An Asymptotic Analysis of Unlabeled k-Trees

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Abstract. In this paper we solve the *asymptotic counting problem* for unlabeled k-trees. By applying a proper singularity analysis of generating functions we show that the numbers U_n of unlabeled k-trees of size n are asymptotically given by $U_n \sim c_k n^{-5/2}(\rho_{1^k})^{-n}$, where $c_k > 0$ and $\rho_{1^k} > 0$ denotes the radius of convergence of the generating function $U(z) = \sum_{n\geq 0} U_n z^n$. Furthermore we prove that the number of *leaves* and more generally the number of *nodes* of given degree satisfy a central limit theorem with mean value and variance that are asymptotically linear in the number of hedra where a hedron is a $(k+1)$-clique in a k-tree.

Keywords: k-tree, k-coding tree, generating function, singularity analysis, degree distribution, Central limit theorem

1 Introduction

A k-tree is – in some sense – a generalization of a tree and can be defined recursively: a k-tree is either a complete graph on k vertices or a graph obtained from a smaller k-tree by adjoining a new vertex together with k edges connecting it to a k-clique of the smaller k-tree. In particular, a 1-tree is a usual tree.

The notion of a k-tree originates from the parameter *tree-width* $\mathrm{tw}(G)$ of a graph G, which is the minimum width among all possible tree decompositions of G, or equivalently, $\mathrm{tw}(G)$ is the minimum k such that G is a subgraph of a k-tree. The concept of tree-width is of central importance to the analysis of graphs with forbidden minors of Robertson and Seymour [20] and received more algorithmic attention due to the general complexity result of Courcelle about monadic second-order logic graph properties decidability for graphs with bounded tree-width [3, 4]. It seems that many NP-hard problems on graphs of bounded tree-width can be solved in polynomial time [13]. A k-tree, as a bounded tree-width graph, is exactly the maximal graph with a fixed tree-width k such that no more edges can be added without increasing its tree width.

Labeled k-trees have been already counted by Beineke, Pippert, Moon and Foata [2, 18, 9] four decades ago that the number B_n of k-trees having n labeled vertices is $B_n = \binom{n}{k}(k(n-k)+1)^{n-k-2}$. Instead the counting problem of unlabeled k-trees is much more difficult. Only the case of 2-trees was already solved by Harary and Palmer [15, 16] and Fowler *et al* [10] by using dissimilarity characteristic theorem. The general case was a long standing open problem and was solved just recently by Gainer-Dewar [11].

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Subsequently both Gessel and Gainer-Dewar [12] simplified the generating function approach for unlabeled k-trees by coloring the vertices of a k-tree in $(k+1)$ colors such that adjacent vertices have different colors. This breaks the symmetry of k-trees and avoids the use of compatible cyclical orientation of each $(k+1)$-clique in a k-tree.

The purpose of this paper is to provide a first asymptotic analysis of k-trees. First we will solve the *asymptotic counting problem* and show (see Theorem 1) the numbers U_n of unlabeled k-trees of size n are asymptotically given by

$$U_n \sim c_k n^{-5/2} (\rho_{1^k})^{-n},$$

where $c_k > 0$ and $\rho_{1^k} > 0$ denotes the radius of convergence of the generating function $U(z) = \sum_{n\ge 0} U_n z^n$. This is in complete accordance with Otter's result for trees [19], Labelle *et al.* result for unlabeled 2-trees [17], and also with the corresponding results for labeled k-trees.

Second we provide a first structural analysis of unlabeled k-trees. We prove that the number of *leaves* and more generally the number of *nodes* of given degree satisfy a central limit theorem with mean value and variance that are asymptotically linear in the number of hedra (Theorems 2 and 3) where a hedron is a $(k+1)$-clique in a k-tree. This is also a natural generalization of corresponding results for (unlabeled) trees, see [6].

Actually we expect that unlabeled k-trees have many asymptotic properties in common with trees. For example, it is very likely that k-trees scaled by $1/\sqrt{n}$ converge weakly to the so-called *continuum random tree* as it holds for unlabeled trees (see [1, 14]). In this case it would follow that the diameter D_n scaled by $1/\sqrt{n}$ has a limiting distribution etc.

However, there are other parameters of interest – like the maximum degree – that cannot characterized by a continuum tree property. Anyway, as in the case of trees (see [5]) we expect that the maximum degree of k-trees should be concentrated at $c\log n$ (for a proper constant $c > 0$). We plan to work on these (and related) questions in a follow-up paper.

The plan of the paper is as follows. In Section 2 we recall the combinatorial background from [12], in particular a system of equations for generating functions. This system is then used to solve the asymptotic counting problem in Section 3. Finally the number of leaves and the number of nodes of given degree are discussed in Sections 4 and 5.

2 Combinatorics of Unlabeled k-Trees

Here we shall use the terminology introduced in [11, 12] to state the system of equations for the generating functions. Let $g \in \mathfrak{S}_m$ be a permutation of $\{1, 2, \cdots, m\}$ that has ℓ_i cycles of size i, $1 \le i \le k$, in its cyclic decomposition. Then its cycle type $\lambda = (1^{\ell_1}, 2^{\ell_2}, \cdots, k^{\ell_k})$ is a partition of $m = \ell_1 + 2\ell_2 + \cdots + k\ell_k$. (In what follows we will denote by $\lambda \vdash m$ that λ is a partition of m.) Furthermore $z_\lambda = 1^{\ell_1}\ell_1! 2^{\ell_2}\ell_2! \cdots k^{\ell_k}\ell_k!$ denotes the number of permutations of cycle type λ.

