THE DEGREE DISTRIBUTION OF RANDOM PLANAR GRAPHS

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Random Planar Graphs

"History"

$\mathcal{R}_n$ ... **labelled planar graphs** with n vertices with uniform distribution

X_n ... number of **edges** is a random planar graph with n vertices

Denise, Vasconcellos, Welsh (1996)

$$\boxed{\mathbb{P}\{X_n > \frac{3}{2}n\} \to 1, \quad \mathbb{P}\{X_n < \frac{5}{2}n\} \to 1}.$$

(Note that $0 \le e \le 3n$ for all planar graphs.)

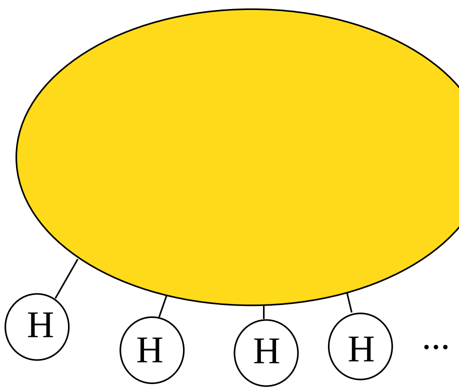
Random Planar Graphs

"History"

McDiarmid, Steger, Welsh (2005)

$$\boxed{\mathbb{P}\{H \text{ appears in } \mathcal{R}_n \text{ at least } \alpha n \text{ times}\} \to 1}$$

H ... any fixed planar graph, $\alpha > 0$ sufficiently small.



Random Planar Graphs

Consequences:

$$\boxed{\mathbb{P}\{\text{There are} \geq \alpha n \text{ vertices of degree } k\} \to 1}$$

$k > 0$ a given integer, $\alpha > 0$ sufficiently small.

$$\boxed{\mathbb{P}\{\text{There are} \geq C^n \text{ automorphisms}\} \to 1}$$

for some $C > 1$.

Random Planar Graphs

Further Results:

$$\boxed{\mathbb{P}\{\mathcal{R}_n \text{ is connected}\} \ge \gamma > 0}$$

[McDiarmid+Reed]

$$\boxed{\mathbb{E}\,\Delta_n = \Theta(\log n)}$$

Δ_n ... maximum degree in $\mathcal{R}_n$

Random Planar Graphs

The number of planar graphs

[Bender, Gao, Wormald (2002)]

b_n ... number of **2-connected** labelled planar graphs

$$\boxed{b_n \sim c \cdot n^{-\frac{7}{2}} \gamma_2^n \, n!}, \quad \gamma_2 = 26.18...$$

[Gimenez+Noy (2005)]

g_n .... number of all labelled planar graphs

$$\boxed{g_n \sim c \cdot n^{-\frac{7}{2}} \gamma^n \, n!}, \quad \gamma = 27.22...$$

Random Planar Graphs

Precise distributional results

[Gimenez+Noy (2005)]

- X_n satisfies a **central limit theorem**:

$$\mathbb{E}\, X_n \sim 2.21... \cdot n, \quad \mathbb{V}\, X_n \sim c \cdot n.$$

$$\mathbb{P}\{|X_n - 2.21... \cdot n| > \varepsilon n\} \le e^{-\alpha(\varepsilon)\cdot n}$$

- **Connectedness:**

$$\mathbb{P}\{\mathcal{R}_n \text{ is connected}\} \to e^{-\nu} = 0.96...$$

$$\text{number of components of } \mathcal{R}_n =: C_n \to 1 + Po(\nu).$$

Random Planar Graphs

Degree Distribution

Theorem [D.+Giménez+Noy]

Let $p_{n,k}$ be the probability that a random node in a random planar graph $\mathcal{R}_n$ has degree k. Then the limit

$$\boxed{p_k := \lim_{n\to\infty} p_{n,k}}$$

exists. The probability generating function

$$\boxed{p(w) = \sum_{k\ge 1} p_k w^k}$$

can be explicitly computed; $\boxed{p_k \sim c\, k^{-\frac{1}{2}} q^k}$ for some $c > 0$ and $0 < q < 1$.

p_1	p_2	p_3	p_4	p_5	p_6
0.0367284	0.1625794	0.2354360	0.1867737	0.1295023	0.0861805

Random Planar Graphs

- Implicit equation for $D_0(y,w)$:

$$1+D_0=(1+y\boxed{w})\exp\left(\frac{\sqrt{S}(D_0(t-1)+t)}{4(3t+1)(D_0+1)}-\right.$$
$$\left.-\frac{D_0^2(t^4-12t^2+20t-9)+D_0(2t^4+6t^3-6t^2+10t-12)+t^4+6t^3+9t^2}{4(t+3)(D_0+1)(3t+1)}\right),$$

where $t=t(y)$ satisfies $y+1=\frac{1+2t}{(1+3t)(1-t)}\exp\left(-\frac{1}{2}\frac{t^2(1-t)(18+36t+5t^2)}{(3+t)(1+2t)(1+3t)^2}\right).$
and $S=(D_0(t-1)+t)(D_0(t-1)^3+t(t+3)^2)$.

- Explicit expressions in terms of $D_0(y,w)$ (**SEVERAL PAGES !!!!**):

$$B_0(y,w),\ B_2(y,w),\ B_3(y,w)$$

- Explict expression for $p(w)$:

$$\boxed{p(w)=-e^{B_0(1,w)-B_0(1,1)}B_2(1,w)+e^{B_0(1,w)-B_0(1,1)}\frac{1+B_2(1,1)}{B_3(1,1)}B_3(1,w)}$$

Random Planar Graphs

Conjecture for maximum degree Δ_n

$$\frac{\Delta_n}{\log n} \to \frac{1}{\log(1/q)} \qquad \text{in probability}$$

and

$$\mathbb{E}\,\Delta_n \sim \frac{\log n}{\log(1/q)}$$

where $q = 0.6734506...$ appear in the asymptotics of $p_k \sim c\,k^{-\frac{1}{2}}q^k$;
$1/\log(1/q) = 2.529464248...$

Degree Distribution

$X_n^{(k)}$ ... **number of vertices of degree** k in a random labelled planar graph of size n

$p_{n,k}$ ... **probability that a random vertex** in a random labelled planar graph of size n **has degree** k

$\widehat{p}_{n,k}$ ... **probability that the root vertex** in a random labelled vertex rooted planar graph of size n **has degree** k

- $\boxed{p_{n,k} = \widehat{p}_{n,k}}$

- $\boxed{\mathbb{E}\, X_n^{(k)} = n\, p_{n,k}}$

Degree Distribution

Generating functions for counting planar graphs

$b_{n,m}$ ... number of **2-connected labelled planar** graphs with n vertices and m edges, $b_n = \sum_m b_{n,m}$

$$B(x,y) = \sum_{n,m} b_{n,m} \frac{x^n}{n!} y^m$$

$c_{n,m}$ ... number of **connected labelled planar** graphs with n vertices and m edges, $c_n = \sum_m c_{n,m}$

$$C(x,y) = \sum_{n,m} c_{n,m} \frac{x^n}{n!} y^m$$

$g_{n,m}$ ... number of **all labelled planar** graphs with n vertices and m edges, $g_n = \sum_m g_{n,m}$

$$G(x,y) = \sum_{n,m} g_{n,m} \frac{x^n}{n!} y^m$$

Degree Distribution

Generating functions for counting planar graphs

$$
\begin{aligned}
G(x,y) &= \exp\left(C(x,y)\right), \\
\frac{\partial C(x,y)}{\partial x} &= \exp\left(\frac{\partial B}{\partial x}\left(x\frac{\partial C(x,y)}{\partial x}, y\right)\right), \\
\frac{\partial B(x,y)}{\partial y} &= \frac{x^2}{2}\,\frac{1+D(x,y)}{1+y}, \\
\frac{M(x,D)}{2x^2D} &= \log\left(\frac{1+D}{1+y}\right) - \frac{xD^2}{1+xD}, \\
M(x,y) &= x^2y^2\left(\frac{1}{1+xy} + \frac{1}{1+y} - 1 - \frac{(1+U)^2(1+V)^2}{(1+U+V)^3}\right), \\
U &= xy(1+V)^2, \\
V &= y(1+U)^2.
\end{aligned}
$$

Degree Distribution

Asymptotic enumeration of planar graphs

$$b_n = b \cdot \rho_1^{-n} \, n^{-\frac{7}{2}} n! \left(1 + O\left(\frac{1}{n}\right)\right),$$
$$c_n = c \cdot \rho_2^{-n} \, n^{-\frac{7}{2}} n! \left(1 + O\left(\frac{1}{n}\right)\right),$$
$$g_n = g \cdot \rho_2^{-n} \, n^{-\frac{7}{2}} n! \left(1 + O\left(\frac{1}{n}\right)\right)$$

$$\rho_1 = 0.03819...,$$
$$\rho_2 = 0.03672841...,$$
$$b = 0.3704247487... \cdot 10^{-5},$$
$$c = 0.4104361100... \cdot 10^{-5},$$
$$g = 0.4260938569... \cdot 10^{-5}$$

Degree Distribution

Generating functions for the degree distribution of planar graphs

$C^\bullet = \frac{\partial C}{\partial x}$ ... GF, where one vertex is marked but not counted

w ... additional variable that *counts* the **degree of the marked vertex**

Generating functions:

$G^\bullet(x,y,w)$	**all rooted** planar graphs
$C^\bullet(x,y,w)$	**connected rooted** planar graphs
$B^\bullet(x,y,w)$	**2-connected rooted** planar graphs
$T^\bullet(x,y,w)$	**3-connected rooted** planar graphs

Note that $G^\bullet(x,y,1) = \dfrac{\partial G}{\partial x}(x,y)$ etc.

Degree Distribution

more precisely ....

