Asymptotic Methods of Enumeration and Applications to Markov Chain Models

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Queueing theory without limits: transient and asymptotic analysis

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References

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“Philosophy”

- Reformulation of the problem in terms of generating functions (coefficients encode the probabilistic distibution of the problem)

- Analysis of singularities and structure of generating functions

- Asymptotics for coefficients of generating functions

- Interpretation as probabilistic limiting distribution

Contents

Part I

Part II

Discrete Quasi Birth and Death Processes

A discrete quasi birth and death process (QBD) is a discrete Markov process X_n on the non-negative integers with transition matrix of the form

$$\mathbf{P} = \begin{pmatrix} \mathbf{B} & \mathbf{A}_0 & \mathbf{0} & \mathbf{0} & \cdots & \\ \mathbf{A}_2 & \mathbf{A}_1 & \mathbf{A}_0 & \mathbf{0} & \cdots & \\ \mathbf{0} & \mathbf{A}_2 & \mathbf{A}_1 & \mathbf{A}_0 & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_2 & \mathbf{A}_1 & \mathbf{A}_0 & \cdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots \end{pmatrix},$$

where $\mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_2$, and $\mathbf{B}$ are square matrices of order m.

Problem: distribution of X_n ?   (encoded in powers of $\mathbf{P}$)

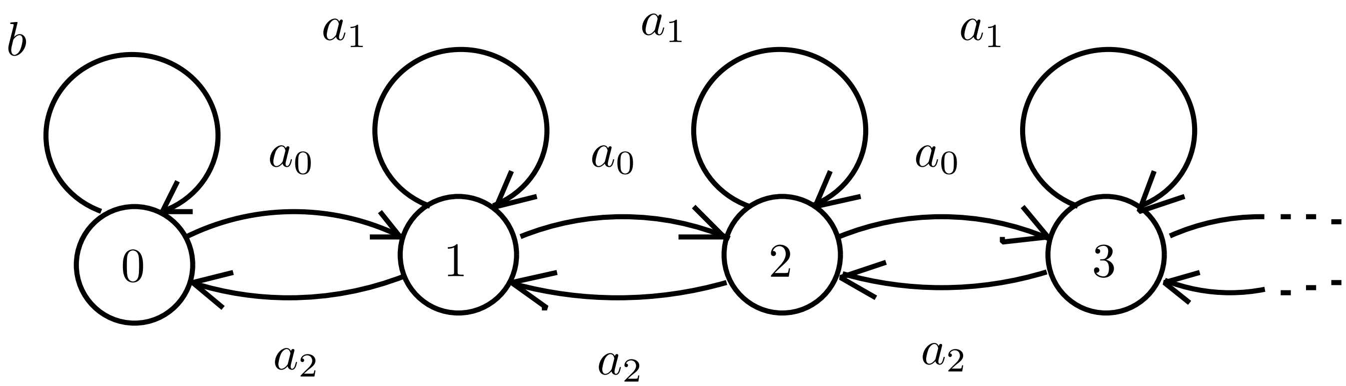
$$\boxed{\mathbf{P}^n = \Big(\mathbf{Pr}(X_n = v \,|\, X_n = w)\Big)_{v,w\geq 0}}$$

Random Walk on Non-negative Integers

$m = 1$:

$$\mathbf{P} = \begin{pmatrix} b & a_0 & 0 & 0 & \cdots & \\ a_2 & a_1 & a_0 & 0 & \cdots & \\ 0 & a_2 & a_1 & a_0 & 0 & \cdots \\ 0 & 0 & a_2 & a_1 & a_0 & \cdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots \end{pmatrix},$$

Interpretation as random walk on non-negative integers:

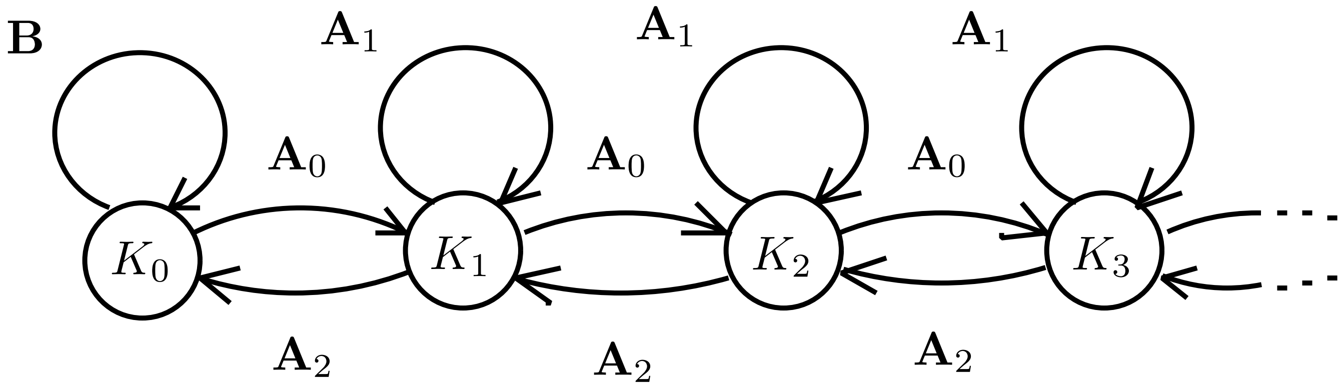


Random Walk on Graphs

$m > 1$:

$$\mathbf{P} = \begin{pmatrix} \mathbf{B} & \mathbf{A}_0 & \mathbf{0} & \mathbf{0} & \cdots & \\ \mathbf{A}_2 & \mathbf{A}_1 & \mathbf{A}_0 & \mathbf{0} & \cdots & \\ \mathbf{0} & \mathbf{A}_2 & \mathbf{A}_1 & \mathbf{A}_0 & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_2 & \mathbf{A}_1 & \mathbf{A}_0 & \cdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

$\mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_2$, and $\mathbf{B}$ transition probability matrices between graphs $K_0, K_1, K_2, \ldots$



Matrix Powers

With

$$p_{w,v} = \mathbf{Pr}\{X_{k+1} = v \,|\, X_k = w\} \qquad (k \geq 0)$$

we have

$$\mathbf{P} = (p_{w,v})_{w,v \geq 0}.$$

Consequently, for

$$p_{w,v}^{(n)} = \mathbf{Pr}\{X_n = v \,|\, X_0 = w\}$$

we have

$$\mathbf{P}^n = (p_{w,v}^{(n)})_{w,v \geq 0}$$

Generating Functions

With

$$
\begin{aligned}
M_{w,v}(x) &= \sum_{n\geq 0} p_{w,v}{}^{(n)} \cdot x^n \\
&= \sum_{n\geq 0} \mathbf{Pr}\{X_n = v \mid X_0 = w\} \cdot x^n
\end{aligned}
$$

we get

$$
\begin{aligned}
\mathbf{M}(x) &= (M_{w,v}(x))_{w,v\geq 0} \\
&= \mathbf{I} + \mathbf{P}x + \mathbf{P}^2x^2 + \cdots = (\mathbf{I} - x\mathbf{P})^{-1}.
\end{aligned}
$$

Generating Functions

Lemma 1 *Let* $N(x)$ *denote the (analytic) solution with* $N(0) = 1$ *of the equation*

$$\boxed{N(x) = 1 + xa_1 N(x) + x^2 a_0 N(x) a_2 N(x)},$$

that is,

$$N(x) = \frac{1 - xa_1 - \sqrt{(1 - xa_1)^2 - 4x^2 a_0 a_2}}{2x^2 a_0 a_2}.$$

Then

$$\boxed{M_{0,\ell}(x) = \left(1 - xb - x^2 a_0\, N(x) a_2\right)^{-1} (xa_0 N(x))^\ell}.$$

Recall: $M_{0,\ell}(x) = \sum_{n \ge 0} \mathbf{Pr}\{X_n = \ell \,|\, X_0 = 0\}\, x^n$

The General Case

Consider the $m \times m$ submatrices $\boxed{\mathbf{M}_{k,\ell}(x) = (M_{v,w}(x))_{v \in K_k, w \in K_\ell})}$.

Lemma 2 *Let* $\mathbf{N}(x)$ *denote the (analytic) solution with* $\mathbf{N}(0) = \mathbf{I}$ *of the matrix equation*

$$\boxed{\mathbf{N}(x) = \mathbf{I} + x\mathbf{A}_1\, \mathbf{N}(x) + x^2 \mathbf{A}_0\, \mathbf{N}(x)\, \mathbf{A}_2\, \mathbf{N}(x)}.$$

Then

$$\boxed{\mathbf{M}_{0,\ell}(x) = \left(\mathbf{I} - x\mathbf{B} - x^2 \mathbf{A}_0\, \mathbf{N}(x)\, \mathbf{A}_2\right)^{-1} (x\mathbf{A}_0\, \mathbf{N}(x))^\ell}.$$

Remark. The entries of $\mathbf{N}(x)$ satisfy a system of (quadratic) equations.

One-Dimensional Discrete QBD's

Theorem 1 *Suppose that* a_0, a_1, a_2 *and* b *are positive numbers with*

$$a_0 + a_1 + a_2 = b + a_0 = 1$$

and let X_n *be the discrete QBD on the non-negative integers with transition matrix*

$$\mathbf{P} = \begin{pmatrix} b & a_0 & 0 & 0 & \cdots & \\ a_2 & a_1 & a_0 & 0 & \cdots & \\ 0 & a_2 & a_1 & a_0 & 0 & \cdots \\ 0 & 0 & a_2 & a_1 & a_0 & \cdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

1. *If* $a_0 < a_2$ *then we have*

$$\boxed{\lim_{n\to\infty} \mathbf{Pr}\{X_n = \ell \,|\, X_0 = 0\} = \frac{a_2 - a_0}{a_2}\left(\frac{a_0}{a_2}\right)^{\ell} \quad (\ell \geq 0).}$$

that is, X_n *is positive recurrent and converges to the (geometric) stationary distribution.*

2. *If* $a_0 = a_2$ *then* X_n *is null recurrent and* $X_n/\sqrt{2a_0 n}$ *converges weakly to the absolute normal distribution:*

$$\boxed{\mathbf{Pr}\{X_n = \ell \,|\, X_0 = 0\} = \frac{1}{\sqrt{n a_0 \pi}} \exp\left(-\frac{\ell^2}{4a_0 n}\right) + \mathcal{O}\left(\frac{1}{n}\right)},$$

uniformly for all $\ell \le C\sqrt{n}$ *as* $n \to \infty$.