A hedron is a $(k+1)$-clique in a k-tree and a front is a k-clique in a k-tree. According to the inductive construction of a k-tree, the number of vertices in a k-tree having n hedra is $k + 1 + (n-1) = n + k$. A colored hedron-labeled k-tree is a k-tree that has each vertex colored from the set $\{1', 2', \cdots, (k+1)'\}$ so that any two adjacent vertices are colored differently, and each hedron is labeled with a distinct number from $\{1, 2, \cdots, n\}$. The only automorphism that preserves hedra and colors of a colored hedron-labeled k-tree is the identity automorphism, for which we can ignore the colors of vertices. k-coding trees have labeled black vertices and colored vertices. Each edge connects a labeled black vertex with a colored vertex from colors $\{1, 2, \cdots, k+1\}$. To construct a k-coding tree from a colored hedron-labeled k-tree, we color each front of a hedron with a distinct color from $\{1, 2, \cdots, k+1\}$. The corresponding k-coding tree has each black vertex labeled with i representing each hedron of the k-tree with label i and each j-colored vertex representing each front of the k-tree with color j. We connect a black vertex with a colored vertex if and only if the corresponding hedron contains the corresponding front. As a result, a colored hedron-labeled k-tree is bijective to a k-coding tree. Under the action of $\mathfrak{S}_n$ and $\mathfrak{S}_{k+1}$, the orbits of colored hedron-labeled k-tree, which are unlabeled k-trees are bijective to the orbits of unlabeled k-coding trees under the action $\mathfrak{S}_{k+1}$. See Figure 1 for an example. In [12] the following system of

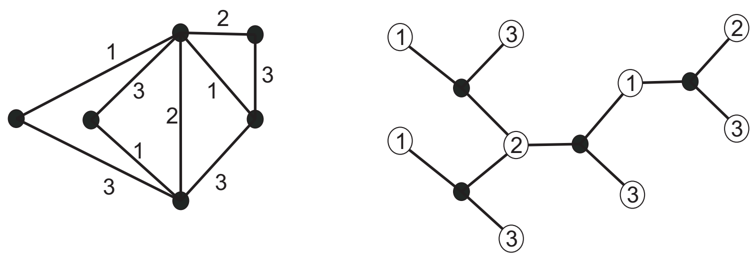

Fig. 1: The bijection between an unlabeled colored 2-tree (left) and an unlabeled 2-coding tree (right).

equations was set up that determines the generating function $U(z)$ for unlabeled k-trees by

$$U(z) = B(z) + C(z) - E(z),$$

where

$$B(z) = \sum_{\lambda \vdash k+1} \frac{B_\lambda(z)}{z_\lambda} \quad (1) \qquad B_\lambda(z) = z \prod_i C_{\lambda^i}(z^i) \quad (4)$$

$$C(z) = \sum_{\mu \vdash k} \frac{C_\mu(z)}{z_\mu} \quad (2) \qquad \bar{B}_\mu(z) = z \prod_i C_{\mu^i}(z^i) \quad (5)$$

$$E(z) = \sum_{\mu \vdash k} \frac{\bar{B}_\mu(z) C_\mu(z)}{z_\mu} \quad (3) \qquad C_\mu(z) = \exp\left[\sum_{m=1}^{\infty} \frac{\bar{B}_{\mu^m}(z^m)}{m}\right]. \quad (6)$$

Here $B(z)$ is the generating function for color-orbits of black-rooted unlabeled k-coding trees, $C(z)$ is the generating function for color-orbits of colored-rooted unlabeled k-coding trees and $E(z)$ is the generating function for color-orbits of unlabeled k-coding trees rooted at an edge. We call an unlabeled k-coding tree an j-reduced black-rooted tree if it is a black-rooted tree with all the neighbors of the root are colored by $\{1, 2, \cdots, j-1, j+1 \cdots, k+1\}$. $\bar{B}(z)$ is the generating function for j-reduced black-rooted tree. For any $\pi \in \mathfrak{S}_{k+1}$, $B_\lambda(z)$ is the generating function for black-rooted tree that are fixed by π where π has cycle type λ. For any $\sigma \in \mathfrak{S}_k$, $\bar{B}_\mu(z)$ (resp. $C_\mu(z)$) is the generating function for reduced black-rooted tree (resp. colored-rooted tree) that are fixed by σ where σ has cycle type μ. Here $\bar{B}_\mu(z) = \bar{B}_\lambda(z)$, $C_\mu(z) = C_\lambda(z)$ if λ is obtained from μ by adding a part 1. $C_{\lambda^i}(z)$ is the generating function for color-rooted trees that are fixed by π^i and λ^i denotes the cycle type of permutation π^i where $\pi \in \mathfrak{S}_{k+1}$ has cycle type λ and i is a part of λ. Finally, the above products over i range over all parts i of λ or μ, respectively, that is, if i is contained m times in λ then it appears m times in the product.

3 Asymptotics of unlabeled k-trees

Let $U_n = [z^n]U(z)$ denote the number of unlabeled k-trees of size n. Then we have the following asymptotic property.

Theorem 1 *The numbers of unlabeled k-trees are asymptotically given by*

$$U_n = \frac{1}{k!}\frac{[3-k+(k-1)(k\rho_{1^k})^{-1/k}]}{2\sqrt{2\pi}k^3\rho_{1^k}^{-1/2}}\left[\frac{m'(\rho_{1^k})}{m(\rho_{1^k})}\right]^{3/2}(k\rho_{1^k})^{2-2/k}n^{-5/2}(\rho_{1^k})^{-n}(1+O(n^{-1})). \quad (7)$$

where $m(z) = z\exp\left[k\sum_{m=2}^{\infty}\bar{B}_{1^k}(z^m)/m\right]$, $\bar{B}_{1^k}(z) = m(z)e^{k\bar{B}_{1^k}(z)}$ and ρ_{1^k} is the unique real positive solution of $m(z) = (ek)^{-1}$.

Proof: For $\mu = 1^k$, from eq. (5) and (6), we have

$$\bar{B}_{1^k}(z) = z\exp\left[k\sum_{m=1}^{\infty}\frac{\bar{B}_{1^k}(z^m)}{m}\right] = \exp(k\bar{B}_{1^k}(z))\cdot z\cdot\exp\left[k\sum_{m=2}^{\infty}\frac{\bar{B}_{1^k}(z^m)}{m}\right].$$