$g^{\bullet}_{n,m,k}$ ... number of **vertex rooted labelled planar graphs** with $n+1$ vertices, m edges, where the (uncounted and unlabelled) **root vertex has degree** k.

$$G^{\bullet}(x,y,w) = \sum_{n,m,k} g^{\bullet}_{n,m,k} \frac{x^n}{n!} y^m w^k$$

$$\sum_k g^{\bullet}_{n-1,m,k} = n\, g_{n,m}, \qquad \boxed{p_{n,k} = \frac{g^{\bullet}_{n-1,m,k}}{n\, g_{n,m}}}$$

$$\sum_{k\ge 1} p_{n,k} w^k = \frac{1}{n\, g_{n,m}} \sum_{k\ge 1} g^{\bullet}_{n-1,m,k} w^k = \frac{[x^{n-1}]G^{\bullet}(x,1,w)}{[x^{n-1}]G^{\bullet}(x,1,1)}$$

$$\Longrightarrow \quad \boxed{p(w) = \sum_{k\ge 1} p_n w^k = \lim_{n\to\infty} \frac{[x^{n-1}]G^{\bullet}(x,1,w)}{[x^{n-1}]G^{\bullet}(x,1,1)}}$$

Degree Distribution

$$
\begin{aligned}
G^\bullet(x,y,w) &= \exp\left(C(x,y,1)\right) C^\bullet(x,y,w),\\
C^\bullet(x,y,w) &= \exp\left(B^\bullet\left(xC^\bullet(x,y,1),y,w\right)\right),\\
w\frac{\partial B^\bullet(x,y,w)}{\partial w} &= xyw\exp\left(S(x,y,w)+\frac{1}{x^2D(x,y,w)}T^\bullet\left(x,D(x,y,1),\frac{D(x,y,w)}{D(x,y,1)}\right)\right)\\
D(x,y,w) &= (1+yw)\exp\left(S(x,y,w)+\frac{1}{x^2D(x,y,w)}\times\right.\\
&\qquad \left.\times\, T^\bullet\left(x,D(x,y,1),\frac{D(x,y,w)}{D(x,y,1)}\right)\right)-1\\
S(x,y,w) &= xD(x,y,1)\left(D(x,y,w)-S(x,y,w)\right),\\
T^\bullet(x,y,w) &= \frac{x^2y^2w^2}{2}\left(\frac{1}{1+wy}+\frac{1}{1+xy}-1-\right.\\
&\qquad \left. -\frac{(u+1)^2\left(-w_1(u,v,w)+(u-w+1)\sqrt{w_2(u,v,w)}\right)}{2w(vw+u^2+2u+1)(1+u+v)^3}\right),\\
u(x,y) &= xy(1+v(x,y))^2,\quad v(x,y)=y(1+u(x,y))^2.
\end{aligned}
$$

Degree Distribution

with polynomials $w_1 = w_1(u, v, w)$ and $w_2 = w_2(u, v, w)$ given by

$$\begin{aligned} w_1 = & -uvw^2 + w(1 + 4v + 3uv^2 + 5v^2 + u^2 + 2u + 2v^3 + 3u^2v + 7uv) \\ & + (u + 1)^2(u + 2v + 1 + v^2), \end{aligned}$$

$$\begin{aligned} w_2 = & u^2v^2w^2 - 2wuv(2u^2v + 6uv + 2v^3 + 3uv^2 + 5v^2 + u^2 + 2u + 4v + 1) \\ & + (u + 1)^2(u + 2v + 1 + v^2)^2. \end{aligned}$$

Planar Maps vs. Planar Graphs

Whitney's Theorem

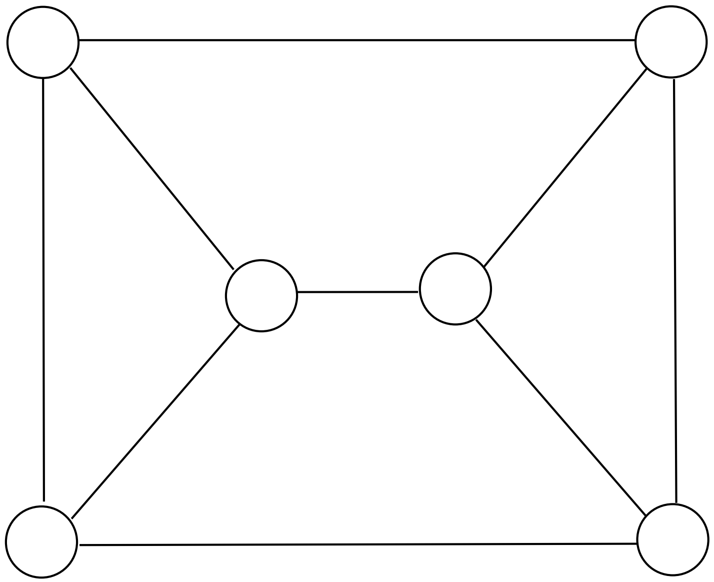
Every 3-connected planar graph has a unique embedding into the plane.

$\Longrightarrow$ The counting problem of **rooted 3-connected planar maps** is equivalent to the counting problem of **rooted (labelled) 3-connected planar graphs** (despite of a factor $(n-1)!$)

Furthermore, the counting problem of **rooted 3-connected planar maps** is equivalent to the counting problem of **rooted simple quadrangulations**.