3. *If* $a_0 > a_2$ *then* X_n *is non recurrent and*

$$\boxed{\frac{X_n - (a_0 - a_2)n}{\sqrt{(a_0 + a_2 - (a_0 - a_2)^2)n}} \to N(0,1).}$$

More precisely

$$\begin{aligned}&\mathbf{Pr}\{X_n = \ell \,|\, X_0 = 0\}\\ =\;& \frac{1}{\sqrt{2\pi(a_0 + a_2 - (a_0 - a_2)^2)n}} \exp\left(-\frac{(\ell - (a_0 - a_2)n)^2}{2(a_0 + a_2 - (a_0 - a_2)^2)n}\right) + \mathcal{O}\left(\frac{1}{n}\right)\end{aligned}$$

uniformly for all $\ell \ge 0$ *with* $|\ell - (a_0 - a_2)n| \le C\sqrt{n}$ *as* $n \to \infty$.

Remark. With a little bit more effort it can be shown that in the case $a_0 = a_2$ the *normalized* discrete processes

$$\left(\frac{X_{\lfloor tn \rfloor}}{\sqrt{2a_0 n}}, t \geq 0\right)_{n \geq 1}$$

converges weakly to a reflected Brownian motion as $n \to \infty$; and for $a_0 < a_2$ the processes

$$\left(\frac{X_{\lfloor tn \rfloor} - t(a_0 - a_2)n}{\sqrt{(a_0 + a_2 - (a_0 - a_2)^2)n}}, t \geq 0\right)_{n \geq 1}$$

converges weakly to the standard Brownian motion.

General Discrete QBD's

Theorem 2 *Let* $\mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_2$ *and* $\mathbf{B}$ *be square matrices of order* m *with non-negative elements with such that* $(\mathbf{B} + \mathbf{A}_0)\mathbf{1} = \mathbf{1}$ *and* $(\mathbf{A}_0 + \mathbf{A}_1 + \mathbf{A}_2)\mathbf{1} = \mathbf{1}$, *and let*

$$\mathbf{P} = \begin{pmatrix} \mathbf{B} & \mathbf{A}_0 & \mathbf{0} & \mathbf{0} & \cdots & \\ \mathbf{A}_2 & \mathbf{A}_1 & \mathbf{A}_0 & \mathbf{0} & \cdots & \\ \mathbf{0} & \mathbf{A}_2 & \mathbf{A}_1 & \mathbf{A}_0 & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_2 & \mathbf{A}_1 & \mathbf{A}_0 & \cdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

denote the is a transition matrix of a discerte QBD X_n. *Furthermore suppose that the matrices* $\mathbf{A}_1$ *is primitive irreducible, that no row of* $\mathbf{A}_0$ *is zero, and that* $\mathbf{A}_2$ *is non-zero.*

Let x_0 *denote the radius of convergence of the entries of* $\mathbf{N}(x)$ *and let* x_1 *denote the radius of convergence of the entries of* $\mathbf{M}_{0,0}(x)$.

1. *If* $x_0 > 1$ *and* $x_1 = 1$ *then* X_n *is positive recurrent and for all* $v \geq 0$ *and* $w_0 \in K_0$ *we have*

$$\boxed{\lim_{n\to\infty} \mathbf{Pr}\{X_n = v \,|\, X_0 = w_0\} = p_v,}$$

where $(p_v)_{v\geq 0}$ *is the (unique) stationary distribution of* X_n.

Set

$$\mathbf{R} = \mathbf{A}_0 \cdot \mathbf{N}(1).$$

Then all eigenvalues of $\mathbf{R}$ *have moduli* < 1 *and we have*

$$\boxed{\mathbf{p}_{\ell+1} = \mathbf{p}_\ell \mathbf{R},}$$

in which $\mathbf{p}_\ell = (p_v)_{v\in K_\ell}$.

2. *If* $x_0 = x_1 = 1$ *then* X_n *is null recurrent and there exist* $\rho_{v'} > 0$ $(v' \in V(K))$ *and* $\eta > 0$ *such that*

$$\mathbf{Pr}\{X_n = v \,|\, X_0 = w_0\} = \rho_{\tilde{v}} \sqrt{\frac{1}{n\pi}} \exp\left(-\frac{\ell^2}{4\eta n}\right) + \mathcal{O}\left(\frac{1}{n}\right) \quad (v \in V(K_\ell)).$$

uniformly for all $\ell \le C\sqrt{n}$*) as* $n \to \infty$*. (*$\tilde{v}'$ *denotes the node in* K *that corresponds to* v *from* K_ℓ*)).*

3. *If* $x_1 > 1$ *then* X_n *is non recurrent and there exist* $\tau_{v'} > 0$ $(v' \in V(K))$, $\mu > 0$ *and* $\sigma > 0$ *such that*

$$\mathbf{Pr}\{X_n = v \,|\, X_0 = w_0\} = \frac{\tau_{\tilde{v}}}{\sqrt{n}} \exp\left(-\frac{(\ell - \mu n)^2}{2\sigma^2 n}\right) + \mathcal{O}\left(\frac{1}{n}\right) \quad (v \in V(K_\ell)).$$

uniformly for all $\ell \ge 0$ *with* $|\ell - \mu n| \le C\sqrt{n}$ *as* $n \to \infty$*.*

Continuous Quasi Birth and Death Processes

A continuous quasi birth and death process is a continuous time Markov process $X(t)$ on the non-negative integers with generator

$$\mathbf{Q} = \begin{pmatrix} \mathbf{B} & \mathbf{A}_0 & \mathbf{0} & \mathbf{0} & \cdots & \\ \mathbf{A}_2 & \mathbf{A}_1 & \mathbf{A}_0 & \mathbf{0} & \cdots & \\ \mathbf{0} & \mathbf{A}_2 & \mathbf{A}_1 & \mathbf{A}_0 & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_2 & \mathbf{A}_1 & \mathbf{A}_0 & \cdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

$\mathbf{A}_0$, $\mathbf{A}_2$: non-negative entries

$\mathbf{B}$, $\mathbf{A}_1$: non-negative off-diagonal elements, the diagonal elements are stricly negative, and the row sums in $\mathbf{Q}$ are all equal to zero:

$$(\mathbf{B} + \mathbf{A}_0)\mathbf{1} = \mathbf{0} \quad \text{and} \quad (\mathbf{A}_0 + \mathbf{A}_1 + \mathbf{A}_2)\mathbf{1} = \mathbf{0}.$$

With

$$q_{w,v}^{(t)} = \mathbf{Pr}\{X(t) = v \,|\, X(0) = w\}.$$

we have

$$\exp(\mathbf{Q}t) = (q_{w,v}^{(t)})_{w,v\geq 0}$$

By use of the Laplace transform (instead of generating functions)

$$\boxed{\widehat{M}_{w,v}(s) = \int_0^\infty \mathbf{Pr}\{X(t) = v \,|\, X(0) = w\}\, e^{-st}\, dt}$$

we get

$$\begin{aligned}\widehat{\mathbf{M}}(s) &= (\widehat{M}_{w,v}(s))_{w,v\geq 0} \\ &= (s\mathbf{I} - \mathbf{Q})^{-1}\end{aligned}$$

$\widehat{\mathbf{M}}(s)$ has almost the same representation as $\mathbf{M}(x)$ in the discrete case. This is reflected by the following property for the submatrices

$$\widehat{\mathbf{M}}_{k,\ell}(s) = \left(\widehat{M}_{w,v}(s)\right)_{w\in K_k, v\in K_\ell}.$$

Lemma 3 *Let* $\widehat{\mathbf{N}}(s)$ *by characterized by* $\lim_{s\to\infty} s\widehat{\mathbf{N}}(s) = \mathbf{I}$ *and by the matrix equation*

$$\boxed{s\widehat{\mathbf{N}}(s) = \mathbf{I} + \mathbf{A}_1\,\widehat{\mathbf{N}}(s) + \mathbf{A}_0\,\widehat{\mathbf{N}}(s)\,\mathbf{A}_2\,\widehat{\mathbf{N}}(s)}$$

Then

$$\boxed{\widehat{\mathbf{M}}_{0,\ell}(s) = \left(s\mathbf{I} - \mathbf{B} - \mathbf{A}_0\,\widehat{\mathbf{N}}(s)\,\mathbf{A}_2\right)^{-1}\left(\mathbf{A}_0\,\widehat{\mathbf{N}}(s)\right)^{\ell}}.$$

Remark. Note that (formally) $\widehat{\mathbf{N}}(s) := \frac{1}{s}\mathbf{N}\left(\frac{1}{s}\right)$.

One-Dimensional Continuous QBD's

Theorem 3 *Suppose that* q_0 *and* q_2 *are positive numbers,* $q_1 = -q_0 - q_2$ *and* $b_0 = -q_0$*; and let* $X(t)$ *be the continuous QBD on the non-negative integers with generator matrix*

$$\mathbf{P} = \begin{pmatrix} b_0 & q_0 & 0 & 0 & \cdots & \\ q_2 & q_1 & q_0 & 0 & \cdots & \\ 0 & q_2 & q_1 & q_0 & 0 & \cdots \\ 0 & 0 & q_2 & q_1 & q_0 & \cdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

1. *If* $q_0 < q_2$ *then we have*

$$\boxed{\lim_{t\to\infty} \mathbf{Pr}\{X(t) = \ell \,|\, X(0) = 0\} = \frac{q_2 - q_0}{q_2}\left(\frac{q_0}{q_2}\right)^{\ell} \quad (\ell \geq 0),}$$

this is, $X(t)$ *is positive recurrent. The distribution of* $X(t)$ *converges to the stationary distribution.*

2. *If* $q_0 = q_2$ *then* $X(t)$ *is null recurrent and* $X(t)/\sqrt{2q_0 t}$ *converges weakly to the absolute normal distribution:*

$$\boxed{\mathbf{Pr}\{X(t) = \ell \,|\, X(0) = 0\} = \frac{1}{\sqrt{tq_0\pi}} \exp\left(-\frac{t^2}{4q_0 t}\right) + \mathcal{O}\left(\frac{1}{t}\right).}$$

uniformly for all $\ell \leq C\sqrt{t}$ *as* $t \to \infty$.

3. *If* $q_0 > q_2$ *then* $X(t)$ *is non recurrent and*

$$\boxed{\frac{X(t) - (q_0 - q_2)t}{\sqrt{(q_0 + q_2)(q_0 - q_2)^{-2} t}} \to N(0,1).}$$

More precisely

$$\mathbf{Pr}\{X(t) = \ell \,|\, X(0) = 0\}$$
$$= \frac{1}{\sqrt{2\pi(q_0 + q_2)(q_0 - q_2)^{-2} t}} \exp\left(-\frac{(\ell - (q_0 - q_2)t)^2}{2(q_0 + q_2)(q_0 - q_2)^{-2} t}\right) + \mathcal{O}\left(\frac{1}{t}\right)$$

uniformly for all $\ell \geq 0$ *with* $|\ell - (q_0 - q_2)t| \leq C\sqrt{t}$ *as* $t \to \infty$.