Setting

$$m(z) = z\exp\left[k\sum_{m=2}^{\infty}\frac{\bar{B}_{1^k}(z^m)}{m}\right],$$

and $\bar{B}_{1^k}(z) = T(m(z))$ for some power series $T(z)$ we thus obtain $T(z) = z\exp(kT(z))$. Hence, if $W(z)$ denotes the classical tree function that is given by $W(z) = z\exp(W(z))$ it follows that $T(z) = \frac{1}{k}W(kz)$. It is very well known that $W(z)$ has radius of convergence $\rho = 1/e$, that it has a singular expansion of the form

$$W(z) = 1-\sqrt{2}(1-ez)^{1/2}+\frac{2}{3}(1-ez)+\cdots$$

around $z = 1/e$ and that $W(z)$ can be analytically continued to a region of the form $\{z\in\mathbb{C} : |z| < 1/e+\eta\}\setminus[1/e,\infty)$ for some $\eta > 0$. In particular it follows that $T(z)$ has corresponding properties, of course its radius of convergence equals $1/(ke)$. Actually, in what follows we will only need that $T(z)$ is analytic in a so-called Δ-domain

$$\Delta_\alpha(M,\phi) = \{z \mid |z| < M, z\neq\alpha, |\arg(z-\alpha)| > \phi\}$$

where $0 < \phi < \frac{\pi}{2}$. (Analyticity in Δ-domains is used to *transfer* the singular expansion of the generating function into an asymptotic expansion for the coefficients, see [8].) In our case we know that $T(z)$ is analytic in $\Delta_{1/(ke)}(1/(ke)+\eta,\phi)$.

Let ρ_{1^k} be the unique dominant singularity of $\bar{B}_{1^k}(z)$, then we shall show $(4k)^{-1}\le\rho_{1^k}\le(ek)^{-1}$. Since $\bar{B}_{1^k}(z)$, $T(z)$ and $m(z)$ have positive coefficients and $\bar{B}_{1^k}(z) = T(m(z))$, we have $[z^n]\bar{B}_{1^k}(z)\ge[z^n]T(z)$ which indicates the radius of convergence for $\bar{B}_{1^k}(z)$ is at most that for $T(z)$ which is $(ek)^{-1}$. On the other hand, the radius of convergence for $\bar{B}_{1^k}(z)$ is at least that for $M(z)$ where $M(z) = z(1-kM(z))^{-1}$ and accordingly $\rho_{1^k}\ge(4k)^{-1}$.

The next step we shall prove that the dominant singularity $z = \rho_{1^k}$ of $\bar{B}_{1^k}(z)$ is of square root type, too. Since $m(z)$ has radius of convergence $\sqrt{\rho_{1^k}} > \rho_{1^k}$ it follows that it is analytic at $z = \rho_{1^k}$. More precisely the singular expansion of $\bar{B}_{1^k}(z)$ close to $z = \rho_{1^k}$ comes from composing the singular expansion of $T(z)$ at $1/(ek)$ with the analytic expansion of $m(z)$ at $\rho_{1,k}$. In this context we also observe that $m(\rho_{1^k}) = (ek)^{-1}$ and $m'(\rho_{1^k}) > 1$. According to this we get the local expansion

$$\begin{aligned}\bar{B}_{1^k}(z) &= \frac{1}{k} - \frac{\sqrt{2}}{k}\left[1 - \frac{m(z)}{m(\rho_{1^k})}\right]^{1/2} + \frac{2}{3k}\left[1 - \frac{m(z)}{m(\rho_{1^k})}\right] + \sum_{i\geq 3}(-1)^i m_i \left[1 - \frac{m(z)}{m(\rho_{1^k})}\right]^{i/2} \\ &= \frac{1}{k} - \frac{\sqrt{2}}{k}\left[\frac{(\rho_{1^k} - z)m'(\rho_{1^k})}{m(\rho_{1^k})}\right]^{1/2} + \frac{2}{3k}\left[\frac{(\rho_{1^k} - z)m'(\rho_{1^k})}{m(\rho_{1^k})}\right] + O(\rho_{1^k} - z)^{3/2}.\end{aligned}$$

Henceforth $C_{1^k}(z) = z^{-1/k}\bar{B}_{1^k}(z)^{1/k}$ has $z = \rho_{1^k}$ as dominant singularity of square root type, too, and a local expansion of the form

$$C_{1^k}(z) = (k\rho_{1^k})^{-1/k} + a(\rho_{1^k} - z)^{1/2} + b(\rho_{1^k} - z) + c(\rho_{1^k} - z)^{3/2} + O(\rho_{1^k} - z)^2 \qquad (8)$$

where a, b are given by

$$a = -\frac{\sqrt{2}(k\rho_{1^k})^{(k-1)/k}}{k^2\rho_{1^k}}\left[\frac{m'(\rho_{1^k})}{m(\rho_{1^k})}\right]^{1/2}, \quad b = \frac{3-k}{3k^3}\frac{(k\rho_{1^k})^{(k-1)/k}}{\rho_{1^k}}\left[\frac{m'(\rho_{1^k})}{m(\rho_{1^k})}\right].$$

Actually the functions $\bar{B}_{1^k}(z)$, $C_{1^k}(z) = C_{1^{k+1}}(z)$, and $B_{1^{k+1}}(z)$ have the same radius of convergence ρ_{1^k} (which is a square-root singularity).

Furthermore we observe that for any $k \geq 2$ and $\mu \neq 1^k$, $\bar{B}_\mu(z)$ and $C_\mu(z)$ are analytic at $z = \rho_{1^k}$. Let ρ_μ be the unique dominant singularity of $\bar{B}_\mu(z)$. Since the number of black-rooted trees that are fixed by permutation of type $\mu \neq 1^k$ is less than or equal to those fixed by identity permutation, i.e., $[z^n]\bar{B}_\mu(z) \leq [z^n]\bar{B}_{1^k}(z)$ it follows that $\rho_\mu \geq \rho_{1^k}$. Therefore it remains to prove $\rho_\mu \neq \rho_{1^k}$. In the case μ has exactly j parts of size 1 where $0 < j < k$, then we have

$$\bar{B}_\mu(z) = zC_\mu(z)^j \prod_{i\neq 1} C_{\mu^i}(z^i) \text{ and } C_\mu(z) = \exp(\bar{B}_\mu(z)) \exp\left[\sum_{m=2}^{\infty} \frac{\bar{B}_{\mu^m}(z^m)}{m}\right] \qquad (9)$$

which lead to

$$\bar{B}_\mu(z) = z\prod_{i\neq 1} C_{\mu^i}(z^i)\exp(j\bar{B}_\mu(z))\exp\left[j\sum_{m=2}^{\infty}\frac{\bar{B}_{\mu^m}(z^m)}{m}\right]. \qquad (10)$$

