The sum of digits of primes in $\mathbb{Z}[\mathrm{i}]$

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Abstract

We study the distribution of the complex sum-of-digits function s_q with basis $q = -a \pm \mathrm{i}$, $a \in \mathbb{Z}^+$ for Gaussian primes p. Inspired by a recent result of Mauduit and Rivat [16] for the real sum-of-digits function, we here get uniform distribution modulo 1 of the sequence $(\alpha s_q(p))$ provided $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and q is prime with $a \geq 28$. We also determine the order of magnitude of the number of Gaussian primes whose sum-of-digits evaluation lies in some fixed residue class mod m.

1 Preliminaries and Notation

Let $q = -a \pm \mathrm{i}$ (choose a sign) with $a \in \mathbb{Z}^+$ and denote $Q = |q|^2 = a^2 + 1$. Then every $z \in \mathbb{Z}[\mathrm{i}]$ has a unique finite representation

$$z = \sum_{j=0}^{\lambda-1} \varepsilon_j q^j,$$

where $\varepsilon_j \in \mathcal{N} = \{0, 1, \ldots, Q-1\}$ are the digits in the digital expansion and $\varepsilon_{\lambda-1} \neq 0$ (see [11, 12]). Denote by $s_q(n) = \sum_{j=0}^{\lambda-1} \varepsilon_j$ the sum-of-digits function in $\mathbb{Z}[\mathrm{i}]$. The aim of the present paper is to study the distribution of $s_q(p)$ in arithmetic progressions, where p runs through the Gaussian primes. The corresponding question for the real sum-of-digits function, posed by Gelfond [2] in a paper of 1968, has recently been answered by Mauduit and Rivat [16]. We resort to the method used in their paper, coupled with some known facts and techniques for $s_q(n)$, to get our distribution results. As a drawback, here we have to assume that the base q is prime. On our way we encounter a two-dimensional exponential sum over a disk, which is both linear in the real and imaginary part of the variable (Lemma 5.1). By the similarity to the circle problem the saving here cannot be too large, this – in the end – makes it impossible to cover general composite q. With much more effort one probably may cope with bases q whose smallest prime factor is not less than $|q|^\alpha$, for some number $0 < \alpha < 1$ (see [9]). In this paper, however, we restrict our estimates to the prime q case. By our reasoning we additionally have to assume $a \geq 28$. For later reference set

$$\mathcal{A} := \{a : \quad q = -a \pm \mathrm{i} \ \text{ prime}, \ \ a \geq 28\} = \{36,\ 40,\ 54,\ 56,\ 66,\ 74,\ 84,\ 90,\ 94, \ldots\}.$$

In what follows, let $g(n) = \mathrm{e}(f(n))$ with $f(n) = \alpha s_q(n)$, $\alpha \in \mathbb{R}$ and $\mathrm{e}(x) = \exp(2\pi \mathrm{i} x)$. Moreover, we always assume $m, n \in \mathbb{Z}[\mathrm{i}]$ and that sums run over the Gaussian integers $\mathbb{Z}[\mathrm{i}]$ unless nothing else is stated. The letter p always refers to a Gaussian prime. Denote by ζ_{i} the Dedekind zeta function of $\mathbb{Q}(\mathrm{i})$ defined for $\Re s > 1$ by

$$\Theta(s) := \zeta_{\mathbb{Q}(\mathrm{i})}(s) = \frac{1}{4} \sum_{n \neq 0} \frac{1}{|n|^{2s}}, \tag{1}$$

where summation runs over Gaussian integers $n \neq 0$. We will also make use of the complex Von Mangoldt function defined by $\Lambda_{\mathrm{i}}(\varepsilon p^{\nu}) = \log |p|$, and 0 for all other Gaussian integers n which cannot be written as $n = \varepsilon p^{\nu}$ with ε a unit, a Gaussian prime p and a positive exponent ν. Similarly, in the natural way, one defines the complex Möbius function $\mu_{\mathrm{i}}(n)$. The index "i" is used to stress the fact that we are working in $\mathbb{Z}[\mathrm{i}]$ instead of $\mathbb{Z}$. For a general introduction to Dedekind zeta-functions over number fields we refer to the monograph of Narkiewicz [17, Chapter 7].

2 Main results

The main contribution of the present paper is a non-trivial upper bound for an exponential sum involving both the complex Von Mangoldt function Λ_{i} and the sum-of-digits function s_q.

Theorem 2.1. *Let $a \in \mathcal{A}$. Then for any $\alpha \in \mathbb{R}$ with $(a^2+2a+2)\alpha \notin \mathbb{Z}$ there is $\sigma_q(\alpha) > 0$ such that*

$$\sum_{|n|^2 \leq N} \Lambda_{\mathrm{i}}(n)\, \mathrm{e}(\alpha s_q(n)) \ll N^{1-\sigma_q(\alpha)}, \tag{2}$$

where the implied constant depends only on a and α.

By partial summation and Weyl's criterion we retrieve from (2) the following uniform distribution result.

Theorem 2.2. *Let $a \in \mathcal{A}$. Then the sequence $(\alpha s_q(p))$, running over Gaussian primes p is uniformly distributed modulo 1 if and only if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.*

Theorem 2.1 also allows to determine the correct order of magnitude of the number of Gaussian primes whose sum-of-digits evaluation lies in some fixed residue class. Denote by $\pi_{\mathrm{i}}(N; b, d)$ the number of Gaussian primes $p \equiv b \bmod d$ with $|p|^2 \leq N$.

Theorem 2.3. *Let $a \in \mathcal{A}$ and $b, g \in \mathbb{Z}$, $g \geq 2$. Moreover, set $d = (g, a^2+2a+2)$ and $\delta = (d, 1 \mp \mathrm{i}(a+1))$, where the choice of the sign depends on the sign for $q = -a \pm \mathrm{i}$. Then there exists $\sigma_{q,g} > 0$ such that*

$$\#\left\{p \in \mathbb{Z}[\mathrm{i}] :\ |p|^2 \leq N,\ s_q(p) \equiv b \mod g\right\} = \frac{d}{g}\, \pi_{\mathrm{i}}(N; b, d/\delta) + O_{q,g}(N^{1-\sigma_{q,g}}). \tag{3}$$

For the sake of clearness, we append the straightforward proofs of Theorem 2.2 and 2.3.

Proof of Theorem 2.2. Recall that $g(n)=\mathrm{e}(\alpha s_q(n))$. We first claim that

$$\left|\sum_{|p|^2\le x} g(p)\right| \le \frac{4}{\log x}\max_{t\le x}\left|\sum_{|n|^2\le t}\Lambda_{\mathrm{i}}(n)g(n)\right| + O(\sqrt{x}). \tag{4}$$

By partial summation we have

$$\sum_{\sqrt{x}<|p|^2\le x} g(p) = \frac{2}{\log x}\sum_{\sqrt{x}<|p|^2\le x} g(p)\log|p| + 2\int_{\sqrt{x}}^{x}\sum_{\sqrt{x}<|p|^2\le t}\frac{g(p)\log|p|}{t(\log t)^2}\,\mathrm{d}t$$

and therefore get

$$\left|\sum_{|p|^2\le x} g(p)\right| \le \frac{4}{\log x}\max_{\sqrt{x}<t\le x}\left|\sum_{|p|^2\le t} g(p)\log|p|\right| + O(\sqrt{x}). \tag{5}$$

Moreover, for $\sqrt{x}<t\le x$,

$$\left|\sum_{|n|^2\le t}\Lambda_{\mathrm{i}}(n)g(n) - \sum_{|p|^2\le t} g(p)\log|p|\right| \le \sum_{|p|^2\le\sqrt{x}}\log|p|\cdot\sum_{2\le k\le\left\lfloor\frac{\log x}{2\log|p|}\right\rfloor} 1 = O(\sqrt{x}), \tag{6}$$

where the last equality holds by Hecke's prime number theorem for $\mathbb{Z}[\mathrm{i}]$ (see [8, p.126]). Now (4) follows by (5) and (6). If $\alpha\in\mathbb{Q}$ then $(\alpha s_q(p))$ is certainly not uniformly distributed mod 1. Let $\alpha\in\mathbb{R}\setminus\mathbb{Q}$. Then for any $h\in\mathbb{Z}$, $h\neq 0$, also $(a^2+2a+2)h\alpha\in\mathbb{R}\setminus\mathbb{Q}$, and Theorem 2.1 together with (4) yields

$$\sum_{|p|^2\le x}\mathrm{e}(h\alpha s_q(p)) = O_{q,h\alpha}(x^{1-\sigma_q(h\alpha)}) + O(\sqrt{x}).$$

The statement now follows from Weyl's criterion [13]. □

Proof of Theorem 2.3. Put $p=z_1+\mathrm{i}z_2$ and $g'=g/d$. We start with

$$\#\left\{|p|^2\le N,\ s_q(p)\equiv b \mod g\right\} = \sum_{|p|^2\le N}\frac{1}{g}\sum_{0\le j<g}\mathrm{e}\left(\frac{j}{g}\left(s_q(p)-b\right)\right).$$

Set $J=\{kg' :\ 0\le k<d\}$ and $J'=\{0,1,\dots,g-1\}\setminus J$. Since $d\mid(a^2+2a+2)$ we notice from [3, Corollary 2.3] that

$$s_q(z_1+\mathrm{i}z_2)\equiv z_1\pm(a+1)z_2 \mod d,$$

where the sign depends on the choice of the sign for $q=-a\pm\mathrm{i}$. Thus, for $j=kg'\in J$ we get

$$\mathrm{e}\left(\frac{j}{g}s_q(p)\right) = \mathrm{e}\left(\frac{k}{d}s_q(p)\right) = \mathrm{e}\left(\frac{k}{d}\left(z_1\pm(a+1)z_2\right)\right).$$

From this, we deduce

$$\begin{aligned}
&\sum_{|p|^2\le N}\frac{1}{g}\sum_{j\in J}\mathrm{e}\left(\frac{j}{g}\left(s_q(p)-b\right)\right)\\
&\qquad = \sum_{|p|^2\le N}\frac{1}{g}\sum_{k=0}^{d-1}\mathrm{e}\left(\frac{k}{d}(z_1\pm(a+1)z_2-b)\right)\\
&\qquad = \frac{d}{g}\#\left\{|p|^2\le N,\ p=z_1+\mathrm{i}z_2 :\ z_1\pm(a+1)z_2\equiv b \mod d\right\}.
\end{aligned}$$