General Continuous QBD's

Theorem 4 *Let* $\mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_2$ *and* $\mathbf{B}$ *be square matrices of order* m *such that* $\mathbf{A}_0$ *and* $\mathbf{A}_2$ *are non-negative and the matrices* $\mathbf{B}$ *and* $\mathbf{A}_1$ *have non-negative off-diagonal elements whereas the diagonal elements are stricly negative so that the row sums are all equal to zero:*

$$(\mathbf{B} + \mathbf{A}_0)\mathbf{1} = \mathbf{0} \quad \textit{and} \quad (\mathbf{A}_0 + \mathbf{A}_1 + \mathbf{A}_2)\mathbf{1} = \mathbf{0}$$

and let

$$\mathbf{Q} = \begin{pmatrix} \mathbf{B} & \mathbf{A}_0 & \mathbf{0} & \mathbf{0} & \cdots & \\ \mathbf{A}_2 & \mathbf{A}_1 & \mathbf{A}_0 & \mathbf{0} & \cdots & \\ \mathbf{0} & \mathbf{A}_2 & \mathbf{A}_1 & \mathbf{A}_0 & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_2 & \mathbf{A}_1 & \mathbf{A}_0 & \cdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

denote the generator matrix of of a homogeneous continuous QBD process $X(t)$*. Furthermore suppose that the matrix* $\mathbf{A}_1$ *is primitive irreducible, that no row of* $\mathbf{A}_0$ *is zero, that* $\mathbf{A}_2$ *is non-zero, and that the system of equations for* $\hat{\mathbf{N}}(x)$ *has the same radius of convergence for all entries and the dominant singularity is of squareroot type.*

Let σ_0 *denote the abscissa of convergence of* $\widehat{\mathbf{N}}(s)$ *and let* σ_1 *denote the abscissa of convergence of* $\widehat{\mathbf{M}}_{0,0}(s)$.

1. *If* $\sigma_0 < 0$ *and* $\sigma_1 = 0$ *then* $X(t)$ *is positive recurrent and for all* $v \geq 0$ *we have*

$$\boxed{\lim_{t\to\infty} \mathbf{Pr}\{X(t) = v \,|\, X(0) = w_0\} = p_v},$$

where $(p_v)_{v\geq 0}$ *is the (unique) stationary distribution of* $X(t)$. *Set*

$$\mathbf{R} = \mathbf{A}_0 \cdot \widehat{\mathbf{N}}(0)$$

Then all eigenvalues of $\mathbf{R}$ *have moduli* < 1 *and we have*

$$\mathbf{p}_{\ell+1} = \mathbf{p}_\ell \mathbf{R},$$

in which $\mathbf{p}_\ell = (p_v)_{v\in K_\ell}$.

2. *If* $\sigma_0 = \sigma_1 = 0$ *then* $X(t)$ *is null recurrent and there exist* $\rho_{v'} > 0$ $(v' \in V(K))$ *and* $\eta > 0$ *such that, as* $t \to \infty$,

$$\boxed{\mathbf{Pr}\{X(t) = v \,|\, X(0) = w_0\} = \rho_{\tilde{v}} \sqrt{\frac{1}{t\pi}} \exp\left(-\frac{\ell^2}{4\eta t}\right) + \mathcal{O}\left(\frac{1}{t}\right) \quad (v \in V(K_\ell)).}$$

uniformly for all $\ell \le C\sqrt{t}$ *as* $t \to \infty$.

3. *If* $\sigma_1 > 0$ *then* $X(t)$ *is non recurrent and there exist* $\tau_{v'} > 0$ $(v' \in V(K))$, $\mu > 0$ *and* $\sigma > 0$ *such that*

$$\boxed{\mathbf{Pr}\{X(t) = v \,|\, X(0) = w_0\} = \frac{\tau_{\tilde{v}}}{\sqrt{t}} \exp\left(-\frac{(\ell - \mu t)^2}{2\sigma^2 t}\right) + \mathcal{O}\left(\frac{1}{t}\right) \quad (v \in V(K_\ell))}$$

uniformly for all $\ell \ge 0$ *with* $|\ell - \mu t| \le C\sqrt{t}$ *as* $t \to \infty$.

Remarks

- Very special situation
- Prototype for results that can be expected in more general situations
- Full asymptotic expansions (order of convergence)
- Tail estimates

Contents (2)

Part I

Part II

Generating Functions

- $y(x) = \sum_{n \geq 0} y_n \, x^n$: generating function of sequence y_n

- $R = \left(\limsup_{n \to \infty} \; |y_n|^{1/n} \right)^{-1}$: radius of convergence

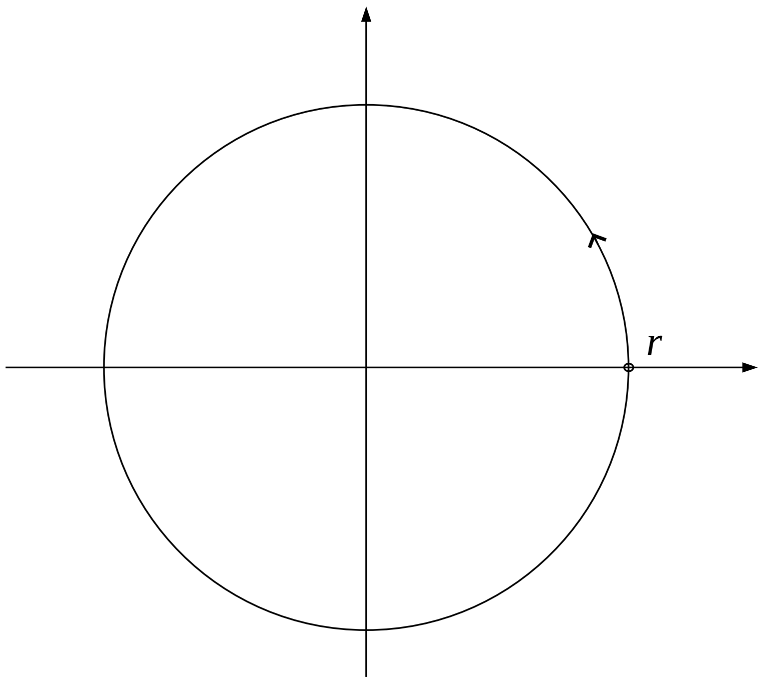
- $y_n \geq 0 \quad \Longrightarrow \quad y(x)$ is singular at $x_0 = R$

- $y_n \leq C_1 \, R^{-n} (1 + \varepsilon)^n$ for all $n \geq 0$

- $y_n \geq C_2 \, R^{-n} (1 - \varepsilon)^n$ for infinitely many $n \geq 0$

Cauchy's formula

$$y_n = \frac{1}{2\pi i} \int_{|x|=r} y(x)\, x^{-n-1}\, dx$$



Notation. $[x^n]\, y(x) = y_n$

Cauchy's formula

Remark.

$$y_n \geq 0 \Longrightarrow y_n \leq \min_{0<r<R} y(r) r^{-n}$$

$$y_n r^n \leq \sum_{k \geq 0} y_k r^k = y(r) \Longrightarrow y_n \leq y(r) r^{-n}$$

Algebraic Singularities

Lemma 4 *Suppose that*

$$\boxed{y(x) = (1-x)^{-\alpha}}.$$

Then

$$\boxed{y_n = (-1)^n \binom{-\alpha}{n} = \frac{n^{\alpha-1}}{\Gamma(\alpha)} + \mathcal{O}\left(n^{\alpha-2}\right).}$$

Proof.

Cauchy's formula:

$$(-1)^n \binom{-\alpha}{n} = \frac{1}{2\pi i} \int_\gamma (1-x)^{-\alpha} \, x^{-n-1} \, dx.$$

$\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$:

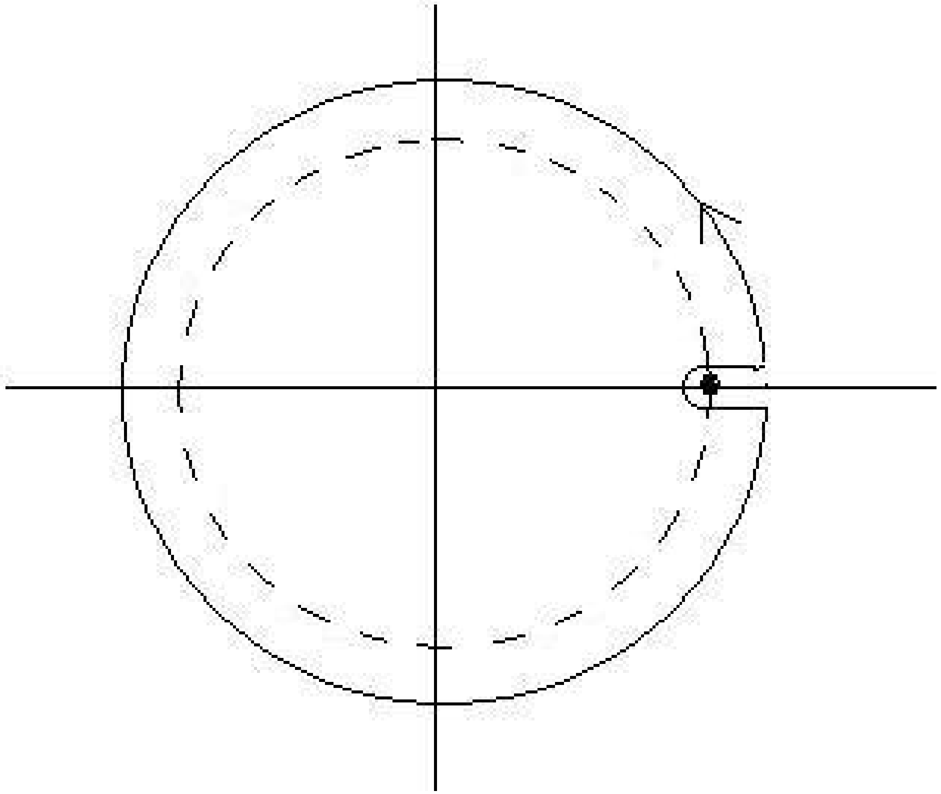
$$\gamma_1 = \left\{ x = 1 + \frac{t}{n} \,\middle|\, |t| = 1, \Re t \le 0 \right\}$$

$$\gamma_2 = \left\{ x = 1 + \frac{t}{n} \,\middle|\, 0 < \Re t \le \log^2 n, \Im t = 1 \right\}$$

$$\gamma_3 = \overline{\gamma_2}$$

$$\gamma_4 = \left\{ x \,\middle|\, |x| = \left| 1 + \frac{\log^2 n + i}{n} \right|, \arg(1 + \frac{\log^2 n + i}{n}) \le |\arg(x)| \le \pi \right\}.$$