By setting $\bar{B}_\mu(z) = y$, it follows that $(\rho_\mu, \bar{B}_\mu(\rho_\mu))$ is the unique solution of

$$\begin{aligned}M(z,y) &= z\prod_{i\neq 1} C_{\mu^i}(z^i)\exp(jy)\exp\left[j\sum_{m=2}^{\infty}\frac{\bar{B}_{\mu^m}(z^m)}{m}\right] \\ M_y(z,y) &= jz\prod_{i\neq 1} C_{\mu^i}(z^i)\exp(jy)\exp\left[j\sum_{m=2}^{\infty}\frac{\bar{B}_{\mu^m}(z^m)}{m}\right] = 1,\end{aligned}$$

and consequently $\bar{B}_\mu(\rho_\mu) = 1/j$. Recall that $\bar{B}_{1^k}(\rho_{1^k}) = 1/k$, thus, we have $k\bar{B}_{1^k}(\rho_{1^k}) = j\bar{B}_\mu(\rho_\mu) = 1$. If $\rho_\mu = \rho_{1^k}$, then $k\bar{B}_{1^k}(\rho_{1^k}) > j\bar{B}_\mu(\rho_{1^k}) = 1$, which contradicts the relation $k\bar{B}_{1^k}(\rho_{1^k}) = 1$. Therefore we can conclude that $\rho_{1^k} < \rho_\mu$ and from eq. (9), eq. (10), $C_\mu(z)$ also has dominant singularity ρ_μ. In the case μ has no part of size 1, then $\bar{B}_\mu(z)$ is a product of $C_{\mu^i}(z^i)$ where $i \geq 2$ and μ^i has part of size 1. Consequently we have $\rho_\mu > \min\{\rho_{\mu^i} : i \in \mu\} > \rho_{1^k}$. Now we can conclude for any $k \geq 2$ and $\mu \neq 1^k$, $\rho_{1^k} < \rho_\mu$, namely $\bar{B}_\mu(z)$ and $C_\mu(z)$ are analytic at $z = \rho_{1^k}$.

Summing up, since $B_{1^{k+1}}(z) = zC_{1^k}(z)^{k+1}$ has a square-root singularity at $z = \rho_{1^k}$ and B_λ for any $\lambda \neq 1^{k+1}$ is analytic at ρ_{1^k}, the dominant term in the singular expansion of $U(z)$ comes from

$$\frac{B_{1^{k+1}}(z)}{z_{1^{k+1}}} + \frac{C_{1^k}(z)}{z_{1^k}} - \frac{C_{1^k}(z)\bar{B}_{1^k}(z)}{z_{1^k}} \quad = \quad \frac{-kzC_{1^k}(z)^{k+1}}{(k+1)!} + \frac{C_{1^k}(z)}{k!}.$$

All the other terms are all analytic at $z = \rho_{1^k}$. Together with the singular expansion of $C_{1^k}(z)$ shown in eq. (8), we can calculate the constant for the term $(\rho_{1^k} - z)^{1/2}$ in the singular expansion of $U(z)$, which is $\frac{-k\rho_{1^k}}{(k+1)!}\binom{k+1}{1}\frac{a}{k\rho_{1^k}} + \frac{a}{k!} = 0$. Similarly the constant for the term $(\rho_{1^k} - z)^{3/2}$ in the singular expansion of $U(z)$ is

$$\frac{-k\rho_{1^k}}{(k-1)!}a(b + \frac{k-1}{3!}a^2) = \frac{\sqrt{2}}{k!}\frac{[3-k+(k-1)(k\rho_{1^k})^{-1/k}]}{3k^3\rho_{1^k}}\left[\frac{m'(\rho_{1^k})}{m(\rho_{1^k})}\right]^{3/2}(k\rho_{1^k})^{(2k-2)/k}$$

which is positive since $\rho_{1^k} \leq (ek)^{-1} < k^{-1}(1 - \frac{2}{k-1})^{-k}$. Now we have derived the singular expansion of $U(z)$ at $z = \rho_{1^k}$:

$$\begin{aligned} U(z) \quad &= \quad U(\rho_{1^k}) + \frac{\sqrt{2}}{k!}\frac{[3-k+(k-1)(k\rho_{1^k})^{-1/k}]}{3k^3\rho_{1^k}}\left[\frac{(\rho_{1^k}-z)m'(\rho_{1^k})}{m(\rho_{1^k})}\right]^{3/2}(k\rho_{1^k})^{2-2/k} \\ &+ \quad c_1(\rho_{1^k} - z) + c_2(\rho_{1^k} - z)^2 + O(\rho_{1^k} - z)^{5/2}. \end{aligned} \tag{11}$$

By applying transfer theorem [5, Corollary 2.15], we get eq. (7) and the proof is complete. □

4 Leaves of unlabeled k-trees

As we have explained in Section 2 unlabeled k-trees are bijective to the orbits of unlabeled k-coding tree under the action $\mathfrak{S}_{k+1}$. We call a black node a *leaf* if only one of its colored neighbor connects with other black nodes. It is obvious that this notion corresponds to the only meaningful notion of a *leaf* of a k-tree.

In the sequel we shall weight each black node by z and each leaf by w. Let $U(z, w)$ be the generating function for unlabeled color-orbits of unlabeled k-coding trees, then we have:

Theorem 2 *Let X_n be the random variable associated with the number of leaves of k-coding trees, that is*

$$\mathbb{P}(X_n = r) = \frac{[z^n w^r]U(z,w)}{[z^n]U(z,1)}.$$

Then there exists positive constants μ and σ^2 such that $\mathbb{E}(X_n) = \mu\, n + O(1)$ and $\mathbb{V}ar(X_n) = \sigma^2\, n + O(1)$. Furthermore X_n satisfies a central limit theorem of type

$$\frac{X_n - \mathbb{E}(X_n)}{\sqrt{\mathbb{V}ar(X_n)}} \longrightarrow N(0,1).$$