Denote the last quantity by $\pi_{\mathrm{i}}'(N;b,d)$. Then, by including $j \in J'$,

$$\#\left\{|p|^2 \le N,\ s_q(p) \equiv b \mod g\right\} = \frac{d}{g}\,\pi_{\mathrm{i}}'(N;b,d) + \frac{1}{g}\sum_{j\in J'} \mathrm{e}\left(-\frac{bj}{g}\right) \sum_{|p|^2\le N} \mathrm{e}\left(\frac{j}{g}s_q(p)\right).$$

First, observe that $z_1 \pm (a+1)z_2 \equiv b \bmod d$ implies $z_2 \mp (a+1)z_1 \equiv \mp b(a+1) \bmod d$, where the simultaneous choice of the signs again only depends on the sign for $q = -a \pm \mathrm{i}$. Now, the identity

$$(z_1 + \mathrm{i}z_2)(1 \mp \mathrm{i}(a+1)) = z_1 \pm (a+1)z_2 + \mathrm{i}\,(z_2 \mp (a+1)z_1)$$

gives $(z_1 + \mathrm{i}z_2)(1 \mp \mathrm{i}(a+1)) \equiv b(1 \mp \mathrm{i}(a+1)) \bmod d$, which is equivalent to

$$(z_1 + \mathrm{i}z_2 - b)\cdot\frac{1 \mp \mathrm{i}(a+1)}{\delta} \equiv 0 \qquad \mod d/\delta,$$

where $\delta = (d, 1 \mp \mathrm{i}(a+1))$ denotes a greatest common divisor in $\mathbb{Z}[\mathrm{i}]$. Since $(1 \mp \mathrm{i}(a+1))/\delta$ and d/δ are coprime, this implies $\pi_{\mathrm{i}}'(N;b,d) = \pi_{\mathrm{i}}(N;b,d/\delta)$, which gives the main term in (3). As for the error term, we distinguish two cases on J'. If $J' = \{\}$, which means that $g \mid (a^2+2a+2)$, then the error term in (3) vanishes. Secondly, let $J' \neq \{\}$ and put $d' = (a^2+2a+2)/d$. Then $(d', g') = 1$ such that for $j = kg' + r \in J'$ we have $(a^2+2a+2)j/g \notin \mathbb{Z}$. Now, using Theorem 2.1 for all $j \in J'$ and taking the minimum of the associated exponents $\sigma_q(j/g)$ gives the statement. $\square$

The paper is organized as follows. In Section 3 we establish a Vaughan-type inequality which is basically the same as the one given for the real case [16]. Then, we use a van der Corput-type inequality for circle rings to split the multiplicative structure of the original exponential sum and get the associated difference process in Section 4. In Section 5 we define a truncated version of the sum-of-digit function, whose periodicity properties are studied with the help of the associated addition automaton [3]. This enables us in Section 6 to use Fourier analysis arguments in the upcoming estimates. The estimates on the trigonometric polynomials are then collected in Section 7, where we establish the bound for the type II-bounds in the Vaughan-type inequality (11). In Section 8 we give the treatment of the type I-sums (10), which in turn allows us to complete the proof of Theorem 2.1.

3 Inequalities à la Vaughan and van der Corput

This section is devoted to establish two inequalities which make up the core of the analytic method. To start with, by Euler's product formula and (1),

$$\Theta(s) = \prod_{\substack{p\in\mathbb{Z}[\mathrm{i}]\\ 0\le \arg p<\pi/2}} \left(1 - \frac{1}{|p|^{2s}}\right)^{-1}, \qquad \Re s > 1, \tag{7}$$

and therefore

$$\frac{1}{\Theta(s)} = \frac{1}{4}\sum_{n\neq 0}\frac{\mu_{\mathrm{i}}(n)}{|n|^{2s}}. \tag{8}$$

Also, taking formal (logarithmic) derivative in (7) (see [8]), we get

$$-\frac{\Theta'(s)}{\Theta(s)} = \frac{1}{2}\sum_{n\neq 0}\frac{\Lambda_{\mathrm{i}}(n)}{|n|^{2s}}, \qquad \Theta'(s) = -\frac{1}{2}\sum_{n\neq 0}\frac{\log|n|}{|n|^{2s}}, \tag{9}$$

for $\Re s > 1$. We now couple the Dirichlet series (8) and (9) with a combinatorial identity reminiscent of Vaughan to deduce

Lemma 3.1. *Let $\beta_1 \in (0, \frac{1}{3})$, $\beta_2 \in (\frac{1}{2}, 1)$ and $g:\ \mathbb{Z}[\mathrm{i}] \to \mathbb{C}$ an arbitrary function. Further suppose that for all complex numbers a_n, b_n with $|a_n|, |b_n| \leq 1$, $n \in \mathbb{Z}[\mathrm{i}]$ and all $M \leq x$ we uniformly have*

$$\sum_{\frac{M}{Q}<|m|^2\leq M}\ \max_{\frac{x}{Q|m|^2}<t\leq\frac{x}{|m|^2}}\left|\sum_{\frac{x}{Q|m|^2}<|n|^2\leq t} g(mn)\right| \leq U \qquad \textit{for} \quad M \leq x^{\beta_1} \quad (\textit{type I}), \tag{10}$$

$$\left|\sum_{\frac{M}{Q}<|m|^2\leq M}\ \sum_{\frac{x}{Q|m|^2}<|n|^2\leq\frac{x}{|m|^2}} a_m b_n g(mn)\right| \leq U \qquad \textit{for} \quad x^{\beta_1} \leq M \leq x^{\beta_2} \quad (\textit{type II}). \tag{11}$$

Then

$$\left|\sum_{\frac{x}{Q}<|n|^2\leq x} \Lambda_{\mathrm{i}}(n) g(n)\right| \ll U(\log x)^2.$$

Remark. It would be natural to consider also the distribution of s_q of Gaussian primes in angular regions. For this purpose, one uses Hecke's character $\chi_h(n) = \exp(4\mathrm{i}h \arg n)$ for $h \in \mathbb{Z}$ as for Hecke's prime number theorem [8]. Of course, the inequality (11) holds also in this case by uniformity in a_m and b_n. However, the internal sum for the type I-sum (10) now takes $\chi_h(n)g(mn)$ as its summands instead of $g(mn)$. As we will see later from the proof, we can choose $\beta_1 \neq 0$ arbitrarily small (of course, thus loosing on the exponent). It seems a difficult task to establish the "type I-estimate" for this mixing of multiplicative and q-additive properties.

Proof. The proof follows the same lines as in the real case [16]. For the sake of completeness we recall the main ingredients. We begin with a Vaughan-type identity in $\mathbb{Z}[\mathrm{i}]$ for $1 \leq u \leq y < |n|^2 \leq x$, namely,

$$-\frac{\Theta'}{\Theta} - F = -\Theta' G - \Theta F G + \Theta\left(\frac{1}{\Theta} - G\right)\left(-\frac{\Theta'}{\Theta} - F\right),$$

where differentiation is with respect to s and

$$F(s) = \frac{1}{2}\sum_{|n|^2\leq u}\frac{\Lambda_{\mathrm{i}}(n)}{|n|^{2s}}, \qquad G(s) = \frac{1}{4}\sum_{|n|^2\leq u}\frac{\mu_{\mathrm{i}}(n)}{|n|^{2s}}.$$

By (8) and (9) we obtain

$$\frac{1}{2}\cdot\sum_{\frac{x}{Q}<|n|^2\leq x}\Lambda_{\mathrm{i}}(n)g(n) = S_1 - S_2 + S_3,$$

where

$$S_1 = \frac{1}{8} \cdot \sum_{\substack{|m|^2 \le u \\ x/Q < |mn|^2 \le x}} \mu_{\mathrm{i}}(m) \, \log(|n|) \, g(mn), \tag{12}$$

$$S_2 = \frac{1}{32} \cdot \sum_{\substack{|m_1|^2 \le u \\ |m_2|^2 \le u \\ x/Q < |m_1 m_2 n|^2 \le x}} \mu_{\mathrm{i}}(m_1) \, \Lambda_{\mathrm{i}}(m_2) \, g(m_1 m_2 n), \tag{13}$$

$$S_3 = \frac{1}{32} \cdot \sum_{\substack{u < |m|^2 \le x \\ u < |n_1|^2 \le x \\ x/Q < |m n_1 n_2|^2 \le x}} \mu_{\mathrm{i}}(m) \, \Lambda_{\mathrm{i}}(n_1) \, g(m n_1 n_2). \tag{14}$$

We choose $y = x/Q$ and $u = x^{\beta_1}$. For (12) we use partial summation (similarly as in the proof of Theorem 2.2) and (10) to establish the bound $U(\log x)^2$ for S_1. Regarding (13), first notice that by (9),

$$\left| \sum_{\substack{|m_1|^2 \le u \\ |m_2|^2 \le u \\ m = m_1 m_2}} \mu_{\mathrm{i}}(m_1) \, \Lambda_{\mathrm{i}}(m_2) \right| \le \sum_{d | m} \Lambda_{\mathrm{i}}(d) \ll \log |m|.$$

We rearrange (13) with respect to sums over m and n, and split the summation over $|m|^2$ according to the powers of Q to get

$$|S_2| \ll (\log x)^2 \max_{M \le u^2} \sum_{\frac{M}{Q} < |m|^2 \le M} \left| \sum_{\frac{x}{Q|m|^2} < |n|^2 \le \frac{x}{|m|^2}} g(mn) \right|. \tag{15}$$