Path of integration

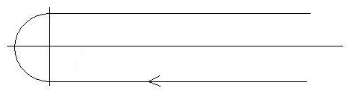


Substitution for $x \in \gamma_1 \cup \gamma_2 \cup \gamma_3$:

$$x = 1 + \frac{t}{n} \Longrightarrow x^{-n-1} = e^{-t}\left(1 + \mathcal{O}\left(\frac{t^2}{n}\right)\right)$$

With Hankel's integral representation for $1/\Gamma(\alpha)$

$$\begin{aligned}\frac{1}{2\pi i}\int_{\gamma_1\cup\gamma_2\cup\gamma_3}(1-x)^{-\alpha}x^{-n-1}\,dx &= \frac{n^{\alpha-1}}{2\pi i}\int_{\gamma'}(-t)^{-\alpha}e^{-t}\,dt \\ &+ \frac{n^{\alpha-2}}{2\pi i}\int_{\gamma'}(-t)^{-\alpha}e^{-t}\cdot\mathcal{O}\left(t^2\right)\,dt \\ &= n^{\alpha-1}\frac{1}{\Gamma(\alpha)} + \mathcal{O}\left(n^{\alpha-2}\right).\end{aligned}$$

$\gamma' = \{t \mid |t| = 1, \Re t \leq 0\} \cup \{t \mid 0 < \Re t \leq \log^2 n, \Im t = \pm 1\}$: 

Lemma 5 (Flajolet and Odlyzko) *Let*

$$y(x) = \sum_{n \geq 0} y_n \, x^n$$

be analytic in a region

$$\Delta = \{x : |x| < x_0 + \eta, \, |\arg(x - x_0)| > \delta\},$$

$x_0 > 0$, $\eta > 0$, $0 < \delta < \pi/2$.

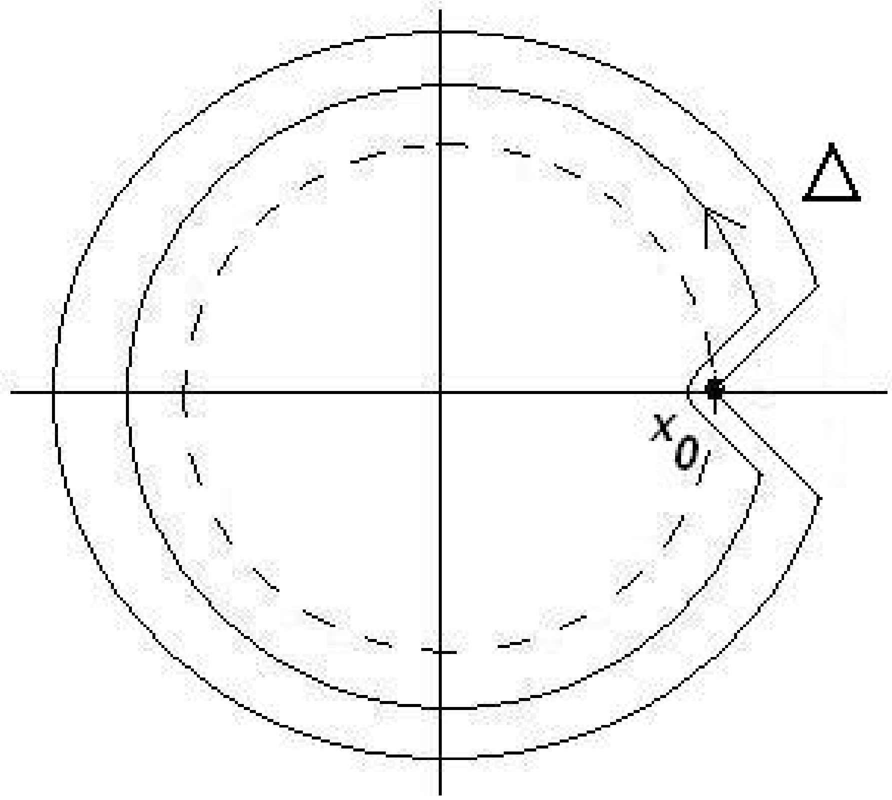
Suppose that for some real α

$$\boxed{y(x) = \mathcal{O}\left((1 - x/x_0)^{-\alpha}\right) \qquad (x \in \Delta).}$$

Then

$$\boxed{y_n = \mathcal{O}\left(x_0^{-n} n^{\alpha-1}\right).}$$

Δ-region and path of integration



Proof

Cauchy's formula:

$$y_n = \frac{1}{2\pi i} \int_\gamma y(x)\, x^{-n-1}\, dx,$$

$\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$:

$$\begin{aligned}
\gamma_1 &= \left\{ x = x_0 + \frac{z}{n} : |z| = 1,\ \delta \le |\arg(z)| \le \pi \right\}, \\
\gamma_2 &= \left\{ x = x_0 + te^{i\delta} : \frac{1}{n} \le t \le \eta \right\}, \\
\gamma_3 &= \left\{ x = x_0 + te^{-i\delta} : \frac{1}{n} \le t \le \eta \right\}, \\
\gamma_4 &= \left\{ x : |x| = \left| x_0 + e^{i\delta}\eta \right|,\ \arg\left(x_0 + e^{i\delta}\eta \right) \le |\arg x| \le \pi \right\}.
\end{aligned}$$

Asymptotic Transfer

Suppose that a function $y(x)$ is analytic in a region of the form Δ and that it has an expansion of the form

$$y(x) = C\left(1 - \frac{x}{x_0}\right)^{-\alpha} + \mathcal{O}\left(\left(1 - \frac{x}{x_0}\right)^{-\beta}\right) \qquad (x \in \Delta),$$

where $\beta < \alpha$. Then we have (as $n \to \infty$)

$$y_n = [x^n]y(x) = C\frac{n^{\alpha-1}}{\Gamma(\alpha)}x_0^{-n} + \mathcal{O}\left(x_0^{-n}n^{\max\{\alpha-2,\beta-1\}}\right).$$

Polar Singularities

Lemma 6 *Suppose that $y(x)$ is a meromorphic function that is analytic at $x = 0$ and has polar singularities at the points $q_1, \ldots, q_r$ in the circle $|x| < R$:*

$$y(x) = \sum_{j=1}^{r} \sum_{k=1}^{\lambda_j} \frac{B_{jk}}{(1 - x/q_j)^k} + T(x),$$

and $T(x)$ is analytic in the region $|x| < R$.

Then for every $\varepsilon > 0$

$$[x^n]\, y(x) = \sum_{j=1}^{r} \sum_{k=1}^{\lambda_j} B_{jk} \binom{n + k - 1}{k - 1} n q_j^{-n} + \mathcal{O}\left(R^{-n}(1 + \varepsilon)^n\right).$$

Systems of Functional Equations

$y_1 = y_1(x), y_2 = y_2(x), \ldots y_N = y_N(x)$ satisfy a system of functional equations:

$$\begin{aligned} y_1 &= F_1(x, y_1, y_2, \ldots, y_N), \\ y_2 &= F_2(x, y_1, y_2, \ldots, y_N), \\ &\vdots \\ y_N &= F_N(x, y_1, y_2, \ldots, y_N). \end{aligned}$$

Problem: What is the singular behaviour of $y_j = y_j(x)$?

Notation: $\mathbf{y} = (y_1, y_2, \ldots, y_N))$, $\mathbf{F}(x, \mathbf{y}) = (F_1(x, \mathbf{y}), \ldots, F_N(x, \mathbf{y}))$

Depencency Graph

$$G_{\mathbf{F}} = (V, E)$$

Vertices: $V = \{y_1, y_2, \ldots, y_N\}$

Edges: $(y_i, y_j) \in E \iff F_i(x, \mathbf{y})$ really depends on y_j.

$G_{\mathbf{F}} = (V, E)$ is strongly connected if and only if no subsystem of $\mathbf{y} = F(x, \mathbf{y})$ can be solved before solving the whole system.

Squareroot Singularities

Lemma 7 *Let* $\mathbf{F}(x, \mathbf{y}) = (F_1(x, \mathbf{y}), \ldots, F_N(x, \mathbf{y}))$ *be analytic functions around* $x = 0$ *and* $\mathbf{y} = \mathbf{0}$ *such that all Taylor coefficients are non-negative, that* $\mathbf{F}(0, \mathbf{y}) \equiv \mathbf{0}$*, that* $\mathbf{F}(x, \mathbf{0}) \not\equiv \mathbf{0}$*, and that there exists* j *with* $\mathbf{F}_{y_j y_j}(x, \mathbf{y}) \not\equiv \mathbf{0}$*. Furthermore assume that the region of convergence of* $\mathbf{F}$ *is large enough such that there exists a non-negative solution*

$$x = x_0, \quad \mathbf{y} = \mathbf{y}_0$$

of the system of equations

$$\begin{aligned} \mathbf{y} &= \mathbf{F}(x, \mathbf{y}), \\ 0 &= \det(\mathbf{I} - \mathbf{F}_{\mathbf{y}}(x, \mathbf{y})), \end{aligned}$$

inside it and that the dependency graph $G_{\mathbf{F}} = (V, E)$ *is strongly connected.*

Then x_0 *is the common radius of convergence of the solutions* $y_1(x), \ldots,$ $y_N(x)$ *of the system of functional equations* $\mathbf{y} = F(x, \mathbf{y})$ *and we have a representation of the form*

$$\boxed{y_j(x) = g_j(x) - h_j(x)\sqrt{1 - \frac{x}{x_0}}}$$

locally around $x = x_0$*, where* $g_j(x)$ *and* $h_j(x)$ *are analytic around* $x = x_0$ *and satisfy*

$$(g_1(x_0), \ldots, g_N(x_0)) = \mathbf{y}_0 \quad \textit{and} \quad (h_1(x_0), \ldots, h_N(x_0))' = \mathbf{b}$$

with the unique solution $\mathbf{b} = (b_1, \ldots, b_N) > \mathbf{0}$ *of*

$$(\mathbf{I} - \mathbf{F}_{\mathbf{y}}(x_0, \mathbf{y}_0))\mathbf{b} = \mathbf{0},$$
$$\mathbf{b}'\mathbf{F}_{\mathbf{y}\mathbf{y}}(x_0, \mathbf{y}_0)\mathbf{b} = -2\mathbf{F}_x(x_0, \mathbf{y}_0).$$

If we further assume that $[x^n]\, y_i(x) > 0$ *for* $n \geq n_0$ *and* $1 \leq j \leq N$ *then* $x = x_0$ *is the only singularity of* $y_j(x)$ *on the circle* $|x| = x_0$ *and we obtain an asymptotic expansion for* $[x^n]\, y_j(x)$ *of the form*

$$\boxed{[x^n]\, y_j(x) = \frac{b_j}{2\sqrt{\pi}} x_0^{-n} n^{-3/2} \left(1 + \mathcal{O}\left(n^{-1}\right)\right).}$$

Idea of the Proof.

