2$, where T_1 is estimated (with the help of (4.8)) by

$$T_1 \ll \sum_{(h_1,h_2)\in H(\lambda,\delta,\delta_1)} |F_\lambda(h_1,\alpha)F_\lambda(-h_2,\alpha)|\, q^\tau\, q^{\nu'(1-2^{2-d})} \ll q^{2\eta_q(\lambda-\delta)} q^{\nu-2^{2-d}\nu'}$$

and T_2 (with the help of Hölder's inequality) by

$$\begin{aligned} T_2 &\leq q^{\nu'(1-(d-1)\,2^{2-d})} \sum_{(h_1,h_2)\in H(\lambda,\delta,\delta_1)} |F_\lambda(h_1,\alpha)F_\lambda(-h_2,\alpha)| \\ &\qquad \times \left(\sum_{1\leq |s_1|,\ldots,|s_{d-2}|\leq q^{\nu'}} \frac{1}{\left|\sin\left(\pi \frac{d!a_d H_1 r_1}{q^{\lambda'-\delta_1-\rho_1}} s_1\cdots s_{d-2}\right)\right|}\right)^{2^{d-2}} \\ &\leq q^{\nu'(1-(d-1)\,2^{2-d})}\, T_3^{1-2^{2-d}}\, T_4^{2^{2-d}}, \end{aligned}$$

where

$$T_3 = \sum_{(h_1,h_2)\in H(\lambda,\delta,\delta_1)} |F_\lambda(h_1,\alpha)F_\lambda(-h_2,\alpha)|$$

and

$$T_4 = \sum_{(h_1,h_2)\in H(\lambda,\delta,\delta_1)} |F_\lambda(h_1,\alpha)F_\lambda(-h_2,\alpha)| \sum_{1\leq |s_1|,\ldots,|s_{d-2}|\leq q^{\nu'}} \frac{1}{\left|\sin\left(\pi \frac{d!a_d H_1 r_1}{q^{\lambda'-\delta_1-\rho_1}} s_1\cdots s_{d-2}\right)\right|}.$$

By (4.8) the term T_3 can be bounded by

$$T_3 \leq q^{2\eta_q(\lambda-\delta)}.$$

In order to handle T_4 we have to be more careful. Since

$$\sum_{h_2\equiv h_1 \bmod q^\delta} |F_\lambda(-h_2,\alpha)| \leq q^{\eta_q(\lambda-\delta)} |F_\delta(h_1,\alpha)|$$

and $|F_\lambda(h_1,\alpha)| \leq |F_\delta(h_1,\alpha)|$ it follows that

$$\begin{aligned} T_4 &\leq q^{\eta_q(\lambda-\delta)} \sum_{\substack{0\leq h_1 < q^\lambda \\ (h_1,q^\lambda)=q^{\delta_1}}} |F_\delta(h_1,\alpha)|^2 \\ &\qquad \times \sum_{1\leq |s_1|,\ldots,|s_{d-2}|\leq q^{\nu'}} \frac{1}{\left|\sin\left(\pi \frac{d!a_d H_1 r_1}{q^{\lambda'-\delta_1-\rho_1}} s_1\cdots s_{d-2}\right)\right|}. \end{aligned}$$

Observe that $\lambda' = \lambda - (d-1)\tau \le \delta$. Hence, due to periodicity and the (already used) inequality $|F_\delta(h_1,\alpha)| \le |F_{\lambda'-\rho_1}(h_1,\alpha)|$ we obtain the upper bound

$$T_4 \le q^{\eta_q(\lambda-\delta)} q^{(d-1)\tau+\rho_1} \sum_{\substack{0\le h'<q^{\lambda'-\rho_1}\\ (h',q^{\lambda'-\rho_1})=q^{\delta_1}}} |F_{\lambda'-\rho_1}(h',\alpha)|^2$$
$$\times \sum_{1\le |s_1|,\ldots,|s_{d-2}|\le q^{\nu'}} \frac{1}{\left|\sin\left(\pi \frac{d! a_d r_1 h'}{q^{\lambda'-\rho_1}} s_1\cdots s_{d-2}\right)\right|}.$$

Finally we apply Lemma 7 and obtain

$$\begin{aligned} T_4 &\ll \nu\, q^{\eta_q(\lambda-\delta)} q^{(d-1)\tau+\rho_1} q^{\lambda'-\delta_1+\nu'\varepsilon} q^{-c_q\,(\lambda'-(d-2)\nu')\|(q-1)\alpha\|^2} \\ &\ll q^{\lambda+\rho_1-\delta_1+\nu\varepsilon+\eta_q(\lambda-\delta)} q^{-c_q\,(\nu+2\rho-\tau)\|(q-1)\alpha\|^2}. \end{aligned}$$