Denote by M_0 the value of M where the maximum in (15) is attained. If $M_0 < u$ then the bound for S_2 follows by using (10); if $u \le M_0 < \sqrt{x}$ then we use (11) to conclude. For $\sqrt{x} \le M_0 \le u^2$ we have $x^{\beta_1} \le x/M_0 \le x^{\beta_2}$ and for x sufficiently large also $x^{\beta_1} \le Qx/M_0 \le x^{\beta_2}$. Interchanging the rôles of m and n in (11) and using the cut-offs $a_m = 0$ if $|m|^2 > M_0$ or $|m|^2 \le M_0/Q$ also yields $S_2 \ll U(\log X)^2$. Finally, for (14), we write

$$S_3 = \frac{\log x}{32} \cdot \sum_{u < |m|^2 \le x/u} \mu_{\mathrm{i}}(m) \sum_{\frac{x}{Q|m|^2} < |n|^2 \le \frac{x}{|m|^2}} \left(\frac{1}{\log x} \sum_{\substack{u < |n_1|^2 \\ |n_2|^2 < |n|^2 \\ n = n_1 n_2}} \Lambda_{\mathrm{i}}(n_1) \right) g(mn)$$

Similarly as before, by splitting the summation over $|m|^2$, we here get the bound $U(\log x)^2$ for S_3 making usage of (11). $\square$

Proposition 3.2. *Let $(a_n)_{n \in \mathbb{Z}[\mathrm{i}]}$ be a sequence of complex numbers. Then for all positive integers N_0, N_1, N_2, N_3 with $N_0 \le N_1 < N_2 \le N_3$ we have*

$$\left| \sum_{\substack{N_1 < |n|^2 \le N_2 \\ n \in \mathbb{Z}[\mathrm{i}]}} a_n \right| \le \int_{-1/2}^{1/2} \min(N_2 - N_1, |\sin \pi \xi|^{-1}) \left| \sum_{\substack{N_0 < |n|^2 \le N_3 \\ n \in \mathbb{Z}[\mathrm{i}]}} a_n \, \mathrm{e}(|n|^2 \xi) \right| \mathrm{d}\xi.$$

Proof. The statement follows by observing

$$\sum_{\substack{N_1<|n|^2\le N_2\\ n\in\mathbb{Z}[\mathrm{i}]}} a_n = \int_{-1/2}^{1/2}\left(\sum_{\substack{N_0<|n|^2\le N_3\\ n\in\mathbb{Z}[\mathrm{i}]}} a_n\,\mathrm{e}(|n|^2\xi)\right)\cdot\left(\sum_{\substack{N_1<n'\le N_2\\ n'\in\mathbb{Z}}}\mathrm{e}(-n'\xi)\right)\,\mathrm{d}\xi,$$

and estimating the geometric series exactly as in Lemme 2 of [16] (see also [6, Lemma 5.2.3]). □

Lemma 3.3. *Let $\beta_1\in(0,\frac{1}{3})$, $\beta_2\in(\frac{1}{2},1)$, $0<\delta<\beta_1$ and suppose that for any numbers $b_n\in\mathbb{C}$ with $|b_n|\le 1$ we have*

$$\sum_{Q^{\mu-1}<|m|^2\le Q^{\mu}}\left|\sum_{Q^{\nu-1}<|n|^2\le Q^{\nu}} b_n g(mn)\right|\le V \qquad \text{for} \qquad \beta_1-\delta\le\frac{\mu}{\mu+\nu}\le\beta_2+\delta. \tag{16}$$

Then for $x>x_0:=\max\left(Q^{1/(1-\beta_2)},Q^{3/\delta}\right)$ and $x^{\beta_1}\le M\le x^{\beta_2}$ we get (11) with $U=\frac{12}{\pi}(1+\log 2x)V$.

Proof. This is Lemme 3 of [16] with q replaced by Q. □

Next, we need a variant of Van der Corput's inequality for circle rings (compare with [10, Lemma 8.17]).

Lemma 3.4. *Let $z_n\in\mathbb{C}$ with $n\in\mathbb{Z}[\mathrm{i}]$ and $A,B,R\in\mathbb{R}$ with $1<A<B$ and $R>1$. Then*

$$\left|\sum_{A<|n|<B} z_n\right|^2\le\frac{C_1}{R^2}\cdot(B-A+R+\min(A,R))\,(B+\max(A,R))\cdot$$
$$\cdot\sum_{|r|<2R}\left(1-\frac{|r|}{2R+1}\right)\sum_{A<|n|,|n+r|<B} z_{n+r}\overline{z_n},$$

where $C_1=\frac{16}{9}\pi(1+\sqrt{2})$.

Proof. Set $z_n=0$ if $|n|\notin(A,B)$. Then for fixed $r\in\mathbb{Z}[\mathrm{i}]$,

$$T:=\sum_{A<|n|<B} z_n=\sum_n z_n=\sum_n z_{n+r}.$$

Put $\hat{R}=\#\{0\le|r|<R\}$ and sum T over $0\le|r|<R$. Then

$$\hat{R}T=\sum_n\sum_{0\le|r|<R} z_{n+r}=\sum_{A-R<|n|<B+R}\ \sum_{0\le|r|<R} z_{n+r}. \tag{17}$$

First, suppose that $A-R\ge 0$. By the classical Gauss estimate for the number of lattice points in a disk (see [18, p.356]) we have

$$\begin{aligned}\#\{n\in\mathbb{Z}[\mathrm{i}]:\quad & A-R<|n|<B+R\}\\ &\le\pi(B+R)^2+2\sqrt{2}\pi(B+R)-\left(\pi(A-R)^2-2\sqrt{2}\pi(A-R)\right)\\ &\le(1+\sqrt{2})\pi(B+A)(B-A+2R).\end{aligned}$$

Applying the Cauchy-Schwarz inequality to (17) therefore gives

$$\begin{aligned}\hat{R}^2|T|^2 &\le \pi(1+\sqrt{2})(B+A)(B-A+2R)\sum_{0\le|r_1|,|r_2|<R}\sum_n z_{n+r_1}\overline{z_{n+r_2}}\\ &= \pi(1+\sqrt{2})(B+A)(B-A+2R)\sum_{|r|<2R} w(r)\sum_n z_{n+r}\overline{z_n},\end{aligned}$$

where

$$w(r) = \#\{(r_1,r_2)\in\mathbb{Z}[\mathrm{i}],\ 0\le|r_1|,|r_2|<R:\quad r_1-r_2=r\} \le (2R+1)(2R+1-|r|).$$

Finally, since $\frac{9}{4}R^2 \le \hat{R}$ and $(2R+1)^2 \le 9R^2$ for all $R>1$, we get the statement with $C_1 = \frac{16}{9}\,\pi(1+\sqrt{2})$. If $R\ge A$ then the factor $(B+A)(B-A+2R)$ has to be replaced by $(B+R)^2$. This completes the proof. □

4 Estimate of type II-sums

We turn back to the sum on the left-hand side of (16). For that purpose, set

$$S = \sum_{Q^{\mu-1}<|m|^2\le Q^\mu}\left|\sum_{Q^{\nu-1}<|n|^2\le Q^\nu} b_n \mathrm{e}(f(mn))\right|.$$

By Cauchy-Schwarz inequality we have

$$|S|^2 \le Q^\mu \sum_{Q^{\mu-1}<|m|^2\le Q^\mu}\left|\sum_{Q^{\nu-1}<|n|^2\le Q^\nu} b_n \mathrm{e}(f(mn))\right|^2.$$

Put $A=|q|^{\nu-1}$, $B=|q|^\nu$, $R=\frac{1}{3}|q|^\rho$ and $z_n = b_n\mathrm{e}(f(mn))$ in Lemma 3.4. Then, since

$$\left(\frac{|q|^\nu-|q|^{\nu-1}}{|q|^\rho}\right)\cdot\left(\frac{|q|^\nu+|q|^{\nu-1}}{|q|^\rho}\right) \ll Q^{\nu-\rho},$$

we get

$$\begin{aligned}|S|^2 \ll Q^{\mu+\nu-\rho} &\sum_{Q^{\mu-1}<|m|^2\le Q^\mu}\sum_{|r|<|q|^\rho}\left(1-\frac{|r|}{|q|^\rho}\right)\cdot\\ &\cdot\sum_{\substack{Q^{\nu-1}<|n|^2\le Q^\nu\\ Q^{\nu-1}<|n+r|^2\le Q^\nu}} b_{n+r}\overline{b_n}\,\mathrm{e}\left(f(m(n+r))-f(mn)\right).\end{aligned}$$

Removing the condition $Q^{\nu-1}<|n+r|^2\le Q^\nu$ gives an error $O(Q^\rho)$ to the internal sum, which in total yields an error of $O(Q^{2\mu+\nu+\rho})$ to $|S|^2$ which is negligible with respect to the $r=0$ instance of the sum above, provided that $\rho<\nu/2$ (note that $2\mu+\nu+\rho<2(\mu+\nu)-\rho$ if and only if $\rho<\nu/2$). Hence,

$$\begin{aligned}|S|^2 \ll Q^{2(\mu+\nu)-\rho} &+ Q^{\mu+\nu}\cdot\\ &\cdot\max_{1\le|r|<|q|^\rho}\sum_{Q^{\nu-1}<|n|^2\le Q^\nu}\left|\sum_{Q^{\mu-1}<|m|^2\le Q^\mu}\mathrm{e}\left(f(m(n+r))-f(mn)\right)\right|. \qquad (18)\end{aligned}$$

As for the next step, we need to show that the second summand is of lower order with respect to the first summand. To that aim, we use a function related to a "truncated version" of the sum-of-digits function for $\mathbb{Z}[\mathrm{i}]$, namely,

$$f_\lambda(z) = \sum_{j=0}^{\lambda-1} f(\varepsilon_j q^j) = \alpha \sum_{j=0}^{\lambda-1} \varepsilon_j,$$

where $\lambda \in \mathbb{Z}$ and $\lambda \geq 0$. To proceed, we have to make clear, what periodicity means for f_λ in the Gaussian integers. Note that Proposition 4.1 and Lemma 4.2 hold true for any $a \geq 1$.