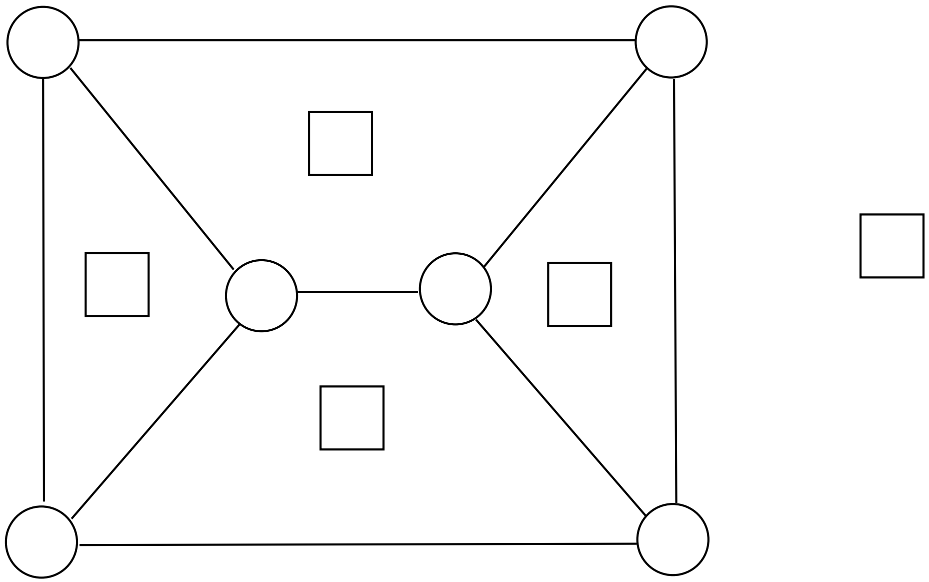
3-Connected Maps

3-connected map – simple quadrangulations



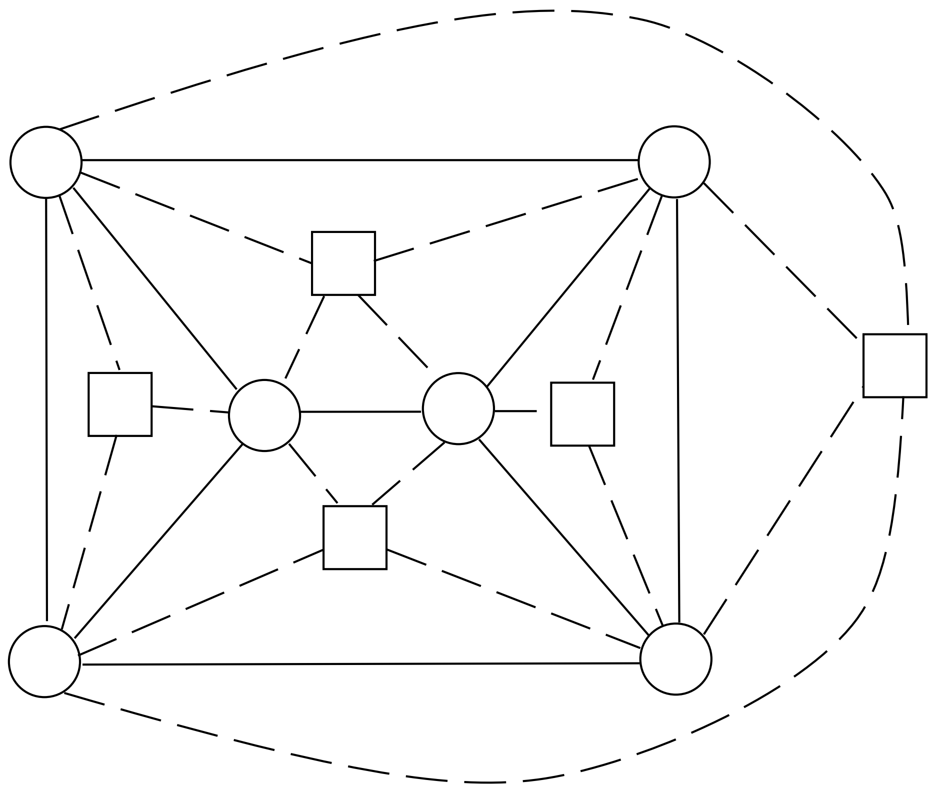
3-Connected Maps

3-connected map – simple quadrangulations



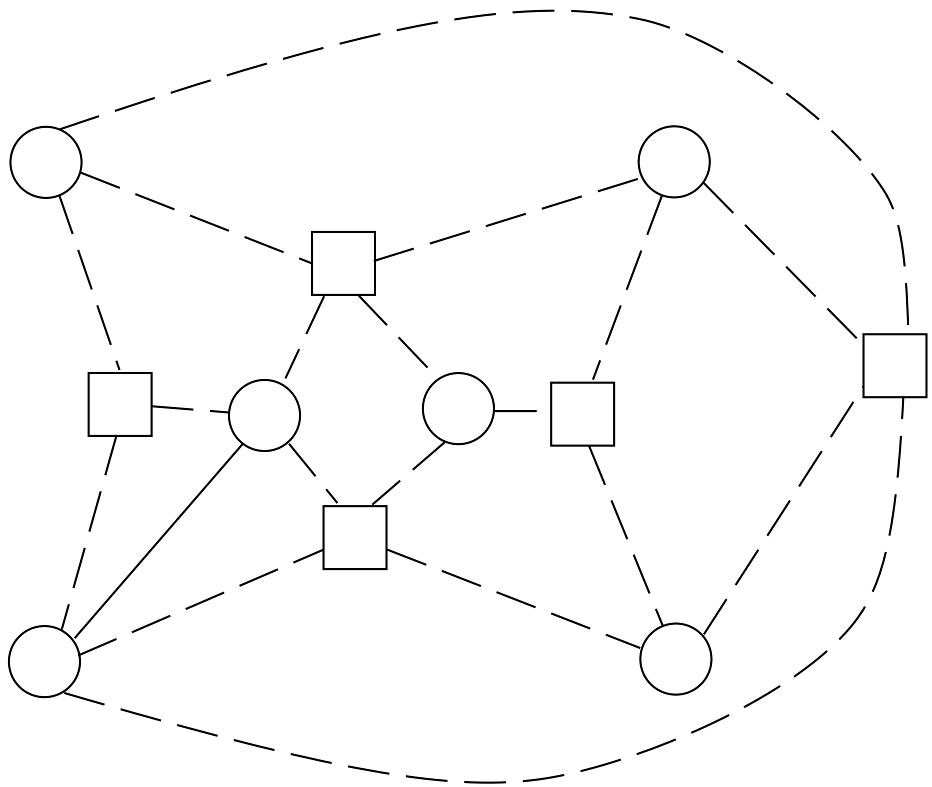
3-Connected Maps

3-connected map – simple quadrangulations



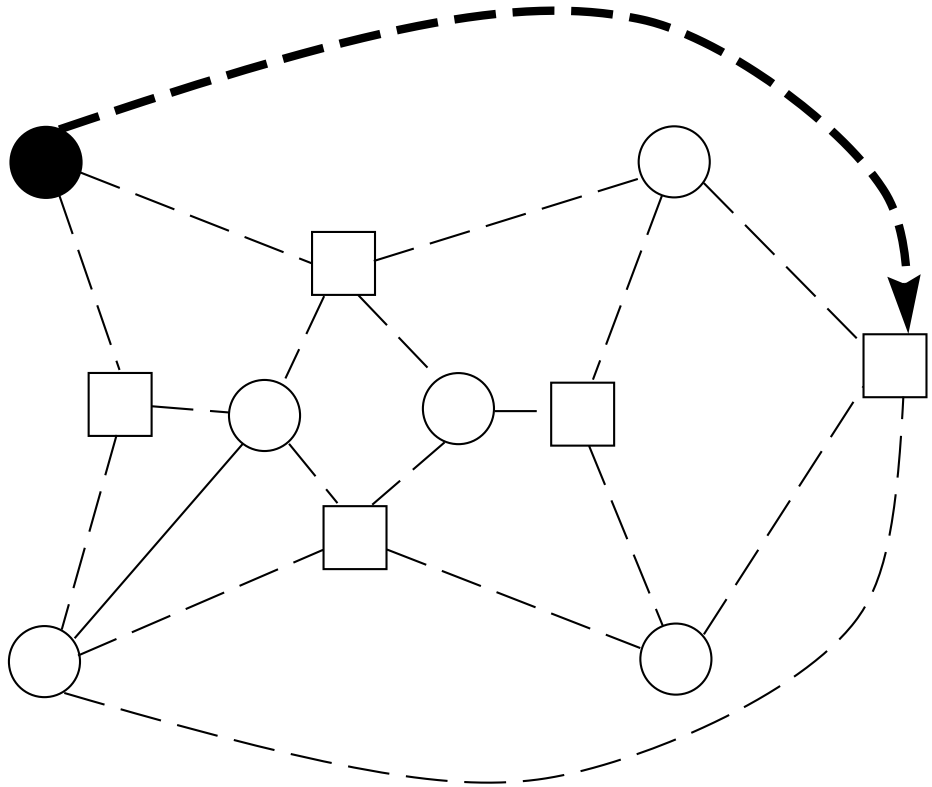
3-Connected Maps

3-connected map – simple quadrangulations



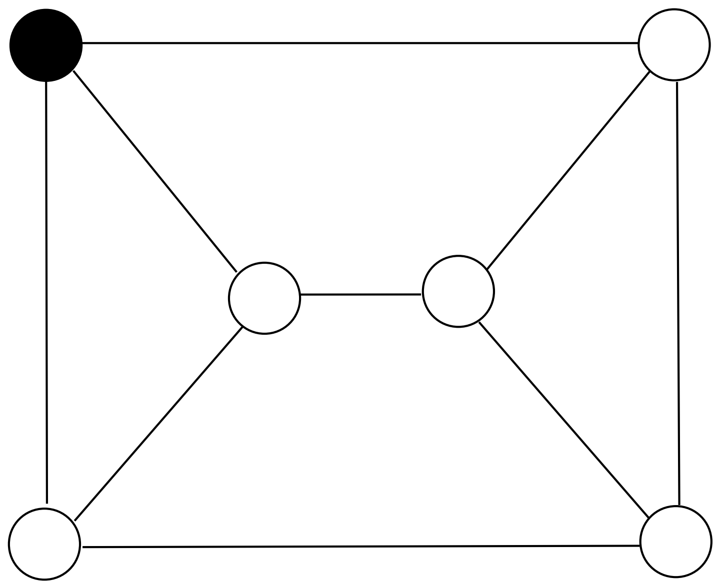
3-Connected Maps

3-connected map – simple quadrangulations



3-Connected Maps

3-connected map – simple quadrangulations



3-Connected Maps

q_{ijk} ... number of simple quadrangulations with
$i+1$ vertices of type 1 ($\circ$),
$j+1$ vertices of type 2 ($\Box$), and with
root vertex of degree $k+1$

$$Q(x,y,w) = \sum_{i,j,k} q_{i,j,k} \cdot x^i \, y^j \, w^k$$

Theorem [Mullin+Schellenberg, D+Gimenez+Noy]

$$\boxed{Q(x,y,w) = xyw \left(\frac{1}{1+wy} + \frac{1}{1+x} - 1 \right) - \frac{UV}{(1+U+V)^3} \cdot W(R,S,w)}$$

with ...

3-Connected Maps

with **algebraic function** $U = U(x,y)$, $V = V(x,y)$ given by

$$\boxed{U = x(V+1)^2}, \qquad \boxed{V = y(U+1)^2}$$

and

$$\boxed{W(U,V,w) = \frac{-w_1(U,V,w) + (U-w+1)\sqrt{w_2(U,V,w)}}{2(V+1)^2(Vw+U^2+2U+1)}}$$

with polynomials $w_1 = w_1(U,V,w)$ and $w_2 = w_2(U,V,w)$ given by

$$\begin{aligned} w_1 =& -UVw^2 + w(1+4V+3UV^2+5V^2+U^2+2U+2V^3+3U^2V+7UV) \\ & + (U+1)^2(U+2V+1+V^2), \\ w_2 =& U^2V^2w^2 - 2wUV(2U^2V+6UV+2V^3+3UV^2+5V^2+U^2+2U+4V+1) \\ & + (U+1)^2(U+2V+1+V^2)^2. \end{aligned}$$

3-Connected Planar Graphs

Corollary By Whitney's theorem:

$$T^{\bullet}(x, y, w) = \frac{xw}{2} Q(xy, y, w).$$

2-Connected Planar Graphs

Planar networks

A **network** N is a (multi-)graph with two distinguished vertices, called its poles (usually labelled 0 and ∞) such that the (multi-)graph $\hat{N}$ obtained from N by adding an edge between the poles of N is 2-connected.

Let M be a network and $X = (N_e, e \in E(M))$ a system of networks indexed by the edge-set $E(M)$ of M. Then $N = M(X)$ is called the **superposition** with core M and components N_e and is obtained by replacing all edges $e \in E(M)$ by the corresponding network N_e (and, of course, by identifying the poles of N_e with the end vertices of e accordingly).

A network N is called an h-**network** if it can be represented by $N = M(X)$, where the core M has the property that the graph $\hat{M}$ obtained by adding an edge joining the poles is 3-connected and has at least 4 vertices. Similarly $N = M(X)$ is called a p-**network** if M consists of 2 or more edges that connect the poles, and it is called an s-**network** if M consists of 2 or more edges that connect the poles in series.

2-Connected Planar Graphs

Planar networks

Trakhtenbrot's canonical network decomposition theorem: any network with at least 2 edges belongs to exactly one of the 3 classes of h-, p- or s-networks. Furthermore, any h-network has a unique decomposition of the form $N = M(X)$, and a p-network (or any s-network) can be uniquely decomposed into components which are not themselves p-networks (or s-networks).

2-Connected Planar Graphs

Planar networks

Lwt $D(x, y, w)$ and $S(x, y, w)$, respectively, the GFs of (planar) networks and series networks, with the same meaning for the variables x, y and w:

Then by a variant of [Walsh (1982)]

$$D(x, y, w) = (1 + yw) \exp\left(S(x, y, w) + \frac{1}{x^2 D(x, y, w)} T^\bullet\left(x, E(x, y), \frac{D(x, y, w)}{D(x, y, 1)}\right)\right.$$

$$S(x, y, w) = xD(x, y, 1)\left(D(x, y, w) - S(x, y, w)\right),$$

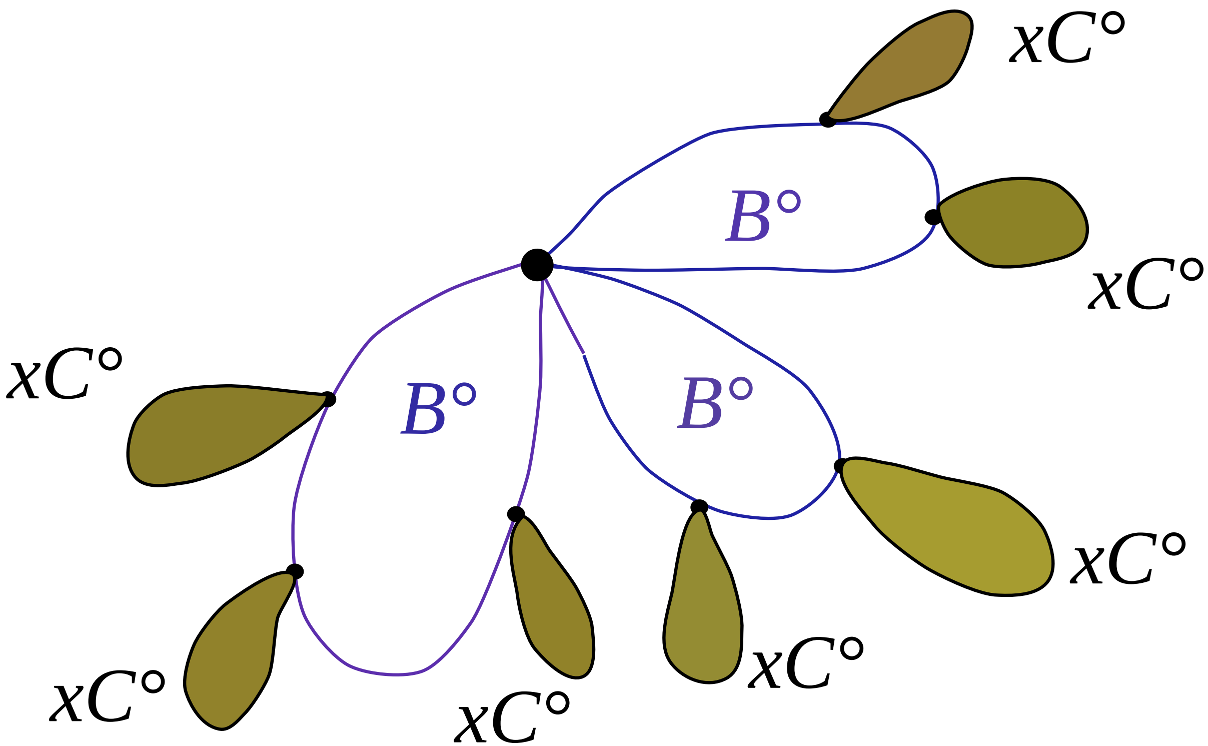
2-Connected Planar Graphs

A planar network with non-adjacent poles is obtained by distinguishing, orienting and then deleting any edge of an arbitrary 2-connected planar graph:

$$w\frac{\partial B^{\bullet}(x,y,w)}{\partial w} = xyw \exp\left(S(x,y,w) + \frac{1}{x^2 D(x,y,w)} T^{\bullet}\left(x, D(x,y,1), \frac{D(x,y,w)}{D(x,y,1)}\right)\right)$$