Proof: Let $B(z,w)$ be the generating function for color-orbits of black-rooted trees, let $C(z,w)$ be the generating function for color-orbits of color-rooted trees and let $E(z,w)$ be the generating function for color-orbits of coding trees rooted at an edge, then according to the dissymmetry theorem and Cauchy-Frobenius theorem, we have

$$U(z,w) = B(z,w) + C(z,w) - E(z,w) \tag{12}$$

where

$$\begin{aligned} B(z,w) &= \sum_{\lambda \vdash k+1} \frac{B_\lambda(z,w)}{z_\lambda},\ C(z,w) = \sum_{\mu \vdash k} \frac{C_\mu(z,w)}{z_\mu},\ B_\lambda(z,w) = z\prod_i C_{\lambda^i}(z^i,w^i) + z(w-1) \\ E(z,w) &= \sum_{\mu \vdash k} \frac{(\bar{B}_\mu(z,w) - zw + z)C_\mu(z,w) + z(w-1)}{z_\mu} \\ \bar{B}_\mu(z,w) &= z\prod_i C_{\mu^i}(z^i,w^i) + z(w-1) \qquad (13) \\ C_\mu(z,w) &= \exp\left[\sum_{m=1}^{\infty} \frac{\bar{B}_{\mu^m}(z^m,w^m)}{m}\right]. \qquad (14) \end{aligned}$$

For $\mu = 1^k$, from eq. (13) and (14), we obtain

$$\bar{B}_{1^k}(z,w) = z\exp\left[k\sum_{m=1}^{\infty} \frac{\bar{B}_{1^k}(z^m,w^m)}{m}\right] + z(w-1). \tag{15}$$

We first show for $\mu \neq 1^k$ and $m \geq 2$, $\bar{B}_\mu(z,w)$ and $\bar{B}_{1^k}(z^m,w^m)$ are analytic if (z,w) is close to $(\rho_{1^k},1)$. For sufficiently small $\epsilon > 0$, we consider $|w| \leq \frac{\rho_{1^k}}{\rho_{1^k}+\epsilon}$ and $|z| \leq \rho_{1^k} + \epsilon$, then for $m \geq 2$,

$$\begin{aligned} |\bar{B}_\mu(z,w)| &\leq \bar{B}_\mu(|zw|,1) \leq \bar{B}_\mu(\rho_{1^k},1) \leq K\rho_{1^k}, \\ |\bar{B}_{1^k}(z^m,w^m)| &\leq \bar{B}_{1^k}(|zw|^m,1) \leq \bar{B}_{1^k}((\rho_{1^k})^m,1) \leq M(\rho_{1^k})^m. \end{aligned}$$

The last inequality holds because $\bar{B}_\mu(z,1)$ and $\bar{B}_{1^k}(z,1)$ are convex for $z \in [0,\rho_{1^k}]$ and $z \in [0,(\rho_{1^k})^2]$, respectively. Now we set

$$F(y,z,w) = z\exp\left[ky + k\sum_{m=2}^{\infty} \frac{\bar{B}_{1^k}(z^m,w^m)}{m}\right] + z(w-1)$$

that $F(y,z,w)$ is analytic for (y,z,w) around $(0,0,0)$ and $F(y,0,w) \equiv 0$, $F(0,z,w) \not\equiv 0$. Furthermore, the coefficients of $F(y,z,w)$ are all non-negative, then $\bar{B}_{1^k}(z,w)$ is the unique solution of $F(y,z,w) = y$ that can be expressed as

$$\bar{B}_{1^k}(z,w) = \alpha(z,w) - \beta(z,w)\left[1 - \frac{z}{\rho_{1^k}(w)}\right]^{1/2},$$

where $\alpha(z,w), \beta(z,w), \rho_{1^k}(w)$ are analytic for $|w-1| \leq \epsilon$, $|z-\rho_{1^k}(w)| < \varepsilon$, $\arg(z-\rho_{1^k}(w))| > \phi$ (for some $\phi \in (0,\pi/2)$) and ϵ is sufficiently small. Furthermore, $\beta(\rho_{1^k}(w),w) \neq 0$ and $\bar{B}_{1^k}(\rho_{1^k}(w),w) =$

$\alpha(\rho_{1^k}(w),w) = k^{-1} + (w-1)\rho_{1^k}(w)$. Since $\bar{B}_{1^k}(z,w) = zC_{1^k}(z,w)^k + z(w-1)$, $C_{1^k}(z,w)$ has a corresponding representation

$$C_{1^k}(z,w) = a(z,w) + b(z,w)\left[1 - \frac{z}{\rho_{1^k}(w)}\right]^{1/2} \tag{16}$$

where $a(z,w), b(z,w)$ are analytic functions around $(z,w) = (\rho_{1^k}, 1)$, $b(\rho_{1^k}(w),w) \neq 0$, $C_{1^k}(\rho_{1^k}(w),w) = a(\rho_{1^k}(w),w) = (k\rho_{1^k}(w))^{-1/k}$. Analogous to $\bar{B}_{1^k}(z,w)$, for $m \geq 2$ and $\mu \neq 1^k$, $C_\mu(z,w)$ and $C_{1^k}(z^m,w^m)$ are analytic if (z,w) is close to $(\rho_{1^k},1)$. Consequently for $z \in U_{\rho_{1^k}}(w)$ and $|w-1| \leq \epsilon$, $U(z,w)$ has the expansion

$$U(z,w) = -\frac{kz}{(k+1)!}C_{1^k}(z,w)^{k+1} + \frac{1}{k!}C_{1^k}(z,w) + H_1(z,w)$$

where $H_1(z,w)$ is analytic function around $(z,w) = (\rho_{1^k},1)$. By substituting $C_{1^k}(z,w)$ by its singular expansion in eq. (16), $U(z,w)$ can be expanded locally around $z = \rho_{1^k}$ and $w = 1$, i.e.,

$$U(z,w) = g(z,w) + \left[-\frac{za^k(z,w)}{(k-1)!} + \frac{1}{k!} + O(\rho_{1^k}(w) - z)\right] b(z,w)\left[1 - \frac{z}{\rho_{1^k}(w)}\right]^{1/2},$$