Proposition 4.1. *The function $f_\lambda(z)$ is periodic with period q^λ, i.e. for any $d \in \mathbb{Z}[\mathrm{i}]$,*

$$f_\lambda(z + dq^\lambda) = f_\lambda(z), \qquad z \in \mathbb{Z}[\mathrm{i}].$$

Proof. This comes from the construction of the digits according to Propositions 2.1 and 2.2 of [3]. Similar to the real case, the digits in the Gaussian number system can be constructed in the way $\varepsilon_0, \varepsilon_1, \ldots,$ where a possible overflow is always carried to the next (higher-placed) digit. For the proof of Proposition 4.1, we have to ensure that with $d = x + \mathrm{i}y$ the addition of xq^λ and $\mathrm{i}yq^\lambda$ to $z = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_N, 0, 0 \ldots)$ does not affect the digits $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{\lambda-1}$. If $x \geq 0$ then a possible overflow $x \geq Q$ at position λ can be cleared by transporting it to the three adjacent higher-placed digits. More specifically, since we have the identity $Q = (a-1)^2 q + (2a-1)q^2 + q^3$, we add $\lfloor (x+\varepsilon_\lambda)/Q \rfloor$ times the carry $((a-1)^2, (2a-1), 1)$ to the positions $(\lambda+1, \lambda+2, \lambda+3)$ of z. On the other hand, if $x < 0$, then by $-Q = q^2 + 2aq$ we add $(a^2, 2a)$ a number of $\lfloor (x+\varepsilon_\lambda)/Q \rfloor$ times to the positions $(\lambda+1, \lambda+2)$. If $y \geq 0$, then by $\mathrm{i}q = aq + q^2$ we add $(a, 1)$ to the digits at positions λ and $\lambda+1$. Finally, if $y < 0$, then we first subtract $(a, 1)$ from these positions and then use carry propagation with the help of $-Q = q^2 + 2aq$ to make digits positive. In all cases, the digits $\varepsilon_0, \varepsilon_1, \ldots \varepsilon_{\lambda-1}$ remain unchanged. This completes the proof. $\square$

The next lemma shows that we may alter the difference process in (18) for the sum-of-digits function with the help of the truncated sum-of-digits function at only a small cost. First, we recall the addition automaton for $q = -a + \mathrm{i}$ obtained in [3, Figure 2]. A similar automaton also exists for $q = -a - \mathrm{i}$. For the sake of clearness we restrict the investigation to the former case.

The addition automaton for $q = -a + \mathrm{i}$ (Figure 1) performs addition by 1 (start at node $\mathbf{P}$), by $-a - \mathrm{i}$ (start at node $\mathbf{R}$) and by $a - 1 + \mathrm{i}$ (start at node $\mathbf{Q}$), respectively. The carry propagation as well as the construction of the corresponding digits of the sum is associated to a walk in the automaton which finishes in one of the two accepting states $[\bullet]$. The labelling $j|k$ means that the automaton reads a digit j and has k as the corresponding output. For illustration, take $a = 36$ and $z = -48852 + 3987\mathrm{i} = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3) = (1296, 100, 0, 1)$, and consider $z + 1$. Then, the corresponding walk is

$$\mathbf{P} \xrightarrow{1296|0} \mathbf{R} \xrightarrow{100|28} -\mathbf{P} \xrightarrow{0|1296} -\mathbf{R} \xrightarrow{1|73} \mathbf{P} \xrightarrow{0|1} [\bullet],$$

thus $z + 1 = (0, 28, 1296, 73, 1)$, which has one more non-zero digit with respect to z. The next lemma ("carry lemma") shows that for most numbers $z \in \mathbb{Z}[\mathrm{i}]$ carry propagation is a "local" phenomenon. For the sequel, set

$$\lambda = \mu + 2\rho. \tag{19}$$

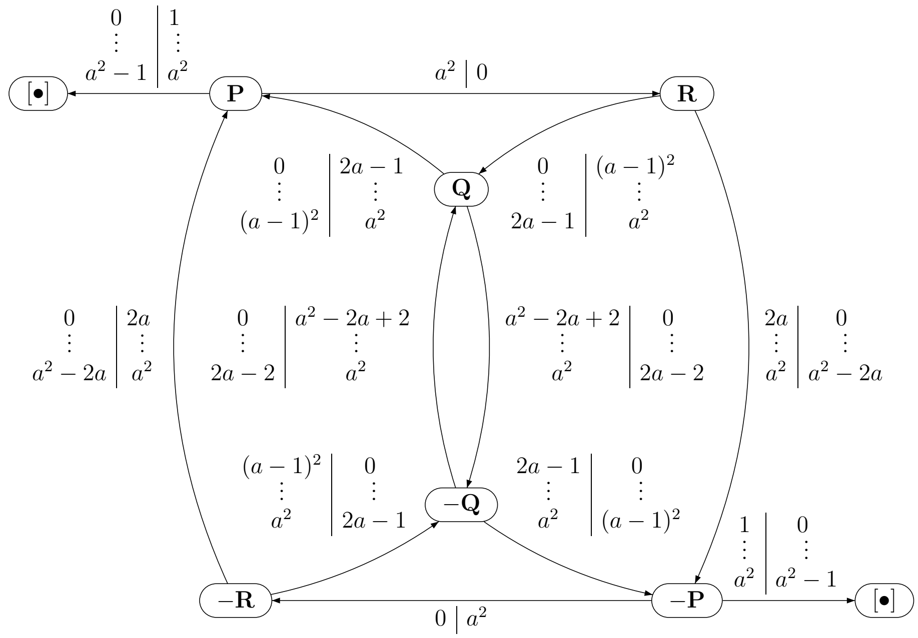

Figure 1: Addition automaton for $z \in \mathbb{Z}[\mathrm{i}]$ with respect to $q = -a + \mathrm{i}$.

Despite the fact, that (19) gives the same value for the margin of truncation as in the real case [16, Lemme 5], here the summation works differently and our reasoning crucially depends on the structure of the automaton given in Figure 1.

Lemma 4.2. *For all integers $\mu > 0$, $\nu > 0$, $0 \le \rho < \nu/2$ and $r \in \mathbb{Z}[\mathrm{i}]$ with $|r|^2 < Q^\rho$ denote by $E(r, \mu, \nu, \rho)$ the number of pairs $(m, n) \in \mathbb{Z}[\mathrm{i}] \times \mathbb{Z}[\mathrm{i}]$ with $Q^{\mu-1} < |m|^2 \le Q^\mu$, $Q^{\nu-1} < |n|^2 \le Q^\nu$ and*

$$f(m(n+r)) - f(mn) \neq f_\lambda(m(n+r)) - f_\lambda(mn). \tag{20}$$

Then for any $\varepsilon > 0$ it holds

$$E(r, \mu, \nu, \rho) \ll_\varepsilon Q^{(\mu+\nu)(1+\varepsilon)-\rho}.$$

Proof. It is a well-known fact [3, Proposition 2.6] that $z \in \mathbb{Z}[\mathrm{i}]$ has

$$2 \log_Q |z| + \theta(z) \tag{21}$$

digits with respect to the digital expansion in $\mathbb{Z}[\mathrm{i}]$, where $\theta : \mathbb{Z}[\mathrm{i}] \to \mathbb{R}$ with $\theta(z) = O(1)$ and the implied constant only depends on a. In the sequel we have to keep track of the carry propagation while performing the addition $mn + mr$. To begin with, since $|mr|^2 < Q^{\mu+\rho}$, we see that the number mr has at most $\mu + \rho + O(1)$ digits. We perform the addition $mn + mr$ in two steps.

(1) *Addition of the lower $\mu + \rho + O(1)$ digits of mn ("lower truncated part") to mr:*

We add mr to the lower truncated version of mn, i.e., where we cut off those digits of mn which are higher placed than digit-position $\mu + \rho + O(1)$. Then by (21) there can only be a carry propagation of $O(1)$ with respect to the digits $> \mu + \rho + O(1)$. Without loss of generality, we denote this overflow by

$$x + \mathrm{i}y = -y(-a - \mathrm{i}) + (x + ay) \qquad \text{with } y < 0, \quad x > -ay$$

(all other cases are similar). Since $x + \mathrm{i}y = O(1)$, also $x = O(1)$ and $y = O(1)$.

(2) *Addition of the higher-placed digits of mn ("upper truncated part") to mr:*

The difference $f(mn+mr)-f(mn)$ differs from $f_\lambda(mn+mr)-f_\lambda(mn)$ if and only if $x+\mathrm{i}y$ gives rise to an overflow which is transported over digit place λ. For the sake of simplicity, write $\lceil mn \rceil$ for the upper truncated version of mn, i.e., where the lower $\mu+\rho+O(1)$ digits equal zero. We use an idea of Grabner, Kirschenhofer and Prodinger [3, Proposition 2.4]. Since addition by 1 and $-a-\mathrm{i}$ can be handled by the automaton, we are led to the following telescoping sum,

$$\begin{aligned} f(\lceil mn \rceil + x + \mathrm{i}y) - f(\lceil mn \rceil) = & \qquad (22)\\ & f(\lceil mn \rceil + x + \mathrm{i}y) - f(\lceil mn \rceil + (a+\mathrm{i}) + x + \mathrm{i}y)\\ &+ f(\lceil mn \rceil + (a+\mathrm{i}) + x + \mathrm{i}y) - f(\lceil mn \rceil + 2(a+\mathrm{i}) + x + \mathrm{i}y) + \dots\\ &+ f(\lceil mn \rceil - (y+1)(a+\mathrm{i}) + x + \mathrm{i}y) - f(\lceil mn \rceil - y(a+\mathrm{i}) + x + \mathrm{i}y)\\ &+ f(\lceil mn \rceil - y(a+\mathrm{i}) + x + \mathrm{i}y) - f(\lceil mn \rceil - y(a+\mathrm{i}) + x + \mathrm{i}y - 1)\\ &+ f(\lceil mn \rceil - y(a+\mathrm{i}) + x + \mathrm{i}y - 1) - f(\lceil mn \rceil - y(a+\mathrm{i}) + x + \mathrm{i}y - 2) + \dots\\ &+ f(\lceil mn \rceil - y(a+\mathrm{i}) + x + \mathrm{i}y - x - ay + 1) - f(\lceil mn \rceil). \end{aligned}$$