$N = 1$ equation: $y = y(x)$ with

$$y = F(x, y).$$

If $F_y(x, y(x)) \neq 1$ then by the implicit function theorem $y(x)$ is not singular. Hence, all singulartities x_0 of $y(x)$ have to satisfy

$$F_y(x_0, y_0) = 1.$$

and also

$$F(x_0, y_0) = y.$$

with $y_0 = y(x_0)$.

By the Weierstrass preparation theorem there exist functions $H(x,y)$, $p(x)$, $q(x)$ which are analytic around $x = x_0$ and $y = y_0$ and satisfy $H(x_0,y_0) \neq 1$, $p(x_0) = q(x_0) = 0$ and

$$y - F(x,y) = H(x,y)((y-y_0)^2 + p(x)(y-y_0) + q(x))$$

locally around $x = x_0$ and $y = y_0$. Consequently

$$\begin{aligned} y(x) &= y_0 - \frac{p(x)}{2} \pm \sqrt{\frac{p(x)^2}{4} - q(x)} \\ &= g(x) - h(x)\sqrt{1 - \frac{x}{x_0}} \end{aligned}$$

Finally we just have to apply the asymptotic transfer property.

Small Powers of Functions

Lemma 8 *Let* $y(x) = \sum_{n \geq 0} y_n x^n$ *be a power series with non-negative coefficents such that there is only one singularity on the circle of convergence* $|x| = x_0 > 0$ *and that* $y(x)$ *can be locally represented as*

$$\boxed{y(x) = g(x) - h(x)\sqrt{1 - \frac{x}{x_0}}},$$

where $g(x)$ *and* $h(x)$ *are analytic functions around* x_0 *with* $g(x_0) > 0$ *and* $h(x_0) > 0$*, and that* $y(x)$ *can be continued analytically to* $|x| < x_0 + \delta$*,* $x \notin [x_0, x_0 + \delta)$ *(for some* $\delta > 0$*). Furthermore, let* $\boxed{\rho(x)}$ *be another power series with non-negative coefficients with radius of convergence* $x_1 > x_0$*.*

Then we have

$$\boxed{[x^n]\rho(x)y(x)^k = \frac{k\rho(x_0)g(x_0)^{k-1}h(x_0)}{2n^{\frac{3}{2}}\sqrt{\pi}x_0^n}\left(\exp\left(-\frac{k^2}{4n}\left(\frac{h(x_0)}{g(x_0)}\right)^2\right) + \mathcal{O}\left(\frac{k}{n}\right)\right)}$$

uniformly for $k \leq C\sqrt{n}$ *as* $n \to \infty$*.*

Proof.

W.l.o.g. $x_0 = 1$

Cauchy's formula:

$$[x^n]\,\rho(x)y(x)^k = \frac{1}{2\pi i}\int_\gamma \rho(x)y(x)^k x^{-n-1}\,dx$$

$\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$:

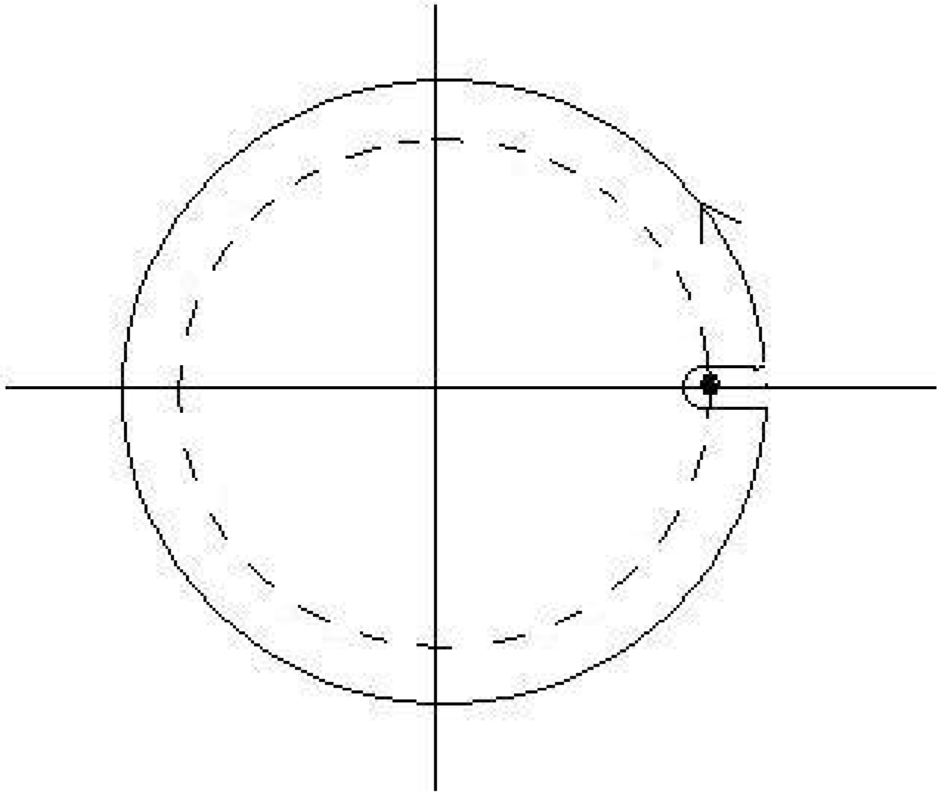
$$\gamma_1 = \left\{ x = 1 + \frac{t}{n} \,\middle|\, |t| = 1, \Re t \leq 0 \right\}$$

$$\gamma_2 = \left\{ x = 1 + \frac{t}{n} \,\middle|\, 0 < \Re t \leq \log^2 n, \Im t = 1 \right\}$$

$$\gamma_3 = \overline{\gamma_2}$$

$$\gamma_4 = \left\{ x \,\middle|\, |x| = \left| 1 + \frac{\log^2 n + i}{n} \right|, \arg(1 + \frac{\log^2 n + i}{n}) \leq |\arg(x)| \leq \pi \right\}.$$

Path of integration



Substitution for $x \in \gamma_1 \cup \gamma_2 \cup \gamma_3$:

$$x = 1 + \frac{t}{n} \Longrightarrow x^{-n-1} = e^{-t}\left(1 + \mathcal{O}\left(\frac{t^2}{n}\right)\right)$$

Furthermore

$$\begin{aligned}
\rho(x)y(x)^k x^{-(n+1)} &= \rho(x)g(x)^k \left(1 - \frac{h(x)}{g(x)}\sqrt{1-x}\right)^k x^{-(n+1)} \\
&= \rho(1)g(1)^k \exp\left(-\frac{k}{\sqrt{n}}\frac{h(1)}{g(1)}(-t)^{\frac{1}{2}} - t\right) \cdot \\
&\quad \cdot \left(1 + \mathcal{O}\left(\frac{|t|^2}{n}\right) + \mathcal{O}\left(\frac{k|t|}{n}\right) + \mathcal{O}\left(k\frac{|t|^{\frac{3}{2}}}{n^{\frac{3}{2}}}\right)\right).
\end{aligned}$$

By using the formula

$$\frac{1}{2\pi i}\int_{\gamma'} e^{-\lambda\sqrt{-t}-t}\,dt = \frac{\lambda}{2\sqrt{\pi}}e^{-\frac{\lambda^2}{4}} + \mathcal{O}\left(e^{-\log^2 n}\right).$$

with

$$\lambda = \frac{k}{\sqrt{n}}\frac{h(1)}{g(1)}$$

the lemma follows.

$(\gamma' = \{t \mid |t| = 1, \Re t \le 0\} \cup \{t \mid 0 < \Re t \le \log^2 n, \Im t = \pm 1\})$

Lemma 9 *Let* $y(x) = \sum_{n \geq 0} y_n x^n$ *be as above and* $\rho(x)$ *another power series that has the same radius of convergence* x_0. *Assume further that it can be continued analytically to the same region as* $y(x)$, *and that it has a local (singular) represenation as*

$$\boxed{\rho(x) = \frac{\overline{g}(x)}{\sqrt{1 - \frac{x}{x_0}}} + \overline{h}(x)},$$

where $\overline{g}(x)$ *and* $\overline{h}(x)$ *are analytic functions around* x_0 *with* $\overline{g}(x_0) > 0$.

Then we have

$$\boxed{[x^n]\rho(x)y(x)^k = \frac{\overline{g}(x_0)g(x_0)^k}{\sqrt{n\pi}x_0^n}\left(\exp\left(-\frac{k^2}{4n}\left(\frac{h(x_0)}{g(x_0)}\right)^2\right) + \mathcal{O}\left(\frac{k}{n}\right)\right)}$$

uniformly for $k \leq C\sqrt{n}$, *where* $C > 0$ *is an arbitrary constant.*

The **Proof** is almost the same as in the previous lemma. The only difference is that one has to use the formula

$$\frac{1}{2\pi i}\int_{\gamma'}\frac{e^{-\lambda\sqrt{-t}-t}}{\sqrt{-t}}\,dt = \frac{1}{\sqrt{\pi}}e^{-\lambda^2/4} + \mathcal{O}\left(e^{-(\log n)^2}\right).$$

Large Powers of Functions

Lemma 10 *Let* $y(x) = \sum_{n \geq 0} y_n x^n$ *be a power series with non-negative coefficents, moreoever, assume that there exists* n_0 *with* $y_n > 0$ *for* $n \geq n_0$*. Furthermore, let* $\rho(x)$ *be another power series with non-negative coefficients and suppose that, both,* $y(x)$ *and* $\rho(x)$ *have positive radius of convergence* R_1, R_2*. Set*

$$\mu(r) = \frac{ry'(r)}{y(r)}$$

and

$$\sigma^2(r) := r\mu'(r) = \frac{ry'(r)}{y(r)} + \frac{r^2 y''(r)}{y(r)} - \frac{r^2 y'(r)^2}{y(r)^2}$$

and let $h(y)$ *denote the inverse function of* $\mu(r)$*.*

Fix a, b with $0 < a < b < \min\{R_1, R_2\}$, then we have

$$\boxed{[x^n]\,\rho(x)y(x)^k = \frac{1}{\sqrt{2\pi k}} \frac{\rho\left(h\left(\frac{n}{k}\right)\right)}{\sigma\left(h\left(\frac{n}{k}\right)\right)} \frac{y\left(h\left(\frac{n}{k}\right)\right)^k}{h\left(\frac{n}{k}\right)^n} \cdot \left(1 + \mathcal{O}\left(\frac{1}{k}\right)\right)}$$

uniformly for n, k with $\mu(a) \le n/k \le \mu(b)$.