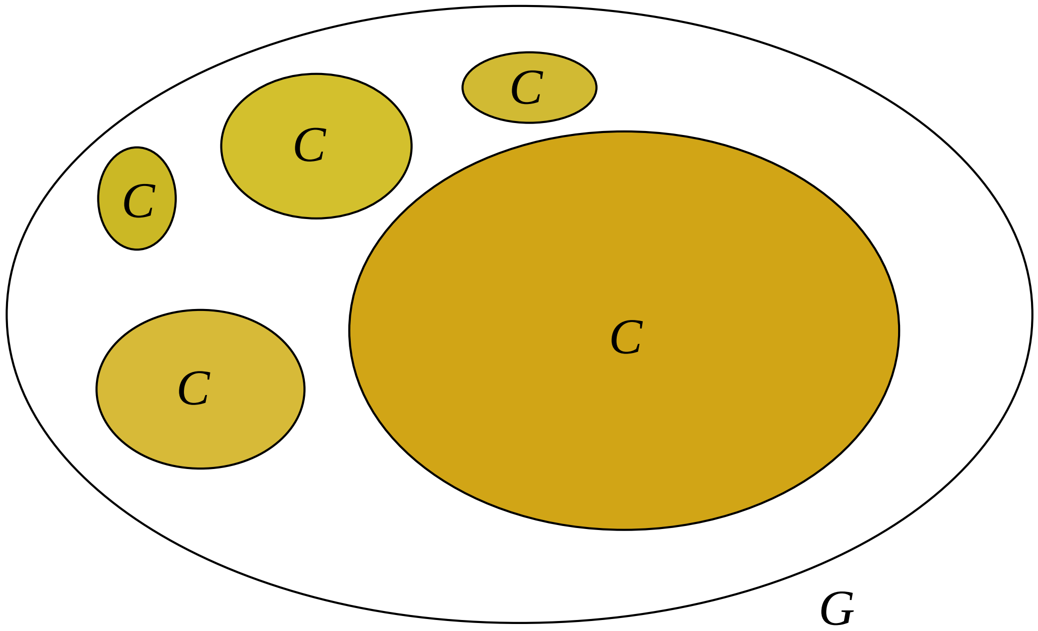
Connected Planar Graphs

$$C^{\bullet}(x, y, w) = e^{B^{\bullet}(xC^{\bullet}(x,y,1),x,w)}$$



All Planar Graphs

$$G^{\bullet}(x, y, w) = \exp\left(C(x, y, 1)\right)\, C^{\bullet}(x, y, w)$$



Asymptotics for Random Planar Graphs

Functional equations

Suppose that $\boxed{A(x,u) = \Phi(x,u,A(x,u))}$, where $\Phi(x,u,a)$ has a power series expansion at $(0,0,0)$ with non-negative coefficients and $\Phi_{aa}(x,u,a) \neq 0$.

Let $x_0 > 0$, $a_0 > 0$ (inside the region of convergence) satisfy the system of equations:

$$\boxed{a_0 = \Phi(x_0, 1, a_0), \quad 1 = \Phi_a(x_0, 1, a_0)}.$$

Then there exists analytic function $g(x,u), h(x,u)$, and $\rho(u)$ such that locally

$$\boxed{A(x,u) = g(x,u) - h(x,u)\sqrt{1 - \frac{x}{\rho(u)}}}.$$

Asymptotics for Random Planar Graphs

Asymptotics for coefficients

$$A(x) = g(x) - h(x)\sqrt{1 - \frac{x}{\rho}} \quad \text{(+ some technical conditions)}$$

$$\Longrightarrow \quad \boxed{[x^n]\, A(x) = \frac{h(\rho)}{2\sqrt{\pi}}\, \rho^{-n}\, n^{-\frac{3}{2}} \left(1 + O\left(\frac{1}{n}\right)\right)}.$$

Similarly:

$$A(x,u) = g(x,u) - h(x,u)\sqrt{1 - \frac{x}{\rho(u)}} \quad \text{(+ some technical conditions)}$$

$$\Longrightarrow \quad [x^n]\, A(x,u) = \frac{h(\rho(u),u)}{2\sqrt{\pi}}\, \rho(u)^{-n}\, n^{-\frac{3}{2}} \left(1 + O\left(\frac{1}{n}\right)\right).$$

Asymptotics for Random Planar Graphs

Asymptotics for coefficients

and

$$A(x) = g(x) + h(x)\left(1 - \frac{x}{\rho}\right)^{\alpha} \quad \text{(+ some technical conditions)}$$

$$\Longrightarrow \quad [x^n]\, A(x) = \frac{h(\rho)}{\Gamma(-\alpha)}\, \rho^{-n}\, n^{-\alpha-1}\left(1 + O\left(\frac{1}{n}\right)\right).$$

Asymptotics for Random Planar Graphs

Singular expansion

$$\begin{aligned} A(x) &= \boxed{g(x) - h(x)\sqrt{1-\frac{x}{\rho}}} \\ &= \left(g_0 + g_1(x-\rho) + g_2(x-\rho)^2 + \cdots\right) \\ &\quad + \left(h_0 + h_1(x-\rho) + h_2(x-\rho)^2 + \cdots\right)\sqrt{1-\frac{x}{\rho}} \\ &= a_0 + a_1\left(1-\frac{x}{\rho}\right)^{\frac{1}{2}} + a_2\left(1-\frac{x}{\rho}\right)^{\frac{2}{2}} + a_3\left(1-\frac{x}{\rho}\right)^{\frac{3}{2}} + \cdots \\ &= \boxed{a_0 + a_1 X + a_2 X^2 + a_3 X^3 + \cdots} \end{aligned}$$

with

$$X = \sqrt{1-\frac{x}{\rho}}.$$

Asymptotics for Random Planar Graphs

Central limit theorem

$$A(x,u) = g(x,u) - h(x,u)\sqrt{1 - \frac{x}{\rho(u)}}.$$

$$\boxed{\mathbb{P}\{X_n = k\} = \frac{[x^n u^k]\, A(x,u)}{[x^n]\, A(x,1)}}$$

(+ some technical conditions).

Then the random variable X_n satisfies a **central limit theorem** with

$$\boxed{\mathbb{E}\, X_n \sim n\mu} \quad \text{and} \quad \boxed{\mathbb{V}\mathrm{ar}\, X_n \sim n\sigma^2},$$

where μ and σ^2 can be computed.

Asymptotics for Random Planar Graphs

$$U(x,y) = xy(1+V(x,y))^2,$$
$$V(x,y) = y(1+U(x,y))^2$$
$$\Longrightarrow \quad U(x,y) = xy(1+y(1+U(x,y))^2)^2$$
$$\Longrightarrow \quad \boxed{U(x,y) = g(x,y) - h(x,y)\sqrt{1-\frac{y}{\tau(x)}}}$$
$$\Longrightarrow \quad V(x,y) = g_2(x,y) - h_2(x,y)\sqrt{1-\frac{y}{\tau(x)}}$$

$$M(x,y) = x^2y^2\left(\frac{1}{1+xy} + \frac{1}{1+y} - 1 - \frac{(1+U)^2(1+V)^2}{(1+U+V)^3}\right)$$
$$!!! \quad \Longrightarrow \quad \boxed{M(x,y) = g_3(x,y) + h_3(x,y)\left(1-\frac{y}{\tau(x)}\right)^{\frac{3}{2}}}$$

due to cancellation of the $\sqrt{1-y/\tau(x)}$-term

Asymptotics for Random Planar Graphs

$$\frac{M(x,D)}{2x^2D} = \log\left(\frac{1+D}{1+y}\right) - \frac{xD^2}{1+xD}$$

!!! $\Longrightarrow$ $$\boxed{D(x,y) = g_4(x,y) + h_4(x,y)\left(1 - \frac{x}{R(y)}\right)^{\frac{3}{2}}}$$

due to interaction of the singularities!!!

$$\frac{\partial B(x,y)}{\partial y} = \frac{x^2}{2}\frac{1+D(x,y)}{1+y},$$

!!! $\Longrightarrow$ $$\boxed{B(x,y) = g_5(x,y) + h_5(x,y)\left(1 - \frac{x}{R(y)}\right)^{\frac{5}{2}}}$$

$\Longrightarrow$ $$\boxed{b_n \sim b \cdot R(1)^{-n} n^{-\frac{7}{2}} n!}$$

Asymptotics for Random Planar Graphs

$$B'(x,y) = g_6(x,y) + h_6(x,y)\left(1 - \frac{x}{R(y)}\right)^{\frac{3}{2}},$$

$$C'(x,y) = e^{B'(xC'(x,y),y)},$$

!!! $\Longrightarrow$ $$\boxed{C'(x,y) = g_7(x,y) + h_7(x,y)\left(1 - \frac{x}{r(y)}\right)^{\frac{3}{2}}}$$

due to interaction of the singularities!!!

$$\Longrightarrow \quad \boxed{C(x,y) = g_8(x,y) + h_8(x,y)\left(1 - \frac{x}{r(y)}\right)^{\frac{5}{2}}}$$

$$\Longrightarrow \quad \boxed{c_n \sim c\, r(1)^{-n} n^{-\frac{7}{2}} n!}$$

Asymptotics for Random Planar Graphs

$$C(x,y) = g_8(x,y) + h_8(x,y)\left(1 - \frac{x}{r(y)}\right)^{\frac{5}{2}}$$

$$\implies \quad G(x,y) = e^{C(x,y)} = g_9(x,y) + h_9(x,y)\left(1 - \frac{x}{r(y)}\right)^{\frac{5}{2}}.$$

$$\implies \quad \boxed{g_n \sim g \cdot r(1)^{-n} n^{-\frac{7}{2}} n!}$$

Asymptotic Degree Distribution

3-connected planar graphs

$$T^\bullet(x,y,w) = \frac{x^2y^2w^2}{2}\left(\frac{1}{1+wy} + \frac{1}{1+xy} - 1 - \right.$$
$$\left. - \frac{(U+1)^2\left(-w_1(U,V,w) + (U-w+1)\sqrt{w_2(U,V,w)}\right)}{2w(Vw+U^2+2U+1)(1+U+V)^3}\right),$$

$$\tilde{u}_0(y) = -\frac{1}{3} + \sqrt{\frac{4}{9} + \frac{1}{3y}}, \quad r(y) = \frac{\tilde{u}_0(y)}{y(1+y(1+\tilde{u}_0(y))^2)^2},$$

$$\tilde{X} = \sqrt{1 - \frac{x}{r(y)}}$$

$$\Longrightarrow \quad \boxed{T^\bullet(x,y,w) = \tilde{T}_0(y,w) + \tilde{T}_2(y,w)\tilde{X}^2 + \tilde{T}_3(y,w)\tilde{X}^3 + O(\tilde{X}^4)}$$

due to cancellation of the $\sqrt{1 - x/r(z)}$-term.