By the previous observation, the number of differences in the sum above is $O(1)$, uniformly for m in the given range. Next, consider (22) with f replaced by f_λ and take an arbitrary summand $f_\lambda(z) - f_\lambda(z+c)$ with $c = a+\mathrm{i}$ or $c = +1$. If (20), then at least in one of the summands the addition $z \mapsto z+c$ gives rise to a carry propagation which is transported over digit-place λ. In order to estimate $E(r,\mu,\nu,\rho)$ we enlarge the set of products $A := mn$ from $Q^{\mu+\nu-2} < |A|^2 \le Q^{\mu+\nu}$ to the set of $A \in \mathbb{Z}[\mathrm{i}]$ whose digital expansion involves $\mu+\nu+O(1)$ digits, i.e., where each of the digits takes values from $\{0,1,\dots,a^2\}$. Consider the addition transducer and take, for instance, $c = a+\mathrm{i}$. Then the number of paths of length M which start from node $[-\mathbf{R}]$ and do not end in one of the two terminal states $[\bullet]$ is $O(|\xi|^M)$, where ξ denotes the eigenvalue with largest modulus of the transition matrix (see [4, Proposition 1]), where the two terminal states are removed. This matrix has characteristic polynomial

$$(u^2+2au+a^2+1)(u-1)(u^3-(2a-1)u^2-(a-1)^2u-a^2-1),$$

such that $|\xi| < (1+\sqrt{2})a$ for $a \ge 2$ and $|\xi| < 2\sqrt{2}-1$ for $a=1$. In particular, we have

$$Q > \xi_a \quad \text{for all } a \ge 1.$$

Therefore, by setting $M=\rho$ and taking into account the number of divisors of A, we conclude that

$$E(r,\mu,\nu,\rho) \ll \frac{Q^{\mu+\nu}}{Q^\rho - \xi_a^\rho} \sum_{A \text{ with } \mu+\nu+O(1) \text{ digits}} \tau(A) \ll_\varepsilon Q^{(\mu+\nu)(1+\varepsilon)-\rho},$$

for $a \ge 1$. This finishes the proof.

$\square$

5 Transformation of S with f_λ

Replacing f by f_λ gives a total error of $O(Q^{2(\mu+\nu)(1+\varepsilon)-\rho})$ to $|S|^2$. Thus,

$$|S|^2 \ll_\varepsilon Q^{2(\mu+\nu)(1+\varepsilon)-\rho} + Q^{\mu+\nu} \max_{1\le|r|<|q|^\rho} S_2(r,\mu,\nu,\rho), \tag{23}$$

where

$$S_2(r,\mu,\nu,\rho) = \sum_{Q^{\nu-1}<|n|^2\le Q^\nu} |S_2'(n)| \tag{24}$$

with

$$S_2'(n) = \sum_{Q^{\mu-1}<|m|^2\le Q^\mu} \mathrm{e}\left(f_\lambda(m(n+r)) - f_\lambda(mn)\right).$$

It remains to show that $S_2(r,\mu,\nu,\rho) \ll Q^{(\mu+\nu)(1+\varepsilon)-\rho}$. Denote by $\mathcal{F}_\lambda = \{\sum_{j=0}^{\lambda-1} \varepsilon_j q^j : \varepsilon_j \in \mathcal{N}\}$ the finite (non-scaled) approximation of the fundamental region of the number system, which is obviously a complete system of residues mod q^λ with $\#\mathcal{F}_\lambda = Q^\lambda$. From these observations we have

$$\sum_{z\in\mathcal{F}_\lambda} \mathrm{e}\left(\frac{1}{2}\,\mathrm{tr}\,(hzq^{-\lambda})\right) = \begin{cases} Q^\lambda, & h \equiv 0 \bmod q^\lambda; \\ 0, & \text{otherwise,} \end{cases} \tag{25}$$

where $\mathrm{tr}\,(z) = z + \bar{z} = 2\Re(z)$. Consequently,

$$S_2'(n) = \sum_{h\in\mathcal{F}_\lambda} \sum_{k\in\mathcal{F}_\lambda} F_\lambda(h,\alpha)\overline{F_\lambda(-k,\alpha)}\ \ S_2''\left(\mu, \frac{(h+k)n+hr}{q^\lambda}\right), \tag{26}$$

where

$$F_\lambda(h,\alpha) = Q^{-\lambda} \sum_{u\in\mathcal{F}_\lambda} \mathrm{e}\left(\alpha s_q(u) - \frac{1}{2}\,\mathrm{tr}\,(huq^{-\lambda})\right) \qquad \text{and}$$

$$S_2''(\mu,\xi) = \sum_{Q^{\mu-1}<|m|^2\le Q^\mu} \mathrm{e}\left(\frac{1}{2}\,\mathrm{tr}\,(m\xi)\right), \quad \xi \in \mathbb{C}.$$

In order to proceed, we first need a tight uniform upper bound for $S_2''(\mu, z/q^\lambda)$ with $z \in \mathbb{Z}[\mathrm{i}]$. To this end, we introduce some more notation,

$$\tau(z) = \max(\|\Re z\|, \|\Im z\|)^{-1}, \quad z \in \mathbb{C}, \tag{27}$$

where, as usual, $\|\cdot\|$ denotes the "distance to the nearest integer"-function. Moreover, we write

$$\{z\} := z \bmod (1+\mathrm{i}), \tag{28}$$

meaning that both the real and imaginary part of $z \in \mathbb{C}$ are reduced modulo 1. Obviously,

$$\tau(z) = \tau(\{z\}) = \tau(\varepsilon z) \quad \text{for} \quad \varepsilon \in \{\pm 1, \pm \mathrm{i}\}. \tag{29}$$

For $|z| \le 1/2$ we have $\frac{|z|}{\sqrt{2}} \le \tau(z)^{-1} \le |z|$ and therefore

$$\tau(\alpha) \ge \sqrt{2}\,|z|\,\tau(z\{\alpha\}), \quad \alpha \in \mathbb{C}. \tag{30}$$

The following lemma lies at the heart of the exponential sum estimate.

Lemma 5.1. *Let $q=-a\pm\mathrm{i}$ with $a\in\mathbb{Z}^+$, $\lambda\geq 0$ and $h\in\mathbb{Z}[\mathrm{i}]$. Then*

$$\sum_{|z|^2<N} \mathrm{e}\left(\frac{1}{2}\operatorname{tr}\left(\frac{hz}{q^\lambda}\right)\right) \ll \min\left(N, \tau\left(\frac{h}{q^\lambda}\right) N^{1/2}\right). \tag{31}$$

Proof. Put $z=z_1+\mathrm{i}z_2$, thus

$$\operatorname{tr}\left(\frac{hz}{q^\lambda}\right) = 2\left(z_1 r - z_2 s\right) \quad \text{with} \quad r=\Re\left(\frac{h}{q^\lambda}\right), \quad s=\Im\left(\frac{h}{q^\lambda}\right).$$

Without loss of generality assume that $\{s\}\neq 0$ (the degenerate cases are trivial). Then

$$\left|\sum_{|z|^2<N} \mathrm{e}\left(\frac{1}{2}\operatorname{tr}\left(\frac{hz}{q^\lambda}\right)\right)\right| = \left|\sum_{\substack{z_1=-\sqrt{N}\\ z_1\in\mathbb{Z}}}^{\sqrt{N}} \sum_{\substack{z_2=-\sqrt{N-z_1^2}\\ z_2\in\mathbb{Z}}}^{\sqrt{N-z_1^2}} \mathrm{e}\left(r z_1 - s z_2\right)\right| \tag{32}$$

$$= \left|\sum_{\substack{z_1=-\sqrt{N}\\ z_1\in\mathbb{Z}}}^{\sqrt{N}} \mathrm{e}(r z_1)\cdot \frac{\sin\left(\pi s\left(2\left\lfloor\sqrt{N-z_1^2}\right\rfloor+1\right)\right)}{\sin(\pi s)}\right| \ll \frac{\sqrt{N}}{|\sin(\pi s)|}.$$

Interchanging r and s and taking the minimum of the two bounds gives the statement. □

Remark. Another bound is obtained by making explicit a classical argument due to Landau [14, 15], namely,

$$\sum_{|z|^2<N} \mathrm{e}\left(\frac{1}{2}\operatorname{tr}\left(\frac{hz}{q^\lambda}\right)\right) \ll \min\left(N, N^{1/3}+\tau\left(\frac{h}{q^\lambda}\right)^{3/2} N^{1/4}\right). \tag{33}$$

However, extracting the main term over arithmetic progressions (which is our next task) here would give a large error term (due to the exponent 3/2), which – as we checked by employing the same approach as in the present paper – would give a more restrictive condition on a. On the other hand, it is the factor $N^{1/2}$ in (31) which indeed makes it impossible to handle general composite digital bases q by this method. For the general q we would need the factor N^ε, $\varepsilon>0$ here, which is not possible by comparing the left hand side of (31) to the classical circle problem.

Rewriting (26) with the aid of (27) now yields

$$|S_2'(n)| \ll \sum_{h\in\mathcal{F}_\lambda}\sum_{k\in\mathcal{F}_\lambda} |F_\lambda(h,\alpha)|\cdot|F_\lambda(-k,\alpha)|\cdot \min\left(Q^\mu, \tau\left(\frac{(h+k)n+hr}{q^\lambda}\right) Q^{\mu/2}\right). \tag{34}$$

In order to estimate further, we need to split off the leading term of the sum over the min-terms over arithmetic progressions in (34). Denote by $d=(c,m)$ the greatest common divisor of $c,m\in\mathbb{Z}[\mathrm{i}]$ multiplied by some (arbitrary) unit $\{\pm1,\pm\mathrm{i}\}$. Recall the notation of (27), (28) and the relation (29).