Proof.

Cauchy's formula:

$$[x^n]\,\rho(x)y(x)^k = \frac{1}{2\pi i}\int_{|x|=r}\rho(x)y(x)^k x^{-n-1}\,dx$$
$$= \rho(x)\frac{1}{2\pi i}\int_{|x|=r} e^{k\log y(x)-n\log x}\, x^{-1}\,dx.$$

$r = h\left(\frac{n}{k}\right)$, that is

$$\boxed{\frac{ry'(r)}{y(r)} = \frac{n}{k}},$$

is given by the saddle point of the function

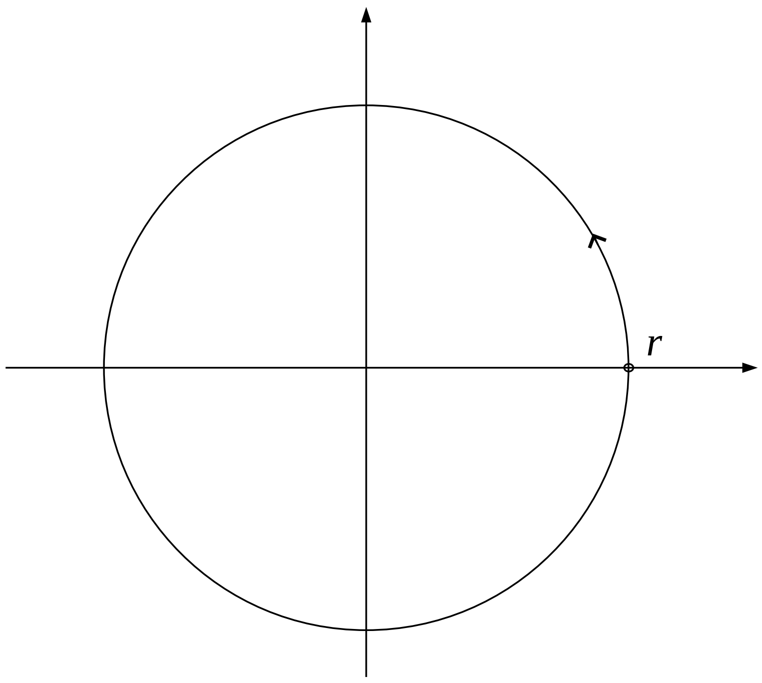
$$x \mapsto k\log y(x) - n\log x.$$

We use the substituion $x = re^{it}$ (for small $|t| \leq k^{-\frac{1}{2}+\eta}$):

$$\rho(x)y(x)^k x^{-n} = \rho(r)y(r)^k r^{-n} e^{-kt^2\sigma^2(r)+\mathcal{O}\left(|t|+k|t|^3\right)}.$$

Consequently

$$\frac{1}{2\pi i} \int\limits_{|t|\leq k^{-\frac{1}{2}+\eta}} \rho(x)y(x)^k x^{-n-1}\,dx = \frac{\rho(r)y(r)^k r^{-n}}{\sqrt{2\pi k\sigma^2(r)}} \cdot \left(1 + \mathcal{O}\left(\frac{1}{k}\right)\right).$$



An Extension

Lemma 11 *Let* $y(x)$ *and* $\rho(x)$ *be as above. Then for every* $0 < r < \min\{R_1, R_2\}$ *we have*

$$\boxed{[x^n]\,\rho(x)y(x)^k = \frac{1}{\sqrt{2\pi k}}\frac{\rho\,(r)}{\sigma\,(r)}\frac{y\,(r)^k}{r^n}\cdot\left(\exp\left(-\frac{(k-n/\mu(r))^2}{2k\sigma^2(r)}\right)+\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)\right)}$$

uniformly for n, k *with* $|k - n/\mu(r)| \le C\sqrt{k}$.

Contents (3)

Part I

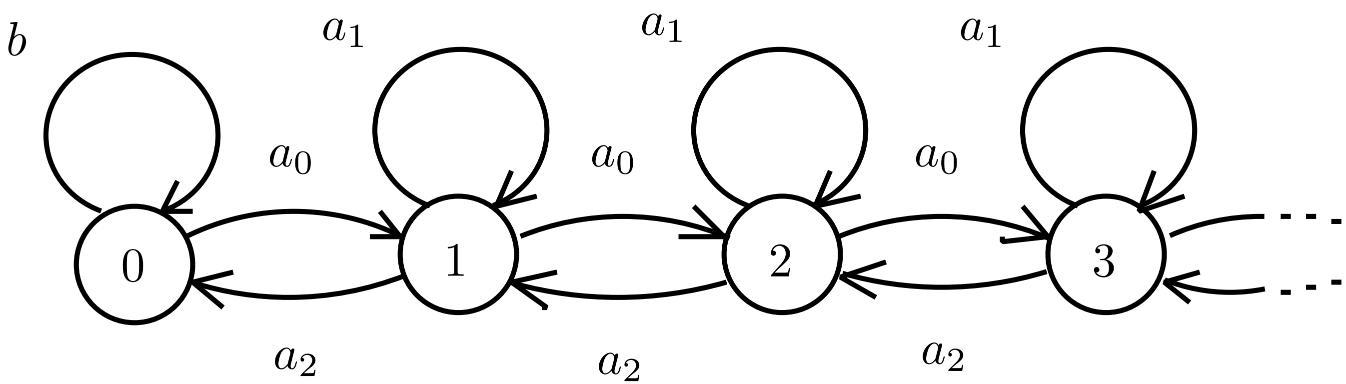
Part II

Random Walk on Non-negative Integers

$m = 1$:

$$\mathbf{P} = \begin{pmatrix} b & a_0 & 0 & 0 & \cdots & \\ a_2 & a_1 & a_0 & 0 & \cdots & \\ 0 & a_2 & a_1 & a_0 & 0 & \cdots \\ 0 & 0 & a_2 & a_1 & a_0 & \cdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots \end{pmatrix},$$

Interpretation as random walk on non-negative integers:



Combinatorial Interpretation

Let h denote a path

$$h = (e_1(h), e_2(h), \ldots, e_n(h))$$

of length n on non-negative integers with edges

$$e_j(h) = (x_{j-1}(h), x_j(h)).$$

Further, denote a weight (or probability) of h by

$$W(h) = \prod_{j=1}^{n} p_{x_{j-1}(h), x_j(h)} = \prod_{j=1}^{n} \mathbf{Pr}\{X_j = x_j(h) \,|\, X_{j-1} = x_{j-1}(h)\}$$

Then

$$\boxed{p_{w,v}^{(n)} = \mathbf{Pr}\{X_n = v \,|\, X_0 = w\} = \sum_h W(h)},$$

where the sum is taken over all paths h of length n with

$$x_0(h) = w \quad \text{and} \quad x_n(h) = v.$$

Generating Functions of Weighted Paths

We then have the interpretation

$$
\begin{aligned}
M_{w,v}(x) &= \sum_{h \text{ path from } w \text{ to } v} W(h) \cdot x^{\text{length}(h)} \\
&= \sum_{n \geq 0} {p_{w,v}}^{(n)} x^n \\
&= \sum_{n \geq 0} \mathbf{Pr}\{X_n = v \,|\, X_0 = w\} \cdot x^n.
\end{aligned}
$$

The calculation of $p_{w,v}^{(n)} = \mathbf{Pr}\{X_n = v \,|\, X_0 = w\}$ can be viewed as a combinatorial enumeration problem of weighted paths of length n and managed with help of generating function techniques.

A First Combinatorial Exercise

Lemma 1 *Let* $N(x)$ *denote the (analytic) solution with* $N(0) = 1$ *of the equation*

$$\boxed{N(x) = 1 + xa_1N(x) + x^2a_0N(x)a_2N(x)},$$

that is,

$$N(x) = \frac{1 - xa_1 - \sqrt{(1 - xa_1)^2 - 4x^2a_0a_2}}{2x^2a_0a_2}.$$

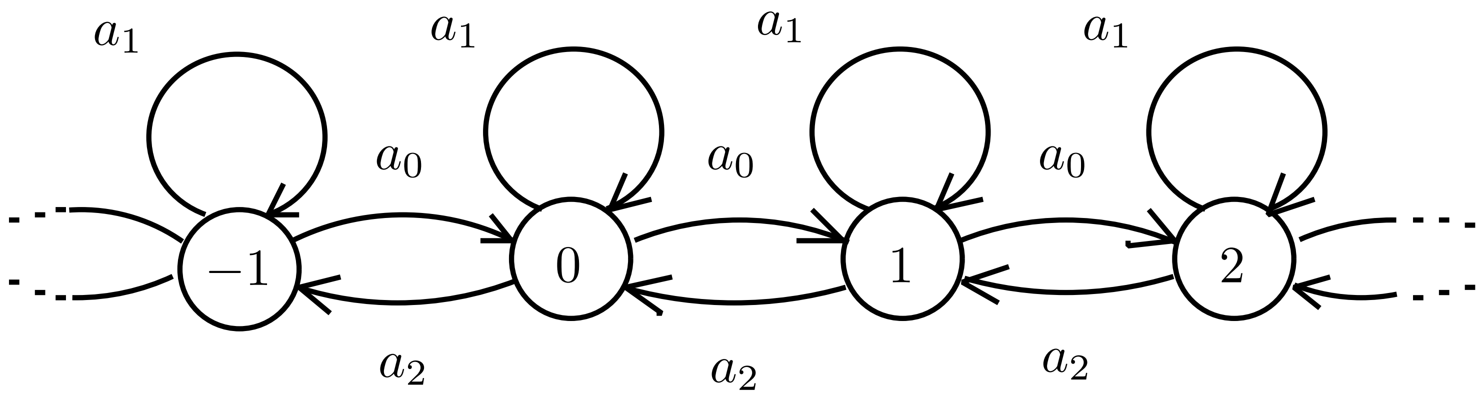
Then

$$\boxed{M_{0,\ell}(x) = \left(1 - xb - x^2a_0\, N(x)a_2\right)^{-1} (xa_0N(x))^\ell}.$$

Recall: $M_{0,\ell}(x) = \sum_{n\geq 0} \mathbf{Pr}\{X_n = \ell \,|\, X_0 = 0\}\, x^n$

Proof.