Since $-\rho_{1^k}(w)\frac{a^k(\rho_{1^k}(w),w)}{(k-1)!} + \frac{1}{k!} = 0$, together with the fact $a(z,w) = a(\rho_{1^k}(w),w) + O(\rho_{1^k}(w) - z)$ and $b(\rho_{1^k}(w),w) \neq 0$, we can conclude

$$U(z,w) = g(z,w) + f(z,w)\left[1 - \frac{z}{\rho_{1^k}(w)}\right]^{3/2} \tag{17}$$

where $f(\rho_{1^k},1) \neq 0$ from Section 3 and therefore $f(\rho_{1^k}(w),w) \neq 0$. By applying [5, Theorem 2.25] to eq. (17), there is a central limit theorem for $(X_n - \mathbb{E}(X_n))/\sqrt{n}$. More precisely there exist μ and σ^2 with $\mathbb{E}(X_n) = \mu\, n + O(1)$ and $\mathbb{V}\mathrm{ar}(X_n) = \sigma^2\, n + O(1)$ where $\mu = -\rho'_{1^k}(1)/\rho_{1^k}(1)$ and $\sigma^2 = -\rho''_{1^k}(1)/\rho_{1^k}(1) + \mu + \mu^2$. By applying Lemma 4 of [7] it actually follows that $\sigma^2 > 0$ and X_n satisfies a central limit theorem as stated. □

5 The degree distribution of unlabeled k-trees

We again refer to the unlabeled k-coding trees and consider here the degree distribution. Clearly every black node in the k-coding tree has degree $k+1$. So we concentrate on the degree distribution of colored nodes. First of all we change the statistics slightly by measuring the size according to the number of colored nodes. Formally the variable x (instead of z) takes care of the number of colored nodes. Now let $U(x)$ be the generating function for unlabeled color-orbits of unlabeled k-coding trees, let $B(x)$ be the generating function for color-orbits of black-rooted trees, let $C(x)$ be the generating function for color-orbits of color-rooted trees and let $E(x)$ be the generating function for color-orbits of unlabeled k-coding trees rooted at an edge, then we have similarly to the above:

$$U(x) = B(x) + C(x) - E(x) \tag{18}$$

where

$$\begin{aligned} B(x) &= \sum_{\lambda\vdash k+1} \frac{B_\lambda(x)}{z_\lambda} & B_\lambda(x) &= \prod_i C_{\lambda^i}(x^i) \\ C(x) &= \sum_{\mu\vdash k} \frac{C_\mu(x)}{z_\mu} & \bar{B}_\mu(x) &= \prod_i C_{\mu^i}(x^i) \\ E(x) &= \sum_{\mu\vdash k} \frac{\bar{B}_\mu(x)C_\mu(x)}{z_\mu} & C_\mu(x) &= x\exp\left[\sum_{m=1}^{\infty} \frac{\bar{B}_{\mu^m}(x^m)}{m}\right]. \end{aligned}$$

In completely the same way as in Section 3, we can find the singular expansion of $U(x)$ given by

$$U(x) = U(\gamma_{1^k}) + c_1(\gamma_{1^k} - x) + r(\gamma_{1^k} - x)^{3/2} + c_2(\gamma_{1^k} - x)^2 + O((\gamma_{1^k} - x)^{5/2})$$

for some positive constant r and some constants c_1, c_2. Furthermore $\bar{B}_{1^k}(\gamma_{1^k}) = 1/k$, it follows that $\rho_{1^k} = \gamma_{1^k}$.

Now we give each colored node of degree d_i with weight u_i. Let $\mathbf{u} = (u_1, \cdots, u_M)$, $\mathbf{m} = (m_1, \cdots, m_M)$ where $m_i \geq 0$ and $\mathbf{d} = (d_1, \cdots, d_M)$ where $d_i > 0$, then the coefficient of $x^n\mathbf{u}^{\mathbf{m}}$ in the generating function $U^{(\mathbf{d})}(x, \mathbf{u})$ is the number of of unlabeled k-trees that there are m_i colored nodes out of n total colored nodes having degree d_i. Then we have

Theorem 3 *Let $Y_{n,\mathbf{d}} = (Y^{(1)}_{n,d_1}, \cdots, Y^{(M)}_{n,d_M})$ be the random vector of the number of colored nodes in an unlabeled k-tree that have degrees $(d_1, \cdots, d_M)$, that is,*

$$\mathbb{P}(Y_{n,\mathbf{d}} = \mathbf{m}) = \frac{[x^n\mathbf{u}^{\mathbf{m}}]U^{(\mathbf{d})}(x, \mathbf{u})}{[x^n]U^{(\mathbf{d})}(x, \mathbf{1})}.$$

Then there exists an M-dimensional vector μ and an $M \times M$ positive semidefinite matrix Σ such that $\mathbb{E}(Y_{n,\mathbf{d}}) = \mu\, n + O(1)$ and $\mathbb{C}\mathrm{ov}(Y_{n,\mathbf{d}}) = \Sigma\, n + O(1)$. Furthermore $Y_{n,\mathbf{d}}$ satisfies a central limit theorem of the form

$$\frac{Y_{n,\mathbf{d}} - \mathbb{E}(Y_{n,\mathbf{d}})}{\sqrt{n}} \longrightarrow N(0, \Sigma).$$

Proof: Let $C^{(\mathbf{d})}(x, \mathbf{u})$ be the generating function for color-orbits of colored-rooted trees that has each colored node of degree d_i weighted by u_i. Let $P^{(\mathbf{d})}(x, \mathbf{u})$ be the generating function for the trees whose root is only connected with the root of a color-orbit of colored node-rooted tree, so that $C^{(\mathbf{d})}(x, \mathbf{1}) = P^{(\mathbf{d})}(x, \mathbf{1})$. Let $B^{(\mathbf{d})}(x, \mathbf{u})$ be the generating function for color-orbits of black-rooted trees that has each node of degree d_i weighted by u_i. $E^{(\mathbf{d})}(x, \mathbf{u})$ be the generating function for color-orbits of unlabeled k-coding trees rooted at an edge that has each node of degree d_i weighted by u_i. Here we introduce $P^{(\mathbf{d})}(x, \mathbf{u})$ to distinguish the case that the colored root has degree d_i for some $1 \leq i \leq M$. Let $Z(\mathfrak{S}_p, \bar{B}^{(\mathbf{d})}_\mu(x, \mathbf{u}))$ represent the generating function for $\mathfrak{S}_p$-orbits of objects counted by $\bar{B}^{(\mathbf{d})}_\mu(x, \mathbf{u})$:

$$\begin{aligned} Z(\mathfrak{S}_p, \bar{B}^{(\mathbf{d})}_\mu(x, \mathbf{u})) &= Z(\mathfrak{S}_p, \bar{B}^{(\mathbf{d})}_\mu(x, \mathbf{u}), \bar{B}^{(\mathbf{d})}_{\mu^2}(x^2, \mathbf{u}^2), \cdots, \bar{B}^{(\mathbf{d})}_{\mu^p}(x^p, \mathbf{u}^p)) \\ &= \sum_{\lambda\vdash p} \frac{1}{z_\lambda} \bar{B}^{(\mathbf{d})}_\mu(x, \mathbf{u})^{\lambda_1} \bar{B}^{(\mathbf{d})}_{\mu^2}(x^2, \mathbf{u}^2)^{\lambda_2} \cdots \bar{B}^{(\mathbf{d})}_{\mu^p}(x^p, \mathbf{u}^p)^{\lambda_p} \end{aligned}$$

where $\lambda = (1^{\lambda_1}, \cdots, p^{\lambda_p})$. Then the generating function $U(x, \mathbf{u})$ for unlabeled k-trees with colored nodes of degree $\mathbf{d}$ is given by

$$U^{(\mathbf{d})}(x, \mathbf{u}) = B^{(\mathbf{d})}(x, \mathbf{u}) + C^{(\mathbf{d})}(x, \mathbf{u}) - E^{(\mathbf{d})}(x, \mathbf{u}) \tag{19}$$

where

$$B^{(\mathbf{d})}(x, \mathbf{u}) = \sum_{\lambda \vdash k+1} \frac{B_\lambda^{(\mathbf{d})}(x, \mathbf{u})}{z_\lambda},\ C^{(\mathbf{d})}(x, \mathbf{u}) = \sum_{\mu \vdash k} \frac{C_\mu^{(\mathbf{d})}(x, \mathbf{u})}{z_\mu},\ B_\lambda^{(\mathbf{d})}(x, \mathbf{u}) = \prod_i P_{\lambda^i}^{(\mathbf{d})}(x^i, \mathbf{u}^i)$$

$$E^{(\mathbf{d})}(x, \mathbf{u}) = \sum_{\mu \vdash k} \frac{\bar{B}_\mu^{(\mathbf{d})}(x, \mathbf{u}) P_\mu^{(\mathbf{d})}(x, \mathbf{u})}{z_\mu},\ \bar{B}_\mu^{(\mathbf{d})}(x, \mathbf{u}) = \prod_i P_{\mu^i}^{(\mathbf{d})}(x^i, \mathbf{u}^i)$$

$$C_\mu^{(\mathbf{d})}(x, \mathbf{u}) = x \exp\left[\sum_{m=1}^{\infty} \frac{\bar{B}_{\mu^m}^{(\mathbf{d})}(x^m, \mathbf{u}^m)}{m}\right] + \sum_{j=1}^{M} x(u_j - 1) Z(\mathfrak{S}_{d_j}, \bar{B}_\mu^{(\mathbf{d})}(x, \mathbf{u})) \tag{20}$$

$$P_\mu^{(\mathbf{d})}(x, \mathbf{u}) = x \exp\left[\sum_{m=1}^{\infty} \frac{\bar{B}_{\mu^m}^{(\mathbf{d})}(x^m, \mathbf{u}^m)}{m}\right] + \sum_{j=1}^{M} x(u_j - 1) Z(\mathfrak{S}_{d_j-1}, \bar{B}_\mu^{(\mathbf{d})}(x, \mathbf{u})). \tag{21}$$

The dominant singularity for $\bar{B}_{1^k}(x, \mathbf{1})$ is ρ_{1^k}. As before, for $\mu \neq 1^k$, $\bar{B}_\mu(x, \mathbf{u})$ and for $m \geq 2$, $\bar{B}_{1^k}(x^m, \mathbf{u}^m)$ are analytic if $(x, \mathbf{u})$ is close to $(\rho_{1^k}, \mathbf{1})$. Next we consider

$$\begin{aligned} S(x, y, \mathbf{u}) &= \Bigg(x e^y \exp\left(\sum_{m=2}^{\infty} \frac{\bar{B}_{1^k}^{(\mathbf{d})}(x^m, \mathbf{u}^m)}{m}\right) \\ &+ \sum_{j=1}^{M} x(u_j - 1) Z(\mathfrak{S}_{d_j-1}, y, \bar{B}_{1^k}^{(\mathbf{d})}(x^2, \mathbf{u}^2), \cdots, \bar{B}_{1^k}^{(\mathbf{d})}(x^{d_j-1}, \mathbf{u}^{d_j-1}))\Bigg)^k. \end{aligned}$$

Since $S(0, y, \mathbf{u}) \equiv 0$, $S(x, 0, \mathbf{u}) \not\equiv 0$ and all coefficients of $S(x, y, \mathbf{1})$ are real and positive, then $y(x, \mathbf{u}) = \bar{B}_{1^k}^{(\mathbf{d})}(x, \mathbf{u})$ is the unique solution of the functional equation $S(x, y, \mathbf{u}) = y$. Furthermore, $(x, y) = (\rho_{1^k}, 1/k)$ is the only solution of $S(x, y, \mathbf{1}) = 0$ and $S_y(x, y, \mathbf{1}) = 1$ with $S_x(\rho_{1^k}, 1/k, \mathbf{1}) \neq 0$, $S_{yy}(\rho_{1^k}, 1/k, \mathbf{1}) \neq 0$. Consequently, $\bar{B}_{1^k}^{(\mathbf{d})}(x, \mathbf{u})$ can be represented as

$$\bar{B}_{1^k}^{(\mathbf{d})}(x, \mathbf{u}) = g(x, \mathbf{u}) - h(x, \mathbf{u}) \left[1 - \frac{x}{\rho_{1^k}(\mathbf{u})}\right]^{1/2} \tag{22}$$

which holds locally around $(x, \mathbf{u}) = (\rho_{1^k}, \mathbf{1})$. In view of $\bar{B}_{1^k}^{(\mathbf{d})}(x, \mathbf{u}) = P_{1^k}^{(\mathbf{d})}(x, \mathbf{u})^k$, $P_{1^k}^{(\mathbf{d})}(x, \mathbf{u})$ also has expansion of square root type, i.e.,