Lemma 5.2. *Let $m \in \mathbb{Z}[\mathrm{i}]$, $|m| \geq 1$, $c \in \mathbb{Z}[\mathrm{i}]$, $b \in \mathbb{C}$, $M_1, M_2 \in \mathbb{R}^+$ and set $d = (c, m)$. Then*

$$\hat{S} := \sum_{|n|^2<|m|^2} \min\left(M_1, \tau\left(\frac{cn+b}{m}\right) M_2\right)$$
$$\ll |d|^2 \min\left(M_1, \tau\left(\frac{\{b/d\}}{m/d}\right) M_2\right) + |m|^2 M_2 \log \frac{|m|}{|d|}. \tag{35}$$

Proof. If $|d| = |m|$ then $\tau\left(\frac{cn+b}{m}\right) = \tau\left(\left\{\frac{b}{d}\right\}\right)$, and the statement is obvious. Let $|d| \neq |m|$, then $1 \leq |d|^2 \leq |m|^2/2$. Set

$$c' = c/d, \quad m' = m/d \quad \text{and} \quad b = b'd + r,$$

where $b' \in \mathbb{Z}[\mathrm{i}]$ and $r \in \mathbb{C}$. It is well-known [7, Theorem 216] that we can choose b' in such a way that $|r| \leq \frac{\sqrt{2}}{2}|d|$. Consider a complete residue system $R = \{\hat{r} \mod m'\}$ with $\#R = |m'|^2$ with each $|\hat{r}| \leq \frac{\sqrt{2}}{2}|m'|$. While completely tessellating the disk $|n|^2 < (|m| + |m'|)^2$ by translates of R, we observe that there are at most $O(|d|^2)$ such translations by plain point counting. Moreover, if n runs through the residue system R, then also $c'n + b'$ does. Therefore,

$$\hat{S} \ll |d|^2 \sum_{n\in R} \min\left(M_1, \tau\left(\frac{n + r/d}{m'}\right) M_2\right). \tag{36}$$

Next, we extract the main term of (36), which corresponds to those values of $n \in R$, where at least one of $\Re\left(\frac{n+r/d}{m'}\right)$ and $\Im\left(\frac{n+r/d}{m'}\right)$ is close to an integer. Observe that since $n \in R$ we have $\left|\frac{n}{m'}\right| \leq \frac{\sqrt{2}}{2}$. Moreover, if $n \in R$ then also $\varepsilon n \in R$ with $\varepsilon \in \{\pm 1, \pm \mathrm{i}\}$, hence we may without loss of generality choose a sign of $d = (c, m)$ in a way that $0 \leq \min\left(\Re\left(\frac{n}{m'}\right), \Im\left(\frac{n}{m'}\right)\right) \leq \frac{1}{2}$. Thus, by $\left|\frac{r}{dm'}\right| \leq \frac{1}{2}$, the general main term comes from the Gaussian integers n, where $\min\left(\Re\left(\frac{n+r/d}{m'}\right), \Im\left(\frac{n+r/d}{m'}\right)\right)$ is most closely to 0 or 1. Denote these values by $\mathfrak{N} = \{n_k : k = 1, \ldots, 5\}$, where n_1 is the value associated to the closest point. Then, by the method of trapezia (note that the points are well-spaced with pairwise distance $\geq 1/|m'|$),

$$\hat{S} \ll |d|^2 \left(\sum_{\substack{n\in R\\ n\in\mathfrak{N}}} + \sum_{\substack{n\in R\\ n\notin\mathfrak{N}}}\right) \min\left(M_1, \tau\left(\frac{n + r/d}{m'}\right) M_2\right)$$
$$\ll |d|^2 \left(\min\left(M_1, \tau\left(\frac{n_1 + r/d}{m'}\right) M_2\right) + |m'|^2 \int_{\frac{1}{|m'|}}^{1-\frac{1}{|m'|}} \frac{M_2}{\sin \pi x}\, \mathrm{d}x\right). \tag{37}$$

For the second summand in (37) we have

$$|d|^2|m'|^2 \int_{\frac{1}{|m'|}}^{1-\frac{1}{|m'|}} \frac{M_2}{\sin \pi x}\, \mathrm{d}x \ll |m|^2 M_2 \log \cot \frac{\pi|d|}{2\,|m|},$$

which gives the second term in (35). For the first term we note by definition of n_1 that

$$\tau\left(\frac{n_1 + r/d}{m'}\right) \leq \tau\left(\frac{r/d}{m'}\right). \tag{38}$$

Since $r/d = b/d + b'$ with $|r/d| \leq \frac{\sqrt{2}}{2}$ implies $r/d = \{b/d\}$, we get the statement of the lemma. $\square$

We turn our attention again back to the sum $S_2(r,\mu,\nu,\rho)$ defined in (24). Recall that we want to show that $S_2(r,\mu,\nu,\rho) \ll Q^{(\mu+\nu)(1+\varepsilon)-\rho}$. Denote by R_d a complete residue system mod d with each element being $\le \frac{\sqrt{2}}{2}|d|$. In order to transform the condition $Q^{\nu-1} < |n|^2 \le Q^\nu$ in (24) into $0 \le |n|^2 < Q^\lambda$, we proceed in a similar way as in the proof of Lemma 5.2: We tessellate the circular ring $Q^{\nu-1} < |n|^2 \le Q^\nu$ by translates of R_{q^λ}; if $\lambda < \nu$ then the number of translates is bounded by

$$\ll \frac{\pi}{Q^\lambda}\left((|q|^\nu + \frac{\sqrt{2}}{2}\,|q|^\lambda)^2 - (|q|^{\nu-1} - \frac{\sqrt{2}}{2}\,|q|^\lambda)^2\right) \ll Q^{\nu-\lambda},$$

while in the case $\lambda \ge \nu$ we estimate the number trivially by 1. By (34) and Lemma 5.1 and 5.2 we therefore obtain

$$\begin{aligned} S_2(r,\mu,\nu,\rho) \ll (1+Q^{\nu-\lambda}) \sum_{d|q^\lambda} \sum_{h\in\mathcal{F}_\lambda} \sum_{\substack{k\in\mathcal{F}_\lambda \\ (h+k,q^\lambda)=d}} & |F_\lambda(h,\alpha)| \cdot |\overline{F_\lambda(-k,\alpha)}| \cdot \\ & \cdot |d|^2 \min\left(Q^\mu, \tau\left(\frac{\{hr/d\}}{q^\lambda/d}\right) Q^{\mu/2}\right) \\ + \lambda(1+Q^{\nu-\lambda})Q^{\lambda+\mu/2}(\log Q) \cdot & \sum_{h\in\mathcal{F}_\lambda}\sum_{k\in\mathcal{F}_\lambda} |F_\lambda(h,\alpha)| \cdot |\overline{F_\lambda(-k,\alpha)}|. \end{aligned}$$

With the notion of

$$G_\lambda(b,d,\alpha) = \sum_{\substack{h\in\mathcal{F}_\lambda \\ h\equiv b \bmod d}} |F_\lambda(h,\alpha)|, \qquad G_\lambda(\alpha) = G_\lambda(0,1,\alpha) = \sum_{h\in\mathcal{F}_\lambda} |F_\lambda(h,\alpha)| \tag{39}$$

this gives

$$\begin{aligned} S_2(r,\mu,\nu,\rho) \ll (1+Q^{\nu-\lambda}) \sum_{d|q^\lambda} |d|^2 \cdot \sum_{b\in R_d} \min\left(Q^\mu, \tau\left(\frac{\{br/d\}}{q^\lambda/d}\right) Q^{\mu/2}\right) \cdot G_\lambda(b,d,\alpha)^2 \\ + \lambda(1+Q^{\nu-\lambda})Q^{\lambda+\mu/2}(\log Q) \cdot G_\lambda(\alpha)^2. \end{aligned} \tag{40}$$

The main task is now to bound each of (39) in (40) for certain $d \mid q^\lambda$ and $b \in R_d$ by means of classical Fourier analysis. Of course, since q is prime we only have to deal with the divisors $d = q^\delta$ with $0 \le \delta \le \lambda$.

6 Fourier analysis of F_λ

In this section we are first concerned with some preliminary results about the functions F_λ. As for a second step, we will focus on estimates of the trigonometric polynomials appearing in (40). To begin with, since $F_\lambda(\cdot,\alpha)$ is periodic with period q^λ and $F_0(h,\alpha) = 1$, we have by induction

$$|F_\lambda(h,\alpha)| = Q^{-\lambda} \prod_{j=1}^{\lambda} \varphi_Q\left(\alpha - \frac{1}{2}\operatorname{tr}(hq^{-j})\right), \tag{41}$$

where

$$\varphi_Q(t) = \begin{cases} |\sin(\pi Q t)|/|\sin(\pi t)|, & t \in \mathbb{R}\setminus\mathbb{Z}; \\ Q, & t \in \mathbb{Z}. \end{cases}$$

First, a few useful observations are in order. From (41) we easily see that for any $0 \leq \lambda_1 \leq \lambda$,

$$|F_\lambda(q^{\lambda_1}b,\alpha)| \leq |F_{\lambda-\lambda_1}(b,\alpha)|. \tag{42}$$

We shall exploit this property in Section 7. By local expansion we also have

$$\frac{1}{Q}\,\varphi_Q(t) \leq \exp(-C'_a\|t\|^2), \tag{43}$$

for some constant C'_a only depending on a. Define

$$\psi_{Q,R}(t) = \frac{1}{Q}\sum_{0\leq r<R}\varphi_Q\left(t+\frac{ar}{R}\right) \qquad \text{and} \qquad \psi_Q(t) = \psi_{Q,Q}(t), \tag{44}$$

where the summation is intended over $r \in \mathbb{Z}$. Note that $\psi_Q(t)$ $(a \geq 1)$ is the same as the function $\psi_Q(t)$ in [16, Lemme 14], since by $(a, a^2+1) = 1$ both $\{r\}$ and $\{ar\}$ run through a complete set of residues mod Q.

Lemma 6.1. *For $a \in \mathbb{Z}^+$, $a \geq 1$ we have*

$$\sum_{0\leq r<Q}\varphi_Q^2(t+\frac{1}{2}\,\mathrm{tr}\,(r/q)) = Q^2.$$

Proof. The statement follows by [16, Lemme 13] and $\frac{1}{2}\,\mathrm{tr}\,(r/q) = -ar/Q$. □

Lemma 6.2. *The function $\psi_Q(t)$ is continuous and periodic with period $1/Q$. Define*

$$\eta_Q = \frac{\log\left(\max_{t\in\mathbb{R}}\psi_Q(t)\right)}{\log Q}, \qquad a \geq 1.$$

Then for $a \geq 27$ we have $0 < \eta_Q \leq \eta_{730} < 0.25$. Moreover, for $3 \leq R \leq Q$ with $R \mid Q$ we have

$$\max_{t\in\mathbb{R}}\psi_{Q,R}(t) \leq R^{\eta_R}, \qquad \textit{where} \quad \eta_R \leq \eta_3 \leq 0.465. \tag{45}$$

Proof. By [16, Lemme 15] we have

$$\max_{t\in\mathbb{R}}\psi_Q(t) = \psi_Q\left(\frac{1}{2Q}\right) \leq \frac{2}{Q\sin\frac{\pi}{2Q}} + \frac{2}{\pi}\log\frac{2Q}{\pi}, \tag{46}$$

for any $Q \in \mathbb{Z}^+$, $Q \geq 2$. It is a straightforward computation to show that for $Q = 27^2+1 = 730$ the bound on the right hand side of (46) implies $\eta_{730} < 0.24957 < 0.25$. This finishes the proof of the first part since for $a \geq 28$ we also have $\eta_Q < \eta_{730}$ by (46). A direct calculation indeed shows that for $a = 26$ the corresponding exponent is $\eta_Q > 0.25027 > 0.25$. The second part of the statement is [16, Lemme 16]. □

Next, we establish a uniform bound for $|F_\lambda(h,\alpha)|$, which will be of particular use in estimating the sum involving $G_\lambda(b,d,\alpha)^2$ with the multiplicative perturbation of the τ-function in (40). Note that this method of proof provides a more direct way to proving the corresponding result in the real case [16, Lemme 22].