Let Y_n be the corresponding random walk on (all) integers:

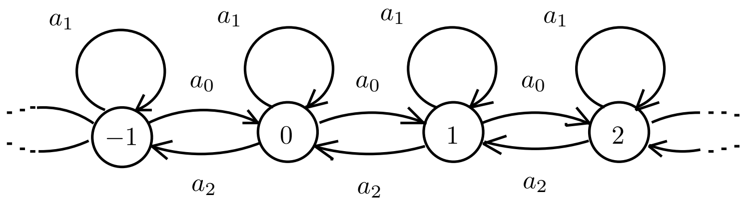


Consider the generating function for non-negative paths of Y_n:

$$N(x) = \sum_{n \geq 0} \mathbf{Pr}\{Y_1 \geq 0,\ Y_2 \geq 0, \ldots,\ Y_{n-1} \geq 0,\ Y_n = 0 \,|\, Y_0 = 0\} \cdot x^n.$$

STEP 1

$$N(x) = 1 + xa_1N(x) + x^2a_0N(x)a_2N(x).$$



- 1 is related to the case $n = 0$.

- If the first step of the path is a loop (with probability a_1) then the remaining part is just a non-negative path from 0 to 0, the corresponding contribution is $a_1x \cdot N(x)$.

- If the first step goes to the right (with probability a_0) then we decompose the path into four parts: into this first step from 0 to the right, into a part from 1 to 1 that is followed by the first step back from 1 to 0, the third part is this step back, and finally into the last part that is again a non-negative path from 0 to 0. Hence, in terms of generating functions this case contributed $a_0x \cdot N(x) \cdot a_2x \cdot N(x)$.

STEP 2 recall: $M_{0,0}(x) = \sum_{n\geq 0} \mathbf{Pr}\{X_n = 0 \,|\, X_0 = 0\}\, x^n$

$$M_{0,0}(x) = 1 + bxM_{0,0}(x) + a_0 x N(x) a_2 x M_{0,0}(x)$$

The same reasoning as in STEP 1. $\Longrightarrow M_{0,0}(x) = \Big(1 - xb - x^2 a_0\, N(x) a_2\Big)^{-1}$

STEP 3 recall: $M_{0,\ell}(x) = \sum_{n\geq 0} \mathbf{Pr}\{X_n = \ell \,|\, X_0 = 0\}\, x^n$

$$M_{0,\ell+1}(x) = M_{0,\ell}(x) a_0 x N(x)$$

All paths from 0 to $\ell+1$ can be divided into three parts. The first part consists of all paths from 0 to ℓ that is followed by the last step from ℓ to $\ell+1$ (which is the second part). And the third part is a *non-negative* path from $\ell+1$ to $\ell+1$. $\Longrightarrow M_{0,\ell}(x) = M_{0,0}(x)(a_0 x N(x))^{\ell}$

The General Case

Consider the $m \times m$ submatrices $\boxed{\mathbf{M}_{k,\ell}(x) = (M_{v,w}(x))_{v \in K_k, w \in K_\ell})}$.

Lemma 2 *Let* $\mathbf{N}(x)$ *denote the (analytic) solution with* $\mathbf{N}(0) = \mathbf{I}$ *of the matrix equation*

$$\boxed{\mathbf{N}(x) = \mathbf{I} + x\mathbf{A}_1\,\mathbf{N}(x) + x^2\mathbf{A}_0\,\mathbf{N}(x)\,\mathbf{A}_2\,\mathbf{N}(x)}.$$

Then

$$\boxed{\mathbf{M}_{0,\ell}(x) = \left(\mathbf{I} - x\mathbf{B} - x^2\mathbf{A}_0\,\mathbf{N}(x)\,\mathbf{A}_2\right)^{-1}(x\mathbf{A}_0\,\mathbf{N}(x))^{\ell}}.$$

The **Proof** is completely the same as in the case $m = 1$.

Contents (4)

Part I

Part II

One-Dimensional Discrete QBD's

Theorem 1 *Suppose that* a_0, a_1, a_2 *and* b *are positive numbers with*

$$a_0 + a_1 + a_2 = b + a_0 = 1$$

and let X_n *be the discrete QBD on the non-negative integers with transition matrix*

$$\mathbf{P} = \begin{pmatrix} b & a_0 & 0 & 0 & \cdots & \\ a_2 & a_1 & a_0 & 0 & \cdots & \\ 0 & a_2 & a_1 & a_0 & 0 & \cdots \\ 0 & 0 & a_2 & a_1 & a_0 & \cdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

1. *If* $a_0 < a_2$ *then we have*

$$\boxed{\lim_{n\to\infty} \mathbf{Pr}\{X_n = \ell \,|\, X_0 = 0\} = \frac{a_2 - a_0}{a_2}\left(\frac{a_0}{a_2}\right)^{\ell} \quad (\ell \geq 0).}$$

that is, X_n *is positive recurrent and converges to the (geometric) stationary distribution.*

2. *If* $a_0 = a_2$ *then* X_n *is null recurrent and* $X_n/\sqrt{2a_0 n}$ *converges weakly to the absolute normal distribution:*

$$\boxed{\mathbf{Pr}\{X_n = \ell \,|\, X_0 = 0\} = \frac{1}{\sqrt{n a_0 \pi}} \exp\left(-\frac{\ell^2}{4 a_0 n}\right) + \mathcal{O}\left(\frac{1}{n}\right)},$$

uniformly for all $\ell \le C\sqrt{n}$ *as* $n \to \infty$.

3. *If* $a_0 > a_2$ *then* X_n *is non recurrent and*

$$\boxed{\frac{X_n - (a_0 - a_2)n}{\sqrt{(a_0 + a_2 - (a_0 - a_2)^2)n}} \to N(0,1).}$$

More precisely

$$\begin{aligned}&\mathbf{Pr}\{X_n = \ell \,|\, X_0 = 0\}\\ =\ & \frac{1}{\sqrt{2\pi(a_0 + a_2 - (a_0 - a_2)^2)n}} \exp\left(-\frac{(\ell - (a_0 - a_2)n)^2}{2(a_0 + a_2 - (a_0 - a_2)^2)n}\right) + \mathcal{O}\left(\frac{1}{n}\right)\end{aligned}$$

uniformly for all $\ell \ge 0$ *with* $|\ell - (a_0 - a_2)n| \le C\sqrt{n}$ *as* $n \to \infty$.

One-Dimensional Discrete QBD's

Recall:

$$M_{0,\ell}(x) = \sum_{n\geq 0} \mathbf{Pr}\{X_n = \ell \mid X_0 = 0\}$$
$$= \left(1 - xb - x^2 a_0 N(x) a_2\right)^{-1} (x a_0 N(x))^\ell$$

with

$$N(x) = 1 + x a_1 N(x) + x^2 a_0 N(x) a_2 N(x)$$

Further

$x_0 =$ radius of convergence of $N(x)$ and
$x_1 =$ radius of convergence of $M_{0,0}(x)$.

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Lemma 12 *Let* $N(x)$ *be given by* $N(x) = 1 + xa_1 N(x) + x^2 a_0 N(x) a_2 N(x)$. *Then we explicitly have*

$$N(x) = \frac{1 - a_1 x - \sqrt{(1 - a_1 x)^2 - 4a_0 a_2 x^2}}{2a_0 a_2 x^2}.$$

The radius of convergence x_0 *is given by*

$$x_0 = \frac{1}{a_1 + 2\sqrt{a_0 a_2}} = \frac{1}{1 - (\sqrt{a}_0 - \sqrt{a}_2)^2}.$$

Furthermore, $N(x)$ *has a local expansion of the form*

$$N(x) = \frac{a_1 + 2\sqrt{a_0 a_2}}{\sqrt{a_0 a_2}} - \left(\frac{a_1 + 2\sqrt{a_0 a_2}}{\sqrt{a_0 a_2}}\right)^{3/2} \cdot \sqrt{1 - (a_1 + 2\sqrt{a_0 a_2})x}$$
$$+ \mathcal{O}\left(1 - (a_1 + 2\sqrt{a_0 a_2})x\right)$$

around its singularity $x = x_0$.

Case 1: $a_0 < a_2$

Lemma 13 *Suppose that* $a_0 < a_2$. *Then* $x_0 > 1$ *but the radius of convergence of* $M_{0,\ell}(x)$ $(\ell \geq 0)$ *is* $x_1 = 1$. *Furthermore*

$$\boxed{\lim_{n\to\infty} \mathbf{Pr}\{X_n = \ell \,|\, X_0 = 0\} = \frac{a_2 - a_0}{a_2}\left(\frac{a_0}{a_2}\right)^{\ell} \quad (\ell \geq 0).}$$

Proof.

$a_0 < a_2$ implies $N(1) = 1/a_2$ and $N'(1) = (1 - a_2 + a_0)/(a_2(a_2 - a_0))$. Thus,

$$1 - bx - a_0 a_2 z^2 N(x) = \frac{a_2}{a_2 - a_0}(1 - x) + \mathcal{O}\left((1 - x)^2\right)$$

and consequently

$$\begin{aligned} M_{0,\ell}(x) &= \left(1 - xb - x^2 a_0\, N(x) a_2\right)^{-1} (x a_0 N(x))^{\ell} \\ &= \frac{a_2 - a_0}{a_2} \left(\frac{a_0}{a_2}\right)^{\ell} \frac{1}{1 - x} + T_\ell(x) \end{aligned}$$

for $|x| < 1/(a_1 + 2\sqrt{a_0 a_2})$.

This directly proves the lemma.

($T_\ell(x)$ is an analytic function that has radius of convergence larger than 1).

Case 2: $a_0 = a_2$

Lemma 14 *Suppose that* $a_0 = a_2$. *Then, both,* $x_0 = 1$ *and the radius of convergence of* $M_\ell(x)$ $(\ell \geq 0)$ *is* $x_1 = 1$.

Furthermore

$$\boxed{\mathbf{Pr}\{X_n = \ell \,|\, X_0 = 0\} = \frac{1}{\sqrt{n a_0 \pi}} \exp\left(-\frac{\ell^2}{4 a_0 n}\right) + \mathcal{O}\left(\frac{\ell}{n^{3/2}}\right).}$$

uniformly for all $\ell \leq C\sqrt{n}$ *as* $n \to \infty$.

Proof.