Asymptotic Degree Distribution

Planar networks

$$D(x,y,w) = (1+yw)\exp\left(S(x,y,w) + \frac{1}{x^2 D(x,y,w)}\times\right.$$
$$\left.\times\, T^\bullet\left(x, D(x,y,1), \frac{D(x,y,w)}{D(x,y,1)}\right)\right) - 1$$
$$S(x,y,w) = xD(x,y,1)\,(D(x,y,w) - S(x,y,w))$$

$\tau(x)$ ... inverse function of $r(y)$

$$D(R(y),y,1) = \tau(R(y))$$

$$X = \sqrt{1 - \frac{x}{R(y)}}$$

$$\Longrightarrow \quad \boxed{D(x,y,w) = D_0(y,w) + D_2(y,w)X^2 + D_3(y,w)X^3 + O(X^4)},$$

Asymptotic Degree Distribution

2-connected planar graphs

$$w\frac{\partial B^{\bullet}(x,y,w)}{\partial w} = xyw \exp\left(S(x,y,w) + \frac{1}{x^2 D(x,y,w)} T^{\bullet}\left(x, D(x,y,1), \frac{D(x,y,w)}{D(x,y,1)}\right)\right)$$

$$\Longrightarrow \quad \boxed{B^{\bullet}(x,y,w) = B_0(y,w) + B_2(y,w)X^2 + B_3(y,w)X^3 + O(X^4)}$$

Remark. All these functions $B_j(y,w)$ can be *explicitly* computed.

Asymptotic Degree Distribution

connected planar graphs

$$C^{\bullet}(x, 1, w) = \exp\left(B^{\bullet}\left(xC'(x), 1, w\right)\right)$$

???

Asymptotic Degree Distribution

Lemma

$$f(x) = \sum_{n\ge 0} \boxed{a_n} \frac{x^n}{n!} = f_0 + f_2 X^2 + f_3 \boxed{X^3} + \mathcal{O}(X^4), \qquad X = \sqrt{1 - \frac{x}{\rho}},$$

$$H(x,z,w) = h_0(x,w) + h_2(x,w) Z^2 + h_3(x,w) \boxed{Z^3} + \mathcal{O}(Z^4),$$

$$Z = \sqrt{1 - \frac{z}{\boxed{f(\rho)}}},$$

$$f_H(x) = H(x, \boxed{f(x)}, w) = \sum_{n\ge 0} \boxed{b_n(w)} \frac{x^n}{n!}$$

$$\Longrightarrow \quad \boxed{\lim_{n\to\infty} \frac{b_n(w)}{a_n} = -\frac{h_2(\rho,w)}{f_0} + \frac{h_3(\rho,w)}{f_3}\left(-\frac{f_2}{f_0}\right)^{3/2}}.$$

Asymptotic Degree Distribution

connected planar graphs

$$C^\bullet(x,1,w) = \exp\left(B^\bullet\left(xC'(x),1,w\right)\right)$$

Application of the lemma with

$$f(x) = xC'(x)$$

and

$$H(x,z,w) = xe^{B^\bullet(z,1,w)}.$$

$$\Longrightarrow \quad p(w) = \lim_{n\to\infty} \frac{b_n(w)}{a_n}$$

$$= \boxed{-e^{B_0(1,w)-B_0(1,1)}B_2(1,w) + e^{B_0(1,w)-B_0(1,1)}\frac{1+B_2(1,1)}{B_3(1,1)}B_3(1,w)}$$

Random Planar Graphs

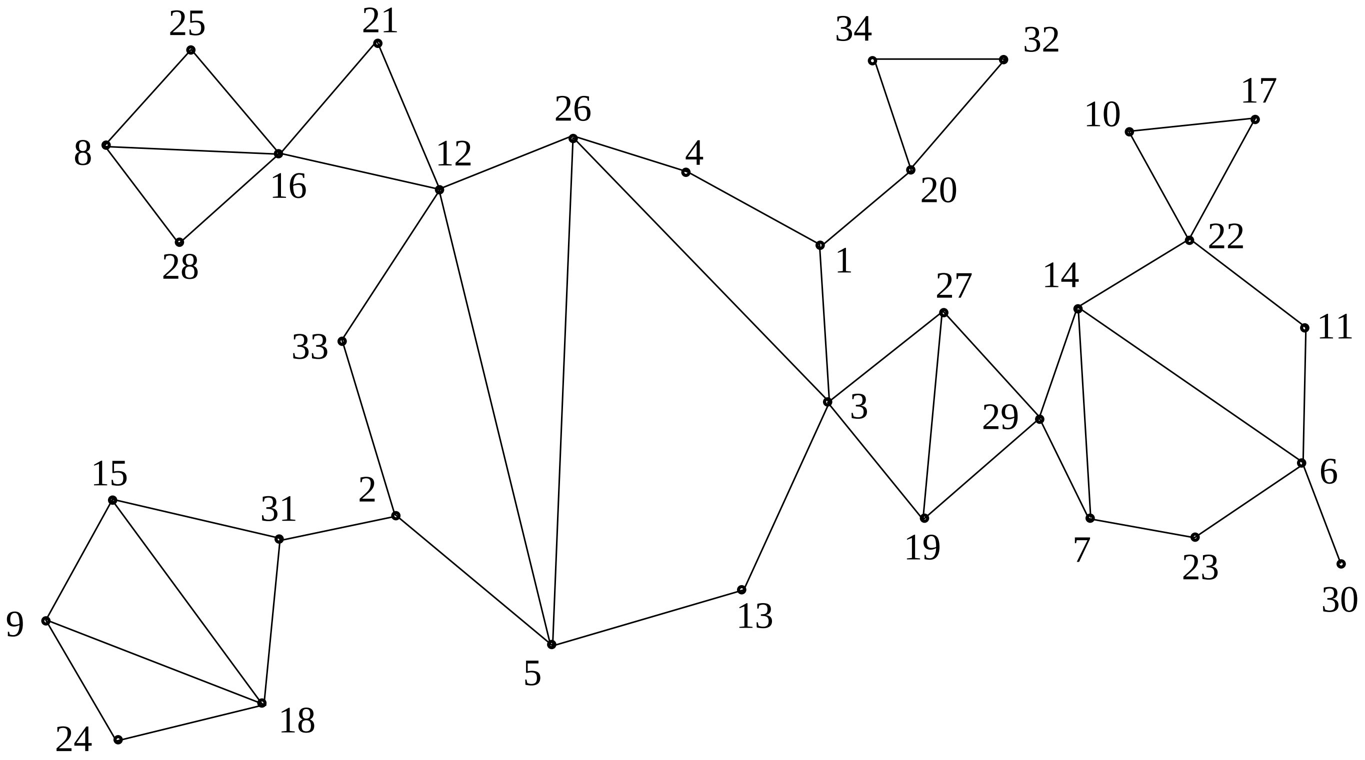
Classes of planar graphs

- **Outerplanar graphs**: all vertices are on the infinite face (equivalently no K_4 and no $K_{2,3}$ as a minor).
- **Series-parallel graphs**: series-parellel extension of a tree or forest (equivalently no K_4 as a minor).
- **Planar graphs**. (no K_5 and no $K_{3,3}$ as a minor)

Remark.

$$\text{outerplanar} \subseteq \text{series-parallel} \subseteq \text{planar}$$

Outerplanar Graphs



All vertices are on the infinite face.

Outerplanar Graphs

Generating functions

b_n ... number of **2-connected labelled outer-planar** graphs with n vertices

$$B(x) = \sum_{n\geq 0} b_n \frac{x^n}{n!}$$

c_n ... number of **connected labelled outer-planar** graphs with n vertices

$$C(x) = \sum_{n\geq 0} c_n \frac{x^n}{n!}$$

g_n ... number of **labelled outer-planar** graphs with n vertices

$$G(x) = \sum_{n\geq 0} g_n \frac{x^n}{n!}$$

Outerplanar Graphs

Generating functions

$$G(x) = e^{C(x)},$$

$$C'(x) = e^{B'(xC'(x))},$$

$$B'(x) = x + \frac{1}{2} x\, A(x),$$

$$A(x) = x(1 + A(x))^2 + x(1 + A(x))A(x)$$

$$= \frac{1 - 3x - \sqrt{1 - 6x + x^2}}{4x}.$$

Outerplanar Graphs

Asymptotic enumeration

$$b_n = b \cdot (3 + 2\sqrt{2})^n \, n^{-\frac{5}{2}} n! \left(1 + O\left(\frac{1}{n}\right)\right),$$
$$c_n = c \cdot \rho^{-n} \, n^{-\frac{5}{2}} \, n! \left(1 + O\left(\frac{1}{n}\right)\right),$$
$$g_n = g \cdot \rho^{-n} \, n^{-\frac{5}{2}} \, n! \left(1 + O\left(\frac{1}{n}\right)\right),$$

$$\rho = y_0 e^{-B'(y_0)} = 0.1365937...,$$
$$y_0 = 0.1707649...\text{satisfies } 1 = y_0 B''(y_0),$$
$$b = \frac{1}{8\sqrt{\pi}} \sqrt{114243\sqrt{2} - 161564} = 0.000175453...,$$
$$c = 0.0069760...,$$
$$g = 0.017657...$$

Outerplanar Graphs

Theorem 1

Let $p_{n,k}$ be the probability that a random node in a random 2-connected, connected or unrestricted **outerplanar graph** with n vertices has degree k. Then the limit

$$\boxed{p_k := \lim_{n\to\infty} p_{n,k}}$$

exists. The probability generating function

$$\boxed{p(w) = \sum_{k\geq 1} p_k w^k}$$

can be explicitly computed.

Outerplanar Graphs

$$p(w) = \sum_{k \geq 1} p_k w^k$$

- **2-connected**

$$p(w) = \frac{2(3 - 2\sqrt{2})w^2}{(1 - (\sqrt{2} - 1)w)^2}$$

- **connected** or **unrestricted**:

$$p(w) = \frac{c_1 w^2}{(1 - c_2 w)^2} \exp\left(c_3 w + \frac{c_4 w^2}{(1 - c_2 w)}\right)$$

(with certain constants $c_1, c_2, c_3, c_4 > 0$).

Outerplanar Graphs

Theorem 2

$X_n^{(k)}$ ... **number of vertices of degree** k in random 2-connected, connected or unrestricted **labelled outerplanar** graphs with n vertices.

$\Longrightarrow$ $X_n^{(k)}$ satisfies a **central limit theorem** with

$$\mathbb{E}\, X_n^{(k)} \sim \mu_k n \quad \text{and} \quad \mathbb{V}\, X_n^{(k)} \sim \sigma_k^2 n.$$

Remark. $\mu_k = p_k$.