Lemma 6.3. *Let $\alpha \in \mathbb{R}$, $\xi \in \mathbb{C}$, $\lambda \in \mathbb{Z}^+$ with $\lambda \geq 3$. Then for all $a \geq 3$ we have*

$$\sum_{j=0}^{\lambda-1} \left\| \alpha - \frac{1}{2} \operatorname{tr}(\xi q^j) \right\|^2 \geq \frac{\lambda - 2}{2(a^2+1)^2} \left\| (a^2+2a+2)\alpha \right\|^2 .$$

Similarly, for $a = 2$ we have

$$\sum_{j=0}^{\lambda-1} \left\| \alpha - \frac{1}{2} \operatorname{tr}(\xi q^j) \right\|^2 \geq \frac{\lambda - 2}{64} \left\| 10\alpha \right\|^2 ,$$

and for $a = 1$,

$$\sum_{j=0}^{\lambda-1} \left\| \alpha - \frac{1}{2} \operatorname{tr}(\xi q^j) \right\|^2 \geq \frac{\lambda - 2}{16} \left\| 5\alpha \right\|^2 .$$

Proof. One can directly check the identity

$$-(1+a^2)(\alpha - \frac{1}{2}\operatorname{tr}\xi) - 2a(\alpha - \frac{1}{2}\operatorname{tr}(\xi q)) = (\alpha - \frac{1}{2}\operatorname{tr}(\xi q^2)) - (a^2+2a+2)\alpha.$$

First let $a \geq 3$. We group the left-hand side sum into three consecutive summands and have for all $x, y \in \mathbb{R}$ (with $\hat{x} = 2ax$, $\hat{y} = (1+a^2)y$),

$$\begin{aligned}
&\|x\|^2 + \|y\|^2 + \left\|2ax + (a^2+1)y - (a^2+2a+2)\alpha\right\|^2 \\
&\quad \geq \frac{1}{4a^2}\|\hat{x}\|^2 + \frac{1}{(a^2+1)^2}\|\hat{y}\|^2 + \frac{1}{4a^2}\left\|\hat{x} + \hat{y} - (a^2+2a+2)\alpha\right\|^2 \\
&\quad \geq \frac{1}{4a^2}\cdot\frac{1}{2}\left\|\hat{y} - (a^2+2a+2)\alpha\right\|^2 + \frac{1}{(a^2+1)^2}\|\hat{y}\|^2 \\
&\quad \geq \frac{1}{(a^2+1)^2}\left\|\hat{y} - (a^2+2a+2)\alpha\right\|^2 + \frac{1}{(a^2+1)^2}\|\hat{y}\|^2 \\
&\quad \geq \frac{1}{2(a^2+1)^2}\left\|(a^2+2a+2)\alpha\right\|^2 .
\end{aligned}$$

The results for $a = 1, 2$ follow in a similar way. □

Corollary 6.4. *Let $a \geq 1$. Then there exists a constant $C_a > 0$ only depending on a such that*

$$|F_\lambda(h,\alpha)| \leq \exp(-C_a \lambda \left\|(a^2+2a+2)\alpha\right\|^2)$$

uniformly for all $h \in \mathbb{Z}[\mathrm{i}]$, $\alpha \in \mathbb{R}$ and integers $\lambda \geq 3$.

Proof. By the bound (43) and the product representation (41) of F_λ we get

$$\begin{aligned}
|F_\lambda(h,\alpha)| &\leq \prod_{j=1}^{\lambda} \exp(-C'_a \|\alpha - \frac{1}{2}\operatorname{tr}(h/q^j)\|^2) \\
&= \exp(-C'_a \sum_{j=0}^{\lambda-1} \|\alpha - \frac{1}{2}\operatorname{tr}(hq^{j-\lambda})\|^2),
\end{aligned}$$

and Lemma 6.3 gives the stated upper bound. □

Corollary 6.5. *Let $a \geq 1$, $\lambda \geq 3$, $\alpha \in \mathbb{R}$ with $(a^2+2a+2)\alpha \notin \mathbb{Z}$ and $b \in \mathbb{Z}[\mathrm{i}]$ with $(b,q)=1$. Then there exists $0 \leq \gamma_Q(\alpha) < 1/2$ such that*

$$\sum_{\substack{h \in \mathcal{F}_\lambda \\ h \not\equiv 0 \ mod\ q}} \tau\left(\frac{bh}{q^\lambda}\right) |F_\lambda(h,\alpha)|^2 \ll Q^{\gamma_Q(\alpha)\lambda}. \tag{47}$$

Proof. We write $bh = h_1 + h_2 q^\lambda$. Since $(b,q)=1$ and $h \not\equiv 0 \bmod q$ we have $|h_1| \geq 1$. Then by (30) we conclude that

$$\tau\left(\frac{bh}{q^\lambda}\right) = \tau\left(\frac{h_1}{q^\lambda}\right) \ll \frac{|q|^\lambda}{|h_1|}.$$

Hence, by writing again h for h_1, the left hand side sum of (47) is bounded by

$$|q|^\lambda \sum_{\substack{h \in \mathcal{F}_\lambda \\ h \not\equiv 0 \bmod q}} \frac{1}{|h|} \cdot |F_\lambda(h,\alpha)|^2.$$

Let $T_M := \{z \in \mathbb{Z}[\mathrm{i}] \setminus \{0\} : \max(|\Re z|, |\Im z|) \leq M\}$. By Lemma 6.7 (see below) we have

$$\begin{aligned}
\sum_{\substack{h \in \mathcal{F}_\lambda \\ h \not\equiv 0 \bmod q}} \frac{1}{|h|} \cdot |F_\lambda(h,\alpha)|^2 &\ll \sum_{\substack{h \in \mathcal{F}_\lambda \\ h \notin T_M}} \frac{1}{|h|} \cdot |F_\lambda(h,\alpha)|^2 + \sum_{h \in T_M} \frac{1}{|h|} \cdot |F_\lambda(h,\alpha)|^2 \\
&\ll \frac{1}{M} \cdot 1 + \sum_{h \in T_M} \frac{1}{|h|} \cdot \exp(-C_a \lambda \|(a^2+2a+2)\alpha\|^2) \\
&\ll \frac{1}{M} + M^2 \cdot \exp(-C_a \lambda \|(a^2+2a+2)\alpha\|^2) \\
&= 2 \cdot \exp(-\frac{1}{3} C_a \lambda \|(a^2+2a+2)\alpha\|^2),
\end{aligned}$$

where we put $M = \exp(\frac{1}{3} C_a \lambda \|(a^2+2a+2)\alpha\|^2)$. □

In the next two lemmas we are concerned with the non-perturbed terms in (40). The proofs are very similar to those in [16], we only give the main steps.

Lemma 6.6. *For $a \geq 1$, $b \in \mathbb{Z}[\mathrm{i}]$, $\alpha \in \mathbb{R}$, $0 \leq \delta \leq \lambda$ we have*

$$G_\lambda(b, q^\delta, \alpha) = \sum_{\substack{h \in \mathcal{F}_\lambda \\ h \equiv b \bmod q^\delta}} |F_\lambda(h,\alpha)| \leq Q^{\eta_Q(\lambda-\delta)} \cdot |F_\delta(b,\alpha)|.$$

In particular,

$$G_\lambda(\alpha) = G_\lambda(0,1,\alpha) \leq Q^{\eta_Q \lambda}.$$

Proof. The proof follows almost literally the lines of the proof of [16, Lemme 17]. We have

$$\sum_{\substack{h\in\mathcal{F}_\lambda\\ h\equiv b \bmod q^\delta}} |F_\lambda(h,\alpha)| = \sum_{0\le r<Q}\sum_{\substack{h\in\mathcal{F}_{\lambda-1}\\ h\equiv b \bmod q^\delta}} \left|F_\lambda(h+rq^{\lambda-1},\alpha)\right|$$
$$= \sum_{\substack{h\in\mathcal{F}_{\lambda-1}\\ h\equiv b \bmod q^\delta}} |F_{\lambda-1}(h,\alpha)|\cdot\frac{1}{Q}\sum_{0\le r<Q}\varphi_Q\left(\alpha-\frac{1}{2}\operatorname{tr}\left(hq^{-\lambda}-\frac{rq}{Q}\right)\right)$$
$$= \sum_{\substack{h\in\mathcal{F}_{\lambda-1}\\ h\equiv b \bmod q^\delta}} |F_{\lambda-1}(h,\alpha)|\cdot\psi_Q(\alpha-\frac{1}{2}\operatorname{tr}(hq^{-\lambda})).$$

The statement now follows by iteration from (45). □

Remark. Regarding the inductive argument in the proof of Lemma 6.6, one may be tempted to couple two consecutive terms in order to get a lower exponent than η_Q. By our numerical experiments, however, this only gives a very small saving and it is by far not possible to include, say, $a\le 18$.