$N(x)$ is not regular at $x = 1$:

$$1 - bx - a_0 a_2 x^2 N(x) = \sqrt{a_0}\,\sqrt{1-x} + \mathcal{O}\left(|1-x|\right).$$

and

$$a_0 x N(x) = 1 - \frac{1}{\sqrt{a_0}}\sqrt{1-x} + \mathcal{O}\left(|1-x|\right).$$

Hence,

$$\boxed{M_{0,\ell}(x) \sim \frac{1}{\sqrt{a_0}\,\sqrt{1-x}}\left(1 - \frac{1}{\sqrt{a_0}}\sqrt{1-x}\right)^{\ell}}$$

and Lemma 9 applies.

Case 3: $a_0 > a_2$

Lemma 15 *Suppose that $a_0 > a_2$. Then X_n satisfies a central limit theorem with mean value*

$$\mathbf{E}\, X_n \sim (a_0 - a_2)n$$

and variance

$$\mathbf{Var}\, X_n \sim (a_0 + a_2 - (a_0 - a_2)^2)n.$$

In particular we have Furthermore

$$\mathbf{Pr}\{X_n = \ell \mid X_0 = 0\} = \frac{1}{\sqrt{2\pi(a_0 + a_2 - (a_0 - a_2)^2)n}} \exp\left(-\frac{(\ell - (a_0 - a_2)n)^2}{2(a_0 + a_2 - (a_0 - a_2)^2)n}\right) + \mathcal{O}\left(\frac{1}{n}\right).$$

uniformly for all $\ell \geq 0$ with $|\ell - (a_0 - a_2)n| \leq C\sqrt{n}$ as $n \to \infty$

Proof.

Both, $x_0 > 1$ and $x_1 > 1$.

We have $N(1) = 1/a_0$ and $N'(1) = (1 - a_0 + a_2)/(a_0(a_0 - a_2))$ which implies that the saddle point $r = 1$.

Hence, Lemma 11 applies for $M_{0,\ell}(x) = M_{0,0}(x)(a_0 x N(x))^{\ell}$.

Note that $\mu(1) = 1/(a_0 - a_2)$ and $\sigma^2(1) = (a_0 + a_2 - (a_0 - a_2)^2)/(a_0 - a_2)$.

General Homogeneous Discrete QBD's

Theorem 2 *Let* $\mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_2$ *and* $\mathbf{B}$ *be square matrices of order* m *with non-negative elements with such that* $(\mathbf{B} + \mathbf{A}_0)\mathbf{1} = \mathbf{1}$ *and* $(\mathbf{A}_0 + \mathbf{A}_1 + \mathbf{A}_2)\mathbf{1} = \mathbf{1}$*, and let*

$$\mathbf{P} = \begin{pmatrix} \mathbf{B} & \mathbf{A}_0 & \mathbf{0} & \mathbf{0} & \cdots & \\ \mathbf{A}_2 & \mathbf{A}_1 & \mathbf{A}_0 & \mathbf{0} & \cdots & \\ \mathbf{0} & \mathbf{A}_2 & \mathbf{A}_1 & \mathbf{A}_0 & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_2 & \mathbf{A}_1 & \mathbf{A}_0 & \cdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

denote the is a transition matrix of a disceрte QBD X_n*. Furthermore suppose that the matrices* $\mathbf{A}_1$ *is primitive irreducible, that no row of* $\mathbf{A}_0$ *is zero, and that* $\mathbf{A}_2$ *is non-zero.*

Let x_0 *denote the radius of convergence of the entries of* $\mathbf{N}(x)$ *and let* x_1 *denote the radius of convergence of the entries of* $\mathbf{M}_{0,0}(x)$*.*

1. *If* $x_0 > 1$ *and* $x_1 = 1$ *then* X_n *is positive recurrent and for all* $v \geq 0$ *and* $w_0 \in K_0$ *we have*

$$\boxed{\lim_{n\to\infty} \mathbf{Pr}\{X_n = v \,|\, X_0 = w_0\} = p_v,}$$

where $(p_v)_{v\geq 0}$ *is the (unique) stationary distribution of* X_n.

Set

$$\mathbf{R} = \mathbf{A}_0 \cdot \mathbf{N}(1).$$

Then all eigenvalues of $\mathbf{R}$ *have moduli* < 1 *and we have*

$$\boxed{\mathbf{p}_{\ell+1} = \mathbf{p}_\ell \mathbf{R},}$$

in which $\mathbf{p}_\ell = (p_v)_{v\in K_\ell}$.

2. *If* $x_0 = x_1 = 1$ *then* X_n *is null recurrent and there exist* $\rho_{v'} > 0$ $(v' \in V(K))$ *and* $\eta > 0$ *such that*

$$\mathbf{Pr}\{X_n = v \,|\, X_0 = w_0\} = \rho_{\tilde{v}} \sqrt{\frac{1}{n\pi}} \exp\left(-\frac{\ell^2}{4\eta n}\right) + \mathcal{O}\left(\frac{1}{n}\right) \quad (v \in V(K_\ell)).$$

uniformly for all $\ell \le C\sqrt{n}$*) as* $n \to \infty$*. (*$\tilde{v}'$ *denotes the node in* K *that corresponds to* v *from* K_ℓ*)).*

3. *If* $x_1 > 1$ *then* X_n *is non recurrent and there exist* $\tau_{v'} > 0$ $(v' \in V(K))$*,* $\mu > 0$ *and* $\sigma > 0$ *such that*

$$\mathbf{Pr}\{X_n = v \,|\, X_0 = w_0\} = \frac{\tau_{\tilde{v}}}{\sqrt{n}} \exp\left(-\frac{(\ell - \mu n)^2}{2\sigma^2 n}\right) + \mathcal{O}\left(\frac{1}{n}\right) \quad (v \in V(K_\ell)).$$

uniformly for all $\ell \ge 0$ *with* $|\ell - \mu n| \le C\sqrt{n}$ *as* $n \to \infty$*.*

General Homogeneous Discrete QBD's

Lemma 16 *Suppose that* $\mathbf{A}_1$ *is a primitive irreducible matrix and let* $\mathbf{N}(x)$ *denote the solution (with* $\mathbf{N}(0) = \mathbf{I}$*) of the matrix equation*

$$\mathbf{N}(x) = \mathbf{I} + x\mathbf{A}_1\,\mathbf{N}(x) + x^2\mathbf{A}_0\,\mathbf{N}(x)\,\mathbf{A}_2\,\mathbf{N}(x).$$

Then all entries of $\mathbf{N}(x)$ *have a common radius of convergence* $x_0 \geq 1$*. Furthermore, there is a local expansion of the form*

$$\boxed{\mathbf{N}(x) = \tilde{\mathbf{N}}_1 - \tilde{\mathbf{N}}_2\sqrt{1 - \frac{x}{x_0}} + \mathcal{O}\left(1 - \frac{x}{x_0}\right)}$$

around its singularity $x = x_0$*, where* $\tilde{\mathbf{N}}_1$ *and* $\tilde{\mathbf{N}}_2$ *are matrices with positive elements.*

Proof.

The equation for $\mathbf{N}(x)$ is a system of m^2 algebraic equation for entries of $\mathbf{N}(x)$.

$\mathbf{B}$ is irreducible (and non-negative). Thus, the so-called *dependency graph* is strongly connected. Consequently, by Lemma 7 all entries of $\mathbf{N}(x)$ have the same finite radius of convergence a squareroot singularity at $x = x_0$ of the above form.

The coefficents of $\mathbf{N}(x)$ are probabilities. Hence $x_0 \geq 1$.

Case 1: $x_0 > 1$ and $x_1 = 1$

$x = 1$ is a regular point of $\mathbf{N}(x)$. $\mathbf{B}+\mathbf{A}_0\,\mathbf{N}(1)\,\mathbf{A}_2$ is primitive irreducible. Thus,

$$f(x) = \det\left(\mathbf{I} - x\mathbf{B} - x^2\mathbf{A}_0\,\mathbf{N}(x)\,\mathbf{A}_2\right)$$

has a simple zero at $x = 1$.

Consequently, all entries of

$$\left(\mathbf{I} - x\mathbf{B} - x^2\mathbf{A}_0\,\mathbf{N}(x)\,\mathbf{A}_2\right)^{-1}$$

have a simple pole at $x = 1$.

Therefore, the limit

$$\lim_{n\to\infty}[x^n]\ \left(\mathbf{I} - x\mathbf{B} - x^2\mathbf{A}_0\,\mathbf{M}(x)\,\mathbf{A}_2\right)^{-1}(x\mathbf{A}_0\,\mathbf{N}(x))^{\ell}$$

exists.

Case 2: $x_0 = x_1 = 1$

$\mathbf{N}(x)$ is singular at $x = 1$ and

$$f(x) = \det\left(\mathbf{I} - x\mathbf{B} - x^2\mathbf{A}_0\,\mathbf{N}(x)\,\mathbf{A}_2\right) = c_1\sqrt{1-x} + \mathcal{O}\left(|1-x|\right),$$

where $c_1 \neq 0$.

Next the largest eigenvalue $\lambda(x)$ of $x\mathbf{A}_0\,\mathbf{N}(x)$ is given by

$$\lambda(x) = 1 - c_2\sqrt{1-x} + \mathcal{O}\left(|1-x|\right).$$

and we have (for some matrix $\mathbf{Q}_1$)

$$(x\mathbf{A}_0\,\mathbf{N}(x))^\ell = \lambda(x)^\ell\mathbf{Q}_1 + \mathcal{O}\left(\lambda(x)^{(1-\eta)\ell}\right).$$

Hence,

$$\left(\mathbf{I} - x\mathbf{B} - x^2\mathbf{A}_0\,\mathbf{M}(x)\,\mathbf{A}_2\right)^{-1}(x\mathbf{A}_0\,\mathbf{N}(x))^\ell \sim \frac{\left(1 - c_2\sqrt{1-x}\right)^\ell}{c_1\sqrt{1-x}}\mathbf{Q}_2$$

and Lemma 9 applies.

Case 3: $x_1 > 1$

Both, $x_0 > 1$ and $x_1 > 1$.

Hence, $\lambda(x)$ is regular at $x = 1$.

Consequently

$$\left(\mathbf{I} - x\mathbf{B} - x^2\mathbf{A}_0\,\mathbf{N}(x)\,\mathbf{A}_2\right)^{-1}(x\mathbf{A}_0\,\mathbf{N}(x))^\ell \sim \lambda(x)^\ell \mathbf{Q}_3$$

and Lemma 11 applies.

Continuous Quasi Birth and Death Processes

Inverse Laplace transform is used instead of Cauchy's formula.
(The technical details are almost the same.)

Thank You!