$$P_{1^k}^{(\mathbf{d})}(x, \mathbf{u}) = s(x, \mathbf{u}) - t(x, \mathbf{u}) \left[1 - \frac{x}{\rho_{1^k}(\mathbf{u})}\right]^{1/2} \tag{23}$$

where $t(\rho_{1^k}(\mathbf{u}), \mathbf{u}) \neq 0$. From eq. (20) and eq. (21), we have

$$C_\mu^{(\mathbf{d})}(x, \mathbf{u}) = P_\mu^{(\mathbf{d})}(x, \mathbf{u}) + \sum_{j=1}^{M} x(u_j - 1) \left[Z(\mathfrak{S}_{d_j}, \bar{B}_\mu^{(\mathbf{d})}(x, \mathbf{u})) - Z(\mathfrak{S}_{d_j-1}, \bar{B}_\mu^{(\mathbf{d})}(x, \mathbf{u}))\right]$$

based on which we shall next compute the dominant term in the singular expansion of $U(x,\mathbf{u})$. For simplicity we will omit variables $(x,\mathbf{u})$ and degree $\mathbf{d}$.

$$\begin{aligned} U(x,\mathbf{u}) &= \frac{P_{1^k}^{k+1}}{(k+1)!} + \frac{C_{1^k}}{k!} - \frac{P_{1^k}^{k+1}}{k!} + M_1 = \frac{P_{1^k}^{k+1}}{(k+1)!} + \frac{1}{k!}(1-P_{1^k}^k)P_{1^k} + \frac{1}{k!}(C_{1^k} - P_{1^k}) + M_1 \\ &= -\frac{kP_{1^k}^{k+1}}{(k+1)!} + \frac{P_{1^k}}{k!} + \frac{1}{k!}\sum_{j=1}^{M} x(u_j-1)\left[Z(\mathfrak{S}_{d_j},\bar{B}_{1^k}) - Z(\mathfrak{S}_{d_j-1},\bar{B}_{1^k})\right] + M_1. \end{aligned}$$

where M_1 is an analytic function around $(x,\mathbf{u}) = (\rho_{1^k},\mathbf{1})$. It is now convenient to write $U(x,\mathbf{u}) = f(x,\mathbf{u}) + h(x,\mathbf{u})\left[1 - \frac{x}{\rho_{1^k}(\mathbf{u})}\right]^{1/2}$. Then by substituting P_{1^k}, $\bar{B}_{1^k}$ with its representation in eq. (23) and eq. (22), we obtain

$$h(x,\mathbf{u}) = \frac{s^k\,t}{(k-1)!} - \frac{t}{k!} + \frac{h}{k!}\sum_{j=1}^{M} x(u_j-1)\left[Z'(\mathfrak{S}_{d_j-1},g,X_2,\cdots,X_{d_j-1}) - Z'(\mathfrak{S}_{d_j},g,X_2,\cdots,X_{d_j})\right]$$

where X_i are analytic functions around $(x,\mathbf{u}) = (\rho_{1^k},\mathbf{1})$ and Z' is the derivative w.r.t. the first variable of $Z(\mathfrak{S}_k,x_1,\cdots,x_k)$, namely $Z'(\mathfrak{S}_k,x_1,\cdots,x_k) = Z(\mathfrak{S}_{k-1},x_1,\cdots,x_{k-1})$. Furthermore, by replacing s,t by $g = s^k$ and $h = ks^{k-1}t$, we can further simplify $h(x,\mathbf{u})$, that is

$$h(x,\mathbf{u}) = \frac{h}{k!}\frac{g-\frac{1}{k}}{g^{1-\frac{1}{k}}} + \frac{h}{k!}\sum_{j=1}^{M} x(u_j-1)\left(Z'(\mathfrak{S}_{d_j-1},g,X_2,\cdots,X_{d_j-1}) - Z'(\mathfrak{S}_{d_j},g,X_2,\cdots,X_{d_j})\right).$$

Now we use the fact that $y = g(\rho_{1^k}(\mathbf{u}),\mathbf{u})$ and $x = \rho_{1^k}(\mathbf{u})$ is the solution of $S(x,y,\mathbf{u}) = y$ and $S_y(x,y,\mathbf{u}) = 1$, which yields

$$\begin{aligned} g(\rho_{1^k}(\mathbf{u}),\mathbf{u}) &= \frac{1}{k} + g(\rho_{1^k}(\mathbf{u}),\mathbf{u})^{\frac{k-1}{k}} \\ &\times \sum_{j=1}^{M} x(u_j-1)\left[Z(\mathfrak{S}_{d_j-1},g,X_2,\cdots,X_{d_j-1}) - Z'(\mathfrak{S}_{d_j-1},g,X_2,\cdots,X_{d_j-1})\right] \end{aligned}$$

and consequently $h(\rho_{1^k}(\mathbf{u}),\mathbf{u}) \equiv 0$ and $U(x,\mathbf{u})$ has a local expansion around $(x,\mathbf{u}) = (\rho_{1^k},\mathbf{1})$ of the form

$$U(x,\mathbf{u}) = w(x,\mathbf{u}) + r(x,\mathbf{u})\left[1 - \frac{x}{\rho_{1^k}(\mathbf{u})}\right]^{3/2}. \tag{24}$$

where $r(\rho_{1^k}(\mathbf{u}),\mathbf{u}) \neq 0$ since $r(\rho_{1^k},\mathbf{1}) = r \neq 0$ and w, r are analytic function around $(x,\mathbf{u}) = (\rho_{1^k},\mathbf{1})$. Thus a central limit theorem follows. More precisely by setting $A(\mathbf{u}) = \log\rho_{1^k}(\mathbf{1}) - \log\rho_{1^k}(\mathbf{u})$, $\mu = (A_{u_j}(\mathbf{1}))_{1\le j\le M}$ and $\Sigma = (A_{u_iu_j}(\mathbf{1}) + \delta_{i,j}A_{u_j}(\mathbf{1}))_{1\le j\le M}$ then $\mathbb{E}(Y_{n,\mathbf{d}}) = \mu\, n + O(1)$ and $\mathbb{C}\mathrm{ov}(Y_{n,\mathbf{d}}) = \Sigma\, n + O(1)$.

□

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