Consequently

$$\begin{aligned} T_2 &\le q^{\nu'(1-(d-1)\,2^{2-d})}\, T_3^{1-2^{2-d}}\, T_4^{2^{2-d}} \\ &\ll q^{\nu'(1-(d-1)\,2^{2-d})+2\eta_q(\lambda-\delta)(1-2^{2-d})+(\lambda+\rho_1-\delta_1+\nu\varepsilon+\eta_q(\lambda-\delta))2^{2-d}} q^{-c_q\,2^{2-d}(\nu+2\rho-\tau)\|(q-1)\alpha\|^2} \\ &= q^{\nu+(2\rho+\rho_1+\nu\varepsilon-\delta_1)2^{2-d}-\tau(1-(d-1)2^{2-d})+2\eta_q(\lambda-\delta)} q^{-c_q\,2^{2-d}(\nu+2\rho-\tau)\|(q-1)\alpha\|^2} \\ &\ll q^{\nu+(3\rho+\nu\varepsilon-\delta_1)2^{2-d}} q^{-c_q\,2^{2-d}(\nu+2\rho)\|(q-1)\alpha\|^2}. \end{aligned}$$

This proves that

$$\begin{aligned} S_6'(r,\nu,\rho,\mu) &= \sum_{\mu<\delta\le\lambda}\ \sum_{0\le\delta_1\le\nu+\tau-\rho_1} (T_1+T_2) \\ &\ll \nu q^{\nu(1-2^{2-d})+(2\eta_q+2^{2-d}/(d-1))(\lambda-\mu)} \\ &\quad + q^{\nu+(3\rho+\nu\varepsilon)2^{2-d}} q^{-c_q\,2^{2-d}(\nu+2\rho)\|(q-1)\alpha\|^2}. \end{aligned} \tag{4.15}$$

4.6. **Completion of the proof of Theorem 1.** We recall that

$$S_1 \ll q^{\nu-\frac{\rho}{2}} + q^{\frac{\nu}{2}} \max_{1\le|r|<q^\rho} |S_2(r,\nu,\rho)|^{1/2}$$

and

$$S_2(r,\nu,\rho) = S_4(r,\nu,\rho) + S_5(r,\nu,\rho,\mu) + S_6(r,\nu,\rho,\mu),$$

where

$$S_6(r,\nu,\rho,\mu) = S_6'(r,\nu,\rho,\mu) + S_6''(r,\nu,\rho,\mu) + S_6'''(r,\nu,\rho,\mu)$$

and $(d-2)\nu+2\rho<\mu\le\lambda=(d-1)\nu+2\rho$. Hence, by (4.3), (4.4), (4.5), (4.12), and (4.15) we obtain:

$$\begin{aligned} S_2(r,\nu,\rho) \ll{}& q^{\nu-\frac12(C_d-2(d-1)\eta_q)\nu} + q^{\nu-(C_d-2\eta_q)(\lambda-\mu)} \\ &+ \nu q^{\nu(1-2^{2-d})+(2\eta_q+2^{2-d}/(d-1))(\lambda-\mu)} + q^{\nu+(3\rho+\nu\varepsilon)2^{2-d}-c_q\,2^{2-d}\|(q-1)\alpha\|^2\nu} \\ &+ \lambda q^{\nu\left(1-\frac14 C_{d-1}\right)} + \lambda\, q^{\nu-\rho C_{d-1}+2\eta_q(\lambda-\mu)-(2d-5)c_q\|(q-1)\alpha\|^2\nu} \\ &+ \lambda^2\, q^{\nu+2(\lambda-\mu+\rho)-2c_q(d-1)\|(q-1)\alpha\|^2\nu}, \end{aligned}$$

provided that (4.1), (4.2), (4.7), (4.9), (4.11), and (4.14) are satisfied. Recall also that $\tau=\lceil\frac{\lambda-\delta}{d-1}\rceil\le\frac{\lambda-\mu}{d-1}+1$.

We now choose

$$\begin{aligned} \rho &= \min\left(\frac{c_q}{12},\,\frac{d-1}{2}c_q,\,\frac{1}{10},\,\frac{C_d-2(d-1)\eta_q}{4(2\eta_q+C_d)}\right)\|(q-1)\alpha\|^2\nu, \\ \lambda-\mu &= \min\left(\rho,\,\frac{(2d-5)c_q}{4\eta_q}\|(q-1)\alpha\|^2\nu,\,\frac{2^{1-d}}{2\eta_q+2^{2-d}/(d-1)}\,\nu,\,\frac{C_{d-1}}{8\eta_q}\,\nu\right), \\ \varepsilon &= \frac{c_q}{4}\,\|(q-1)\alpha\|^2. \end{aligned}$$

This assures that (4.1), (4.2), and (4.9) are automatically satisfied. Since

$$\tau<\frac{\lambda-\mu}{d-1}+1\le\frac{\rho}{d-1}+1\le\rho$$

we also have (4.14). Since $\rho\le\nu/10$ this also implies and (4.7) and (4.11).