Outerplanar Graphs

Theorem 3

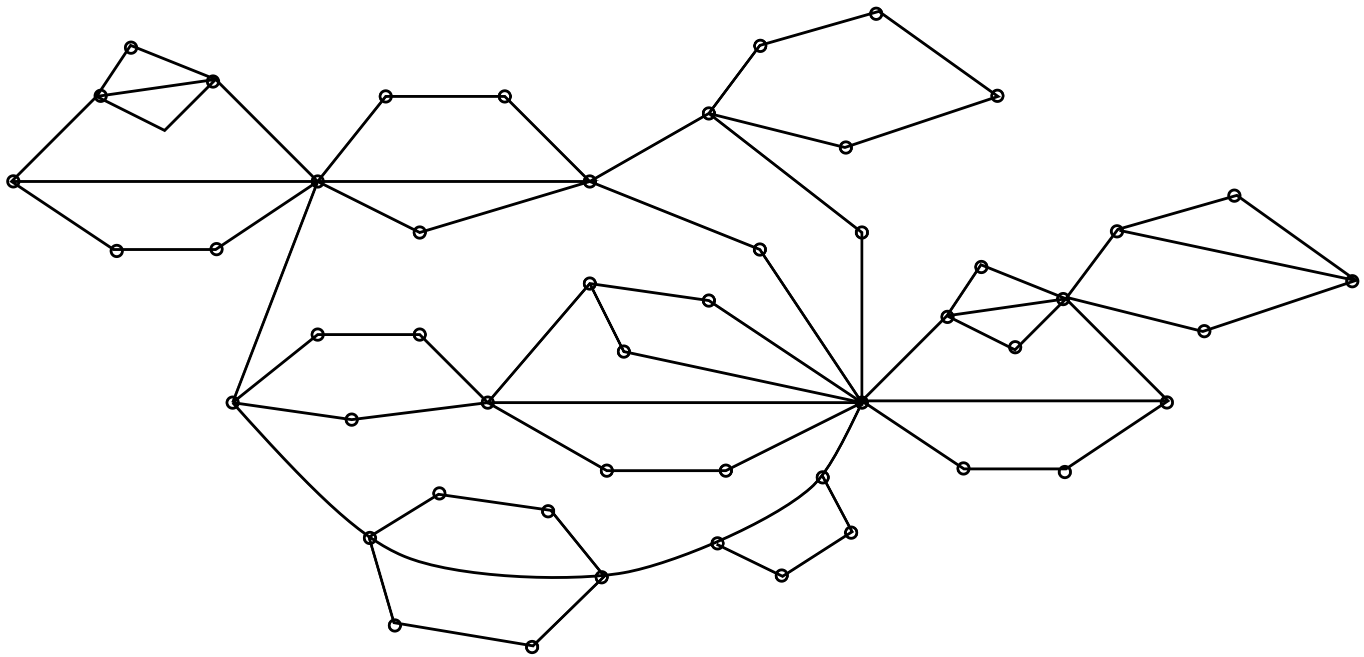
Δ_n ... **maximum degree** in random 2-connected, connected or unrestricted **labelled outerplanar** graphs with n vertices.

$$\Longrightarrow \quad \boxed{\frac{\Delta_n}{\log n} \to c \quad \text{in probability}}$$

$$\mathbb{E}\,\Delta_n \sim c \log n,$$

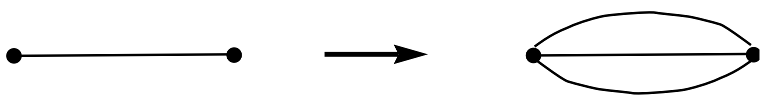
where $c = 1/\log(1/q)$ and $1/q$ in radius of convergence of $p(w)$.

Series-Parallel Graphs



Series-parallel extension of a tree or forest

Series-extension: 

Parallel-extension: 

Series-Parallel Graphs

Generating functions

$b_{n,m}$ ... number of **2-connected labelled series-parallel** graphs with n vertices and m edges, $b_n = \sum_m b_{n,m}$

$$B(x,y) = \sum_{n,m} b_{n,m} \frac{x^n}{n!} y^m$$

$c_{n,m}$ ... number of **connected labelled series-parallel** graphs with n vertices and m edges, $c_n = \sum_m c_{n,m}$

$$C(x,y) = \sum_{n,m} c_{n,m} \frac{x^n}{n!} y^m$$

$g_{n,m}$ ... number of **labelled series-parallel** graphs with n vertices and m edges, $g_n = \sum_m g_{n,m}$

$$G(x,y) = \sum_{n,m} g_{n,m} \frac{x^n}{n!} y^m$$

Series-Parallel Graphs

Generating functions

$$G(x,y) = e^{C(x,y)}$$

$$\frac{\partial C(x,y)}{\partial x} = \exp\left(\frac{\partial B}{\partial x}\left(x\frac{\partial C(x,y)}{\partial x}, y\right)\right),$$

$$\frac{\partial B(x,y)}{\partial y} = \frac{x^2}{2}\,\frac{1+D(x,y)}{1+y},$$

$$D(x,y) = (1+y)e^{S(x,y)} - 1,$$

$$S(x,y) = (D(x,y) - S(x,y))xD(x,y).$$

Series-Parallel Graphs

Asymptotic enumeration

$$b_n = b \cdot \rho_1^{-n} n^{-\frac{5}{2}} n! \left(1 + O\left(\frac{1}{n}\right)\right),$$
$$c_n = c \cdot \rho_2^{-n} n^{-\frac{5}{2}} n! \left(1 + O\left(\frac{1}{n}\right)\right),$$
$$g_n = g \cdot \rho_2^{-n} n^{-\frac{5}{2}} n! \left(1 + O\left(\frac{1}{n}\right)\right),$$

$$\rho_1 = 0.1280038...,$$
$$\rho_2 = 0.11021...,$$
$$b = 0.0010131...,$$
$$c = 0.0067912...,$$
$$g = 0.0076388...$$

Series-Parallel Graphs

Theorem 1

Let $p_{n,k}$ be the probability that a random node in a random 2-connected, connected or unrestricted series-parallel graph with n vertices has degree k. Then the limit

$$\boxed{p_k := \lim_{n\to\infty} p_{n,k}}$$

exists. The probability generating function

$$\boxed{p(w) = \sum_{k\geq 1} p_k w^k}$$

can be explicitly computed.

Series-Parallel Graphs

We just mention the case of

2-connected series-parallel graphs $p(w) = \sum_{k \geq 1} p_k w^k$:

$$\boxed{p(w) = \frac{B_1(1, w)}{B_1(1, 1)}},$$

where $B_1(y, w)$ is given by the following procedure ...

Series-Parallel Graphs

$$
\begin{aligned}
\frac{E_0(y)^3}{E_0(y)-1} &= \left(\log\frac{1+E_0(y)}{1+R(y)} - E_0(y)\right)^2, \\
R(y) &= \frac{\sqrt{1-1/E_0(y)}-1}{E_0(y)}, \\
E_1(y) &= -\left(\frac{2R(y)E_0(y)^2(1+R(y)E_0(y))^2}{(2R(y)E_0(y)+R(y)^2E_0(y)^2)^2+2R(y)(1+R(y)E_0(y))}\right)^{\frac{1}{2}}, \\
D_0(y,w) &= (1+yw)e^{\frac{R(y)E_0(y)}{1+R(y)E_0(y)}D_0(y,w)} - 1, \\
D_1(y,w) &= \frac{(1+D_0(y,w))^{\frac{R(y)E_1(y)D_0(y,w)}{1+R(y)E_0(y)}}}{1-(1+D_0(y,w))^{\frac{R(y)E_0(y)D_0(y,w)}{1+R(y)E_0(y)}}}, \\
B_0(y,w) &= \frac{R(y)D_0(y,w)}{1+R(y)E_0(y)} - \frac{R(y)^2E_0(y)D_0(y,w)^2}{2(1+R(y)E_0(y))}, \\
B_1(y,w) &= \frac{R(y)D_1(y,w)}{1+R(y)E_0(y)} - \frac{R(y)^2E_0(y)D_0(y,w)D_1(y,w)}{1+R(y)E_0(y)} \\
&\quad - \frac{R(y)^2E_1(y)D_0(y,w)(1+D_0(y,w)/2)}{(1+R(y)E_0(y))^2}.
\end{aligned}
$$

Series-Parallel Graphs

Theorem 2

$X_n^{(k)}$ ... **number of vertices of degree** k in random 2-connected, connected or unrestricted **labelled series-parallel** graphs with n vertices.

$\Longrightarrow$ $X_n^{(k)}$ satisfies a **central limit theorem** with

$$\mathbb{E}\, X_n^{(k)} \sim \mu_k n \quad \text{and} \quad \mathbb{V}\, X_n^{(k)} \sim \sigma_k^2 n.$$

Remark. $\mu_k = p_k$.

Series-Parallel Graphs

Theorem 3

Δ_n ... **maximum degree** in random 2-connected, connected or unrestricted **labelled series-parallel** graphs with n vertices.

$$\Longrightarrow \quad \boxed{\frac{\Delta_n}{\log n} \to c \quad \text{in probability}}$$

$$\mathbb{E}\,\Delta_n \sim c \log n,$$

where $c = 1/\log(1/q)$ and $1/q$ in radius of convergence of $p(w)$.

Systems of Functional equations

COMBINATORIAL CENTRAL LIMIT THEOREM

Suppose that a sequence of random variables X_n has distribution

$$\mathbb{P}[X_n = k] = \frac{a_{nk}}{a_n},$$

where the generating function $A(x,u) = \sum_{n,k} a_{n,k} x^n u^k$ is given by

$$\boxed{A(x,u) = \Psi(x, u, A_1(x,u), \ldots, A_r(x,u))}$$

for an analytic function Ψ and the generating functions

$$A_1(x,u) = \sum_{n,k} a_{1;n,k} u^k x^n, \ \ldots \ , A_r(x,u) = \sum_{n,k} a_{r;n,k} u^k x^n$$

satisfy a **non-linear system of equations**

$$\boxed{A_j(x,u) = \Phi_j(x, u, A_1(x,u), \ldots, A_r(x,u))}, \quad (1 \le j \le r).$$

Systems of Functional equations

COMBINATORIAL CENTRAL LIMIT THEOREM (cont.)

Suppose that at least one of the functions $\Phi_j(x,u,a_1,\ldots,a_r)$ is non-linear in $a_1,\ldots,a_r$ and they all have a power series expansion at $(0,0,0)$ with non-negative coefficients.

Let $x_0 > 0$, $\mathbf{a}_0 = (a_{0,0},\ldots,a_{r,0}) > 0$ (inside the region of convergence) satisfy the system of equations: $(\mathbf{\Phi} = (\Phi_1,\ldots,\Phi_r))$

$$\boxed{\mathbf{a}_0 = \mathbf{\Phi}(x_0,1,\mathbf{a}_0), \quad 0 = \det(\mathbb{I} - \mathbf{\Phi}_{\mathbf{a}}(x_0,1,\mathbf{a}_0)}.$$

Suppose further, that the **dependency graph** of the system $\mathbf{a} = \mathbf{\Phi}(x,u,\mathbf{a})$ is **strongly connected** (which means that no subsystem can be solved before the whole system).

Systems of Functional equations

COMBINATORIAL CENTRAL LIMIT THEOREM (cont.)

Then there exists analytic function $g_j(x,u), h_j(x,u)$, and $\rho(u)$ (that is **independent of** j) such that locally

$$\boxed{A_j(x,u) = g_j(x,u) - h_j(x,u)\sqrt{1 - \frac{x}{\rho(u)}}}.$$

and consequently (for some $g(x,u)$, $h(x,u)$)

$$\boxed{A(x,u) = g(x,u) - h(x,u)\sqrt{1 - \frac{x}{\rho(u)}}}.$$

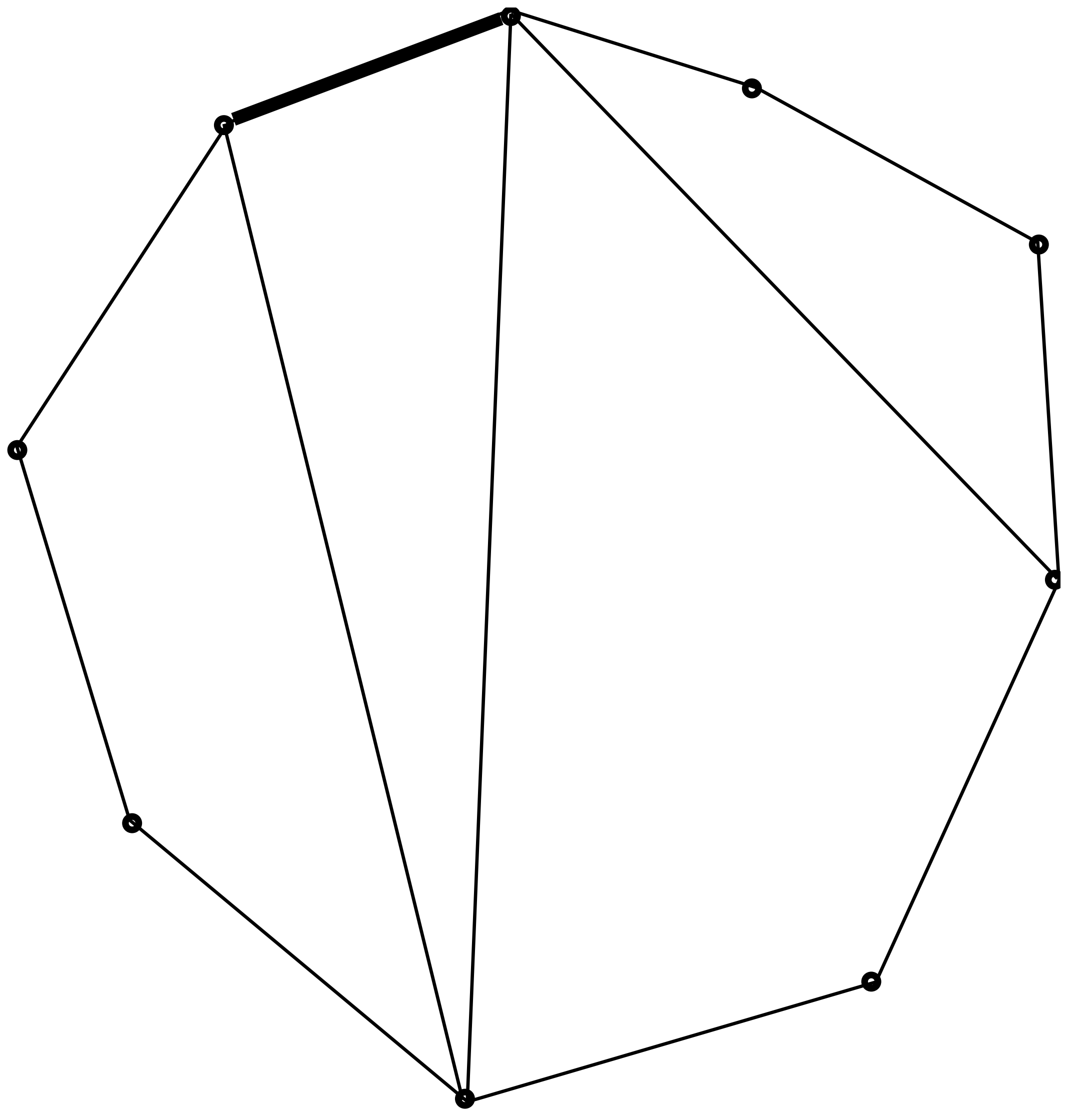
Consequently the random variable X_n satisfies a **central limit theorem** with

$$\boxed{\mathbb{E}\, X_n \sim n\mu} \quad \text{and} \quad \boxed{\mathbb{V}\mathrm{ar}\, X_n \sim n\sigma^2},$$

where μ and σ^2 can be computed.

Nodes of Given Degree

Dissections:



Nodes of Given Degree

- v_2 counts the number of nodes with degree 2,

- v_3 counts the number of nodes with degree 3,

- v counts the number of nodes with degree > 3, and

- in all cases the two **nodes of the rooted edge are are not taken into account.**

Nodes of Given Degree

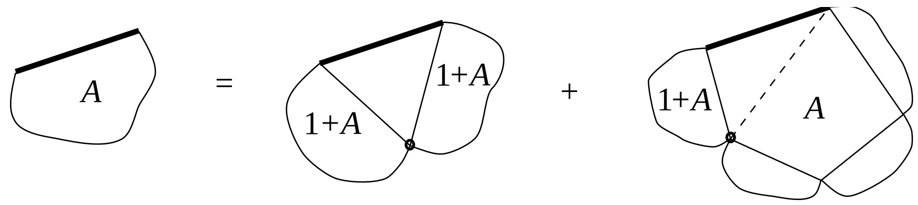
- $A_{ij}(v_2, v_3, v)$ ... generating function of dissections with the properties that the left node of the rooted edge has degree i and right one has degree i, $2 \le i, j \le 3$

- $A_{i\infty}(v_2, v_3, v)$ ... generating function of dissections with the properties that the left node of the rooted edge has degree i and the right has degree > 3,

- $A_{\infty\infty}(v_2, v_3, v)$ ... generating function of dissections with the properties that both nodes of the rooted edge have degree > 3.

Nodes of Given Degree

Lemma 1



$$
\begin{aligned}
A_{22} &= v_2 \\
&\quad + v_2 A_{22} + v_3 A_{23} + v A_{2\infty}, \\
A_{23} &= v_3 A_{22} + v(A_{23} + A_{2\infty}) \\
&= v_2 A_{23} + v_3 A_{33} + v A_{3\infty}, \\
A_{33} &= v(A_{22} + A_{23} + A_{2\infty})^2 \\
&\quad + v(A_{22} + A_{23} + A_{2\infty})(A_{23} + A_{33} + A_{3\infty}), \\
A_{2\infty} &= v_3 A_{23} + v(A_{33} + A_{3\infty}) + v(A_{2\infty} + A_{3\infty} + A_{\infty\infty}) \\
&\quad + v_2 A_{2\infty} + v_3 A_{3\infty} + v A_{\infty\infty}, \\
A_{3\infty} &= v(A_{23} + A_{33} + A_{3\infty})(A_{2\infty} + A_{3\infty} + A_{\infty\infty}) \\
&\quad + v(A_{22} + A_{23} + A_{2\infty})(A_{2\infty} + A_{3\infty} + A_{\infty\infty}), \\
A_{\infty\infty} &= v(A_{23} + A_{33} + A_{3\infty} + A_{2\infty} + A_{3\infty} + A_{\infty\infty})^2 \\
&\quad + v(A_{23} + A_{33} + A_{3\infty} + A_{2\infty} + A_{3\infty} + A_{\infty\infty})(A_{2\infty} + A_{3\infty} + A_{\infty\infty}).
\end{aligned}
$$

Nodes of Given Degree

Remark

All functions $A_{ij}(v_2, v_3, v)$ have a **squareroot singularity** due to the COMBINATORIAL CENTRAL LIMIT THEOREM II!!!

Nodes of Given Degree

- $B_i^\bullet(v_1, v_2, v_3, v)$ ... exponential genenerating functions of 2-connected labelled and marked outer planar graphs, where the root node (that is not counted) has degree i, $1 \le i \le 3$.

- $B_\infty^\bullet(v_1, v_2, v_3, v)$ ... exponential genenerating functions of 2-connected labelled and marked outer planar graphs, where the root node (that is not counted) has degree > 3.

Nodes of Given Degree

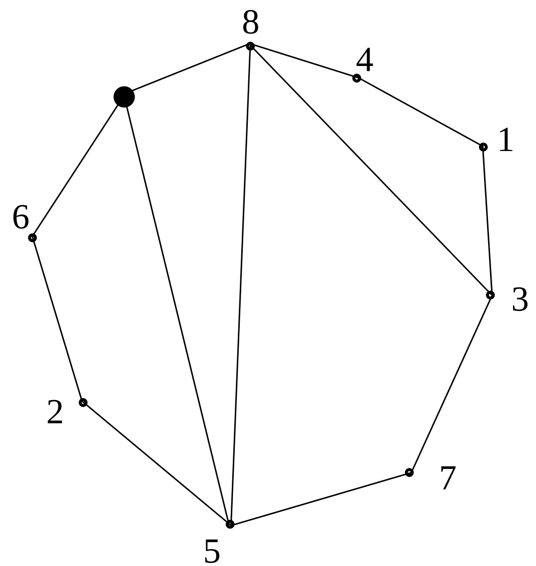
Lemma 2

$$B_1^\bullet(v_1, v_2, v_3, v) = v_1,$$

$$B_2^\bullet(v_1, v_2, v_3, v) = \frac{1}{2}\left(v_2 A_{22} + v_3 A_{23} + v A_{2\infty}\right),$$

$$B_3^\bullet(v_1, v_2, v_3, v) = \frac{1}{2}\left(v_2 A_{23} + v_3 A_{33} + v A_{3\infty}\right),$$

$$B_\infty^\bullet(v_1, v_2, v_3, v) = \frac{1}{2}\left(v_2 A_{2\infty} + v_3 A_{3\infty} + v A_{\infty\infty}\right).$$



Nodes of Given Degree

Remark

All functions $B_i^\bullet(v_1, v_2, v_3, v)$ have a **squareroot singularity** since all functions $A_{ij}(v_2, v_3, v)$ have squareroot singularities!!!

Nodes of Given Degree

- $C_i^\bullet(v_1, v_2, v_3, v)$ ... exponential generating functions of connected labelled and marked outer planar graphs, where the root node (that is not counted) has degree i, $0 \le i \le 3$.

- $C_\infty^\bullet(v_1, v_2, v_3, v)$ ... exponential generating functions of connected labelled and marked outer planar graphs, where the root node (that is not counted) has degree > 3.