Furthermore this choice of parameters assures that there exist a constant $c>0$ depending only on q and d such that uniformly for $1\le|r|\le q^\rho$

$$S_2(r,\nu,\rho)\ll q^{\nu-2c\|(q-1)\alpha\|^2\nu}, \tag{4.16}$$

where the implied constant depends on q, d and α (and without loss of generality we can assume that $2c\|(q-1)\alpha\|^2\nu\le\rho$). Hence

$$S_1\ll q^{\nu-c\|(q-1)\alpha\|^2\nu}. \tag{4.17}$$

This completes the proof of Proposition 1 and consequently the proof of Theorem 1.

5. Proofs of Theorems 2 and 3

5.1. **Proof of Theorem 2.** By a simple discrete Fourier analysis we have

$$\mathrm{card}\{n\le x: s_q(P(n))\equiv a \bmod m\}=\sum_{n\le x}\frac1m\sum_{0\le j<m}e\left(\frac{j}{m}(s_q(P(n))-a)\right).$$

Set $t=(m,q-1)$,$m'=m/t$, $J=\{km':0\le k<t\}$, $J'=\{0,\dots,m-1\}\setminus J=\{km'+r: 0\le k<t,\ 1\le r<m'\}$.

Now observe that $s_q(n) \equiv n \bmod t$ for all divisors $t|q-1$. Hence, if $j = km' \in J$ then

$$e\left(\frac{j}{m}s_q(P(n))\right) = e\left(\frac{km'}{tm'}s_q(P(n))\right) = e\left(\frac{k}{t}s_q(P(n))\right) = e\left(\frac{k}{t}P(n)\right)$$

and consequently

$$\begin{aligned}\sum_{n\le x}\frac{1}{m}\sum_{j\in J} e\left(\frac{j}{m}(s_q(P(n))-a)\right) &= \sum_{n\le x}\frac{1}{m}\sum_{k=1}^{t} e\left(\frac{k}{t}(P(n)-a)\right)\\ &= \frac{t}{m}\sum_{n\le x,\ P(n)\equiv a \bmod t} 1\\ &= \frac{t}{m}\left(\frac{x}{t}+O(1)\right)Q(a,t)\\ &= \left(\frac{x}{m}+O(1)\right)Q(a,t).\end{aligned}$$

Thus,

$$\begin{aligned}\mathrm{card}\{n\le x : s_q(P(n))\equiv a \bmod m\} &= \frac{x}{m}Q(a,t)+O(t)\\ &\quad+\frac{1}{m}\sum_{j\in J'} e\left(\frac{-aj}{m}\right)\sum_{n\le x} e\left(\frac{j}{m}s_q(P(n))\right).\end{aligned}$$

If $J' = \emptyset$ which corresponds to the degenerate case $m|q-1$ then we are done. If $J' \ne \emptyset$ then we set $q' = (q-1)/t$ so that $(q', m) = 1$. Furthermore for $j = km' + r \in J'$ we have

$$\frac{(q-1)j}{m} = \frac{tq'(km'+r)}{tm'} = q'k + \frac{q'r}{m'} \notin \mathbb{Z}.$$

Hence, by Theorem 1 there exists $\sigma_{m,j} > 0$ with

$$\sum_{n\le x} e\left(\frac{j}{m}s_q(P(n))\right) \ll x^{1-\sigma_{m,j}}.$$

This finally implies

$$\mathrm{card}\{n\le x : s_q(P(n))\equiv a \bmod m\} = \frac{x}{m}Q(a,t)+O\left(x^{1-\sigma}\right)$$

with

$$\sigma = \min_{j\in J'}\sigma_{m,j}$$

and completes the proof of Theorem 3.

5.2. **Proof of Theorem 3.** If $\alpha \in \mathbb{Q}$ then $(\alpha s_q(P(n)))_{n\in\mathbb{N}}$ attains only finitely many values modulo 1. Hence, the sequence $(\alpha s_q(P(n)))_{n\in\mathbb{N}}$ is definitely not uniformly distributed modulo 1.

Conversely, if $\alpha \notin \mathbb{Q}$ then $(q-1)h\alpha \notin \mathbb{Z}$. Thus, we can apply Theorem 1 where we formally replace α by $h\alpha$ and observe that there exists $\sigma > 0$ with

$$\sum_{n\le x} e(h\alpha s_q(P(n))) \ll x^{1-\sigma}.$$

Hence, by Weyl's criterion (see [9]) the sequence $(\alpha s_q(P(n)))_{n\in\mathbb{N}}$ is uniformly distributed modulo 1.