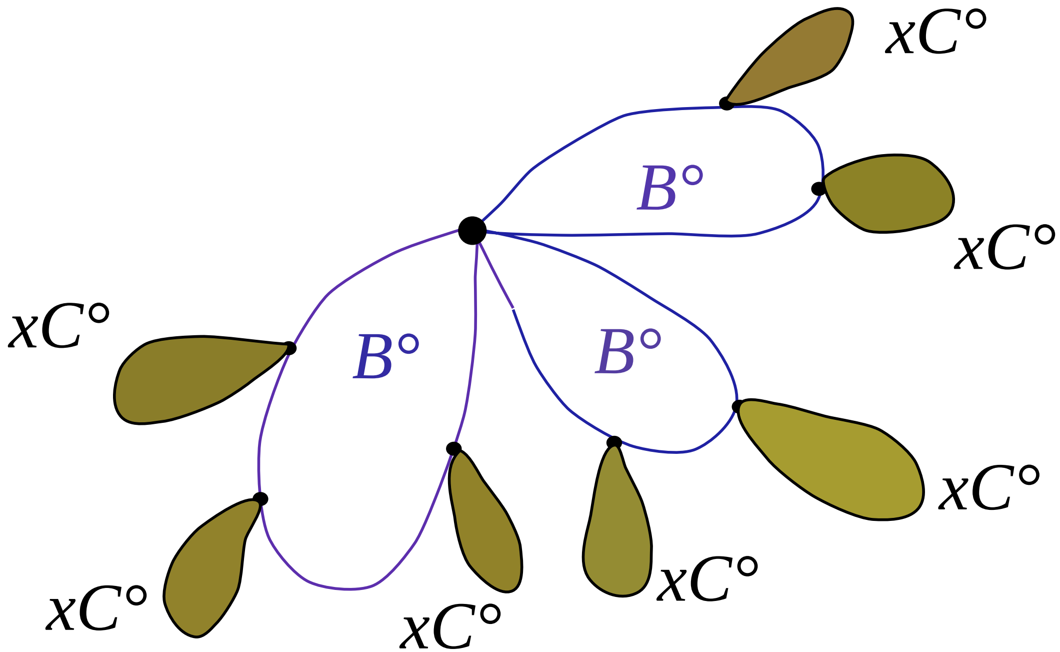
Nodes of Given Degree

Lemma 3

$$
\begin{aligned}
C_0^\bullet(v_1,v_2,v_3,v) &= 1,\\
C_1^\bullet(v_1,v_2,v_3,v) &= B_1^\bullet(W_1,W_2,W_3,W),\\
C_2^\bullet(v_1,v_2,v_3,v) &= \frac{1}{2!}(B_1^\bullet(W_1,W_2,W_3,W))^2 + B_2^\bullet(W_1,W_2,W_3,W),\\
C_3^\bullet(v_1,v_2,v_3,v) &= \frac{1}{3!}(B_1^\bullet(W_1,W_2,W_3,W))^3\\
&\quad + \frac{1}{1!1!}B_1^\bullet(W_1,W_2,W_3,W)B_2^\bullet(W_1,W_2,W_3,W)\\
&\quad + B_3^\bullet(W_1,W_2,W_3,W),\\
C_\infty^\bullet(v_1,v_2,v_3,v) &= e^{B_1^\bullet(W_1,W_2,W_3,W)+B_2^\bullet(\ldots)+B_3^\bullet(\ldots)+B_\infty^\bullet(W_1,W_2,W_3,W)}\\
&\quad - 1 - B_1^\bullet(W_1,W_2,W_3,W) - B_2^\bullet(\ldots) - B_3^\bullet(\ldots)\\
&\quad - \frac{1}{1!}(B_1^\bullet(W_1,W_2,W_3,W))^2 - \frac{1}{3!}(B_1^\bullet(W_1,W_2,W_3,W))^3\\
&\quad - \frac{1}{1!1!}B_1^\bullet(W_1,W_2,W_3,W)B_2^\bullet(W_1,W_2,W_3,W),
\end{aligned}
$$

where on the right hand side

$$
\begin{aligned}
W_1 &= v_1 C_0^\bullet + v_2 C_1^\bullet + v_3 C_2^\bullet + v(C_3^\bullet + C_\infty^\bullet),\\
W_2 &= v_2 C_0^\bullet + v_3 C_1^\bullet + v(C_2^\bullet + C_3^\bullet + C_\infty^\bullet),\\
W_3 &= v_3 C_0^\bullet + v(C_1^\bullet + C_2^\bullet + C_3^\bullet + C_\infty^\bullet),\\
W &= v(C_0^\bullet + C_1^\bullet + C_2^\bullet + C_3^\bullet + C_\infty^\bullet).
\end{aligned}
$$



Nodes of Given Degree

Remark

All functions $C_i^\bullet(v_1, v_2, v_3, v)$ have a **squareroot singularity** due to the COMBINATORIAL CENTRAL LIMIT THEOREM II!!!

Nodes of Given Degree

Counting nodes of degree 3:

$C(v_1, v_2, v_3, v)$ ... exponential generating function of all connected labelled outer planar graphs

$C_{d=3}(x,u)$ ... exponential generating function that counts the number of nodes with x and the number of nodes of degree $d=3$ with u:

$$C_{d=3}(x,u) = C(x,x,xu,x).$$

Also:

$$\frac{\partial C_{d=3}(x,u)}{\partial x} = C_1^\bullet + C_2^\bullet + uC_3^\bullet + C_\infty^\bullet \quad \text{and} \quad \frac{\partial C_{d=3}(x,u)}{\partial u} = xC_3^\bullet$$

Nodes of Given Degree

Central limit theorem

$$\frac{\partial C_{d=3}(x,u)}{\partial x} = C_1^\bullet + C_2^\bullet + uC_3^\bullet + C_\infty^\bullet$$

$$\Longrightarrow \quad \frac{\partial C_{d=3}(x,u)}{\partial x} g(x,y) - h(x,y)\sqrt{1 - \frac{x}{\rho(y)}}$$

$$\Longrightarrow \quad C_{d=3}(x,u) = g_2(x,y) + h_2(x,y)\left(1 - \frac{x}{\rho(y)}\right)^{\frac{3}{2}}$$

$\Longrightarrow$ The number of nodes of degree 3 in outerplanar graphs satisfies a **central limit theorem**.

Maximum Degree

Relation to number of nodes of given degree

Δ_n ... maximum degree

$X_n^{(>k)} = X_n^{(k+1)} + X_n^{(k+2)} + \cdots$ ... **number of nodes of degree** $> k$.

$$\boxed{\Delta_n > k \iff X_n^{(>k)} > 0}$$

Maximum Degree

First moment method

Y ... a discrete random variable on non-negative integers.

$$\Longrightarrow \quad \boxed{\mathbb{P}\{Y > 0\} \leq \min\{1, \mathbb{E}\, Y\}}$$

Second moment method

Y is a non-negative random variable with finite second moment.

$$\Longrightarrow \quad \boxed{\mathbb{P}\{Y > 0\} \geq \frac{(\mathbb{E}\, Y)^2}{\mathbb{E}\,(Y^2)}}$$

Maximum Degree

Asymptotics for moments

$$\boxed{\mathbb{E}\, X_n^{(k)} = n\, p_{n,k}}$$

$$p_{n,k} = \frac{g^\bullet_{n-1,m,k}}{n\, g_{n,m}} = \frac{[x^{n-1}w^k]\, G^\bullet(x,w)}{[x^{n-1}]\, G^\bullet(x,1)}$$

$$\Longrightarrow \quad \mathbb{E}\, X_n^{(>k)} = \mathbb{E}\left(\sum_{\ell>k} X_n^{(\ell)}\right) = n \sum_{\ell>k} p_{n,\ell}.$$

Precise asymptotics for $p_{n,k}$ are needed that are **uniform in** n **and** k.

Maximum Degree

Asymptotics for moments

$p_{n,k,\ell}$ ... **probability that two different randomly selected vertices** (in a random planar graph of size n) **have degrees** k **and** ℓ**.**

$$\boxed{\mathbb{E}\left(X_n^{(k)} X_n^{(\ell)}\right) = n(n-1)\, p_{n,k,\ell}} \qquad (k \neq \ell)$$

$$p_{n,k,\ell} = \frac{[x^{n-2} w^k t^\ell]\, G^{\bullet\bullet}(x,w,t)}{[x^{n-1}]\, G^{\bullet\bullet}(x,1,1)}$$

$$\Longrightarrow \quad \mathbb{E}\,(X_n^{(>k)})^2 = \mathbb{E}\left(\sum_{j>k} X_n^{(j)}\right)^2 = n \sum_{\ell>k} p_{n,\ell} + n(n-1) \sum_{\ell_1,\ell_2>k} p_{n,\ell_1,\ell_1}.$$

Maximum Degree

Bounds for the distribution of Δ_n

$$\frac{n^2\left(\sum_{\ell>k} p_{n,\ell}\right)^2}{n\sum_{\ell>k} p_{n,\ell} + n(n-1)\sum_{\ell_1,\ell_2>k} p_{n,\ell_1,\ell_1}} \le \mathbb{P}\{\Delta_n > k\} \le \min\left\{1, n\sum_{\ell>k} p_{n,\ell}\right\}.$$

$$p_{n,k} \sim c\, k^\alpha q^k$$

$$p_{n,k,\ell} \sim p_{n,k}p_{n,\ell} \sim c^2\,(k\ell)^\alpha q^{k+\ell}$$

$$\Longrightarrow \quad \boxed{\frac{\Delta_n}{\log n} \to \frac{1}{\log(1/q)} \qquad \text{in probability}}$$

Maximum Degree

Generating functions for series-parallel graphs

$$
\begin{aligned}
G^{\bullet\bullet}(x,w,t) &= e^{C(x)} G^{\bullet}(x,w) G^{\bullet}(x,t) + e^{C(x)} C^{\bullet\bullet}(x,w,t), \\
C^{\bullet\bullet}(x,w,t) &= \frac{x}{(xC'(x))'} \frac{\partial}{\partial x} C^{\bullet}(x,w) \frac{\partial}{\partial x} C^{\bullet}(x,t) \\
&\quad + B^{\bullet\bullet}(xC'(x),w,t) C^{\bullet}(x,w) C^{\bullet}(x,t), \\
w\frac{\partial}{\partial w} B^{\bullet\bullet}(x,w,t) &= wte^{S_1(x,w,t})) + we^{S(x,w)} S_2(x,w,t), \\
D_1(x,w,t) &= (1+wt)e^{S_1(x,w,t)} - 1, \\
S_1(x,w,t) &= x(D(x,w) - S(x,w))D(x,t), \\
D_2(x,w,t) &= (1+wt)e^{S_2(x,w,t)}, \\
S_2(x,w,t) &= x(D_2(x,w,t) - S_2(x,w,t))E(x) \\
&\quad + x(D_1(x,w,t) - S_1(x,w,t))D(x,t) \\
&\quad + x(D(x,w) - S(x,w))D_2(x,1,t), \\
D(x,w) &= (1+w)e^{S(x,w)} - 1, \\
S(x,w) &= x(D(x,w) - S(x,w))D(x,1).
\end{aligned}
$$

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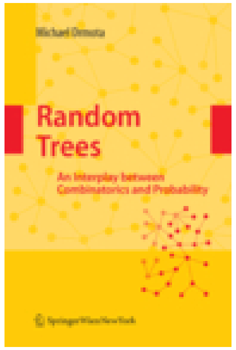

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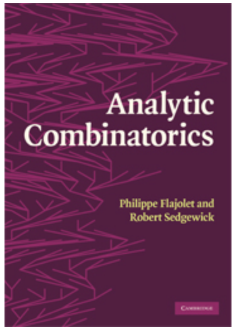

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Thank You!