References

[1] J.-P. Allouche and J. Shallit, *Automatic sequences*, Cambridge University Press, Cambridge, 2003. Theory, applications, generalizations.

[2] N. L. Bassily and I. Kátai, Distribution of the values of q-additive functions on polynomial sequences, *Acta Math. Hungar.* **68**,4 (1995), 353–361.

[3] C. Dartyge and G. Tenenbaum, Sommes des chiffres de multiples d'entiers, *Ann. Inst. Fourier (Grenoble)* **55**,7 (2005), 2423–2474.

[4] C. Dartyge and G. Tenenbaum, Congruences de sommes de chiffres de valeurs polynomiales, *Bull. London Math. Soc.* **38**,1 (2006), 61–69.

[5] H. Davenport and P. Erdős, Note on normal decimals, *Canadian J. Math.* **4** (1952), 58–63.

[6] H. Delange, Sur la fonction sommatoire de la fonction "somme des chiffres", *Enseignement Math. (2)* **21**,1 (1975), 31–47.

[7] M. Drmota, C. Mauduit, and J. Rivat, Primes with an Average Sum of Digits, *Compositio* **145**,2 (2009), 271–292.

[8] M. Drmota and J. Rivat, The sum-of-digits function of squares, *J. London Math. Soc. (2)* **72**,2 (2005), 273–292.

[9] M. Drmota and R. F. Tichy, *Sequences, discrepancies and applications*, vol. 1651 of *Lecture Notes in Mathematics*, Springer-Verlag, Berlin, 1997.

[10] A. O. Gelfond, Sur les nombres qui ont des propriétés additives et multiplicatives données, *Acta Arith.* **13** (1967/1968), 259–265.

[11] H. Iwaniec and E. Kowalski, *Analytic number theory*, vol. 53 of *American Mathematical Society Colloquium Publications*, American Mathematical Society, Providence, RI, 2004.

[12] M. Keane, Generalized Morse sequences, *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **10** (1968), 335–353.

[13] K. Mahler, The Spectrum of an Array and Its Application to the Study of the Translation Properties of a Simple Class of Arithmetical Functions. II: On the Translation Properties of a Simple Class of Arithmetical Functions, *J. of Math. Phys. Mass. Inst. Techn.* **6** (1927), 158–163.

[14] C. Mauduit, Multiplicative properties of the Thue-Morse sequence, *Period. Math. Hungar.* **43**,1-2 (2001), 137–153.

[15] C. Mauduit and J. Rivat, Répartition des fonctions q-multiplicatives dans la suite $([n^c])_{n\in\mathbb{N}}$, $c > 1$, *Acta Arith.* **71**,2 (1995), 171–179.

[16] C. Mauduit and J. Rivat, Propriétés q-multiplicatives de la suite $\lfloor n^c \rfloor$, $c > 1$, *Acta Arith.* **118**,2 (2005), 187–203.

[17] C. Mauduit and J. Rivat, La somme des chiffres des carrés, *Acta Mathematica* **203** (2009), 107–148.

[18] C. Mauduit and J. Rivat, Sur un problème de Gelfond: la somme des chiffres des nombres premiers, *Annals of Mathematics*, à paraître.

[19] H. L. MONTGOMERY, *Ten lectures on the interface between analytic number theory and harmonic analysis*, vol. 84 of *CBMS Regional Conference Series in Mathematics*, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1994.

[20] J. MORGENBESSER, The sum of digits of $\lfloor n^c \rfloor$, *preprint*.

[21] M. PETER, The summatory function of the sum-of-digits function on polynomial sequences, *Acta Arith.* **104**,1 (2002), 85–96.

[22] I. I. PIATETSKI-SHAPIRO, On the distribution of prime numbers in sequences of the form $[f(n)]$, *Mat. Sbornik N.S.* **33(75)** (1953), 559–566.

[23] N. PYTHEAS FOGG, *Substitutions in dynamics, arithmetics and combinatorics*, vol. 1794 of *Lecture Notes in Mathematics*, Springer-Verlag, Berlin, 2002. Edited by V. Berthé, S. Ferenczi, C. Mauduit and A. Siegel.

[24] M. QUEFFELEC, *Substitution Dynamical Systems – Spectral Analysis*, vol. 1294 of *Lecture Notes in Math.*, Springer Verlag, New-York – Berlin, 1987.

[25] W. SIERPIŃSKI, *Elementary theory of numbers*, vol. 31 of *North-Holland Mathematical Library*, North-Holland Publishing Co., Amsterdam, second ed., 1988. Edited and with a preface by Andrzej Schinzel.

[26] N. WIENER, The Spectrum of an Array and Its Applications to the Study of the Translation Properties of a Simple Class of Arithmetical Functions. I: The Spectrum of an Array, *J. of Math. Phys. Mass. Inst. Techn.* **6** (1927), 145–157.