Lemma 6.7. *For $a\ge 1$ we have*

$$\sum_{\substack{h\in\mathcal{F}_\lambda\\ h\equiv b \bmod q^\delta}} |F_\lambda(h,\alpha)|^2 = |F_\delta(b,\alpha)|^2$$

and thus, in particular,

$$\sum_{h\in\mathcal{F}_\lambda}|F_\lambda(h,\alpha)|^2=1.$$

Proof. Similarly to the proof of Lemma 6.6, the formula (41) implies

$$\sum_{\substack{h\in\mathcal{F}_\lambda\\ h\equiv b \bmod q^\delta}} |F_\lambda(h,\alpha)|^2 = \sum_{\substack{h\in\mathcal{F}_{\lambda-1}\\ h\equiv b \bmod q^\delta}} |F_{\lambda-1}(h,\alpha)|^2\cdot\frac{1}{Q^2}\sum_{0\le r<Q}\varphi_Q^2\left(\alpha-\frac{1}{2}\operatorname{tr}\left(hq^{-\lambda}-\frac{rq}{Q}\right)\right)$$

and the statement now follows by Lemma 6.1 and iterating $(\lambda-\delta)$ times. □

7 Final estimates for the type II-sums

We are now in the position to use the estimates from the previous section in a similar way as in the work of Mauduit/Rivat [16, Section 7] to obtain the stated bound of (40). Recall that $\lambda=\mu+2\rho$. For simplicity we set

$$\rho=\xi(\mu+\nu), \tag{48}$$

where we will make $\xi=\xi_Q(\alpha)>0$ explicit only in the last step of our estimates. Recall that by (23) it is sufficient to show $S_2\ll Q^{\mu+\nu-\rho}$.

To begin with, by Lemma 6.6 with $d = q^\delta$ we obtain

$$S_2 \ll (1+Q^{\nu-\lambda}) \sum_{0\le\delta\le\lambda} Q^{\delta+2\eta_Q(\lambda-\delta)} S_3(\delta) + \hat{S}_2, \tag{49}$$

where for brevity

$$\hat{S}_2 = \lambda(1+Q^{\nu-\lambda}) Q^{(1+2\eta_Q)\lambda} (\log Q) \cdot Q^{\mu/2} \tag{50}$$

and

$$S_3(\delta) = \sum_{b\in\mathcal{F}_\delta} |F_\delta(b,\alpha)|^2 \min\left(Q^\mu, \tau\left(q^{\delta-\lambda}\left\{\frac{br}{q^\delta}\right\}\right) Q^{\mu/2}\right).$$

Informally speaking, we will show that S_2, up to some small modification of the exponent, has the order of magnitude of $\hat{S}_2$. More precisely, we will replace $1+2\eta_Q$ by an effectively computable number $< 3/2$. This can be achieved if $\eta_Q < 1/4$, thus the condition $a \ge 28$ comes into the play (compare with Lemma 6.2). We collect the various upcoming estimates from the first summand into this "error" term $\hat{S}_2$ (writing $\hat{S}_2'$, $\hat{S}_2''$ in the sequel).

Using (30) (which holds for $\delta \le \lambda$ and $a \in \mathcal{A}$) we have

$$S_3(\delta) \ll |q|^{\lambda-\delta} \sum_{b\in\mathcal{F}_\delta} |F_\delta(b,\alpha)|^2 \min\left(|q|^{\delta-\lambda+2\mu}, \tau\left(\frac{br}{q^\delta}\right) Q^{\mu/2}\right).$$

We split the sum (49) into two parts according to whether $\delta \le \triangle$ or $\delta > \triangle$. It is easy to see that the terms for $\delta \le \lambda^{1/2}$ contribute less to S_2 than the error term $\hat{S}_2$. Put $\triangle = \lfloor \lambda^{1/2} \rfloor$ and first assume $\delta' = \delta - \triangle > 0$. Then

$$\begin{aligned} S_3(\delta) &\ll |q|^{\lambda-\delta} \sum_{b\in\mathcal{F}_{\delta'}} \sum_{i\in R_{q^\triangle}} |F_\delta(b+iq^{\delta'},\alpha)|^2 \min\left(|q|^{\delta-\lambda+2\mu}, \tau\left(\frac{(b+iq^{\delta'}r)}{q^\delta}\right) Q^{\mu/2}\right) \\ &\ll |q|^{\lambda-\delta} \sum_{b\in\mathcal{F}_{\delta'}} |F_{\delta'}(b,\alpha)|^2 \sum_{i\in R_{q^\triangle}} \min\left(|q|^{\delta-\lambda+2\mu}, \tau\left(\frac{br/q^{\delta'}+ir}{q^\triangle}\right) Q^{\mu/2}\right). \end{aligned}$$

Set $r' = r/(r,q^\triangle)$. With the help of Lemma 5.2, the internal sum can be bounded by

$$\begin{aligned} &|(r,q^\triangle)|^2 \min\left(|q|^{\delta-\lambda+2\mu}, \tau\left(\frac{\{br/q^{\delta'}(r,q^\triangle)\}}{q^\triangle/(r,q^\triangle)}\right) Q^{\mu/2}\right) + Q^{\triangle+\mu/2} \log\left(\frac{Q^\triangle}{|(r,q^\triangle)|^2\pi}\right) \\ &\ll |q|^\triangle \cdot |(r,q^\triangle)| \min\left(|q|^{\delta-\lambda+2\mu-\triangle}|(r,q^\triangle)|, \tau\left(\frac{br}{q^{\delta'}(r,q^\triangle)}\right) Q^{\mu/2}\right) + Q^{\triangle+\mu/2}\log(Q^\triangle) \\ &\ll |q|^{\triangle+\rho} \min\left(|q|^{\mu-\rho-\triangle+\delta}, \tau\left(\frac{br'}{q^{\delta'}}\right) Q^{\mu/2}\right) + Q^{\triangle+\mu/2} \log\left(Q^\triangle\right). \end{aligned}$$

The contribution of the second summand to S_2 is bounded by

$$\begin{aligned} &(1+Q^{\nu-\lambda}) \sum_{\triangle<\delta\le\lambda} Q^{\delta+(1+2\eta_Q)(\lambda-\delta)} \cdot Q^{\triangle+\mu/2} \log(Q^\triangle) \\ &\qquad\qquad \ll \lambda(1+Q^{\nu-\lambda}) Q^{(1+2\eta_Q)\lambda} (\log Q) \cdot Q^{\mu/2+(1-2\eta_Q)\triangle}. \end{aligned}$$

Thus we have

$$\begin{aligned}
S_2 \ll (1+Q^{\nu-\lambda}) \sum_{\triangle<\delta\le\lambda} Q^{\delta+2\eta_Q(\lambda-\delta)} |q|^{\triangle+\rho+\lambda-\delta} \sum_{b\in\mathcal{F}_{\delta'}} |F_{\delta'}(b,\alpha)|^2 \cdot \\
\cdot \min\left(|q|^{\mu-\rho-\triangle+\delta}, \tau\left(\frac{br'}{q^{\delta'}}\right) Q^{\mu/2}\right) + \hat{S}_2' \\
= (1+Q^{\nu-\lambda})|q|^{\rho} \sum_{\triangle<\delta\le\lambda} Q^{\delta+2\eta_Q(\lambda-\delta)} S_4(\delta') + \hat{S}_2',
\end{aligned}$$

where

$$S_4(\delta') = |q|^{\lambda-\delta'} \sum_{b\in\mathcal{F}_{\delta'}} |F_{\delta'}(b,\alpha)|^2 \cdot \min\left(|q|^{\mu-\rho+\delta'}, \tau\left(\frac{br'}{q^{\delta'}}\right) Q^{\mu/2}\right)$$

and

$$\hat{S}_2' = \lambda(1+Q^{\nu-\lambda}) Q^{(1+2\eta_Q)\lambda} (\log Q) \cdot Q^{\mu/2+(1-2\eta_Q)\triangle}.$$

By our choice $\triangle = \lfloor \lambda^{1/2} \rfloor$ we have $(r',q)=1$ (in fact, this holds whenever $\triangle \ge \lfloor \rho \log Q / \log 2 \rfloor$; for our purpose it is sufficient to choose λ reasonably large). Therefore, by rearranging the sum over b we obtain,

$$S_4(\delta') = |q|^{\lambda-\delta'+\mu} \sum_{0\le\theta\le\delta'} \sum_{\substack{b\in\mathcal{F}_{\delta'-\theta} \\ b\not\equiv 0 \bmod q}} |F_{\delta'}(q^{\theta}b,\alpha)|^2 \min\left(|q|^{\delta'-\rho}, \tau\left(\frac{br'}{q^{\delta'-\theta}}\right)\right).$$

In view of Corollary 6.5 we first split off the case $\theta = \delta'$ which yields the summand

$$|q|^{\lambda+\mu-\rho} \left(\varphi_Q(\alpha)/Q\right)^{2\delta'} \le |q|^{\lambda+\mu-\rho-\tau_Q(\alpha)\delta'},$$

for some $\tau_Q(\alpha) > 0$. We look at the impact of this summand contributing to S_2. This yields the term

$$\begin{aligned}
(1+Q^{\nu-\lambda}) \sum_{\triangle<\delta\le\lambda} Q^{\delta+2\eta_Q(\lambda-\delta)} |q|^{\lambda+\mu-\tau_Q(\alpha)\delta'} \\
\le (1+Q^{\nu-\lambda}) Q^{\mu+\rho} \sum_{\triangle<\delta\le\lambda} Q^{\delta+2\eta_Q(\lambda-\delta)-\tau_Q(\alpha)\delta+\tau_Q(\alpha)\triangle} \\
\le (1+Q^{\nu-\lambda}) Q^{\mu+\rho} \left(Q^{\lambda(1-\tau_Q(\alpha))+\tau_Q(\alpha)\triangle} + Q^{\triangle+2\eta_Q(\lambda-\triangle)}\right).
\end{aligned}$$

Put $\tau_Q'(\alpha) = \min(\frac{1}{2} - 2\eta_Q, \tau_Q(\alpha))$. Then

$$S_2 \ll (1+Q^{\nu-\lambda})|q|^{\rho} \sum_{\triangle<\delta\le\lambda} Q^{\delta+2\eta_Q(\lambda-\delta)} S_5(\delta) + \hat{S}_2'',$$

where by (42),

$$\begin{aligned}
S_5(\delta') = |q|^{\lambda-\delta'+\mu} \sum_{0\le\theta<\delta'} \sum_{\substack{b\in\mathcal{F}_{\delta'-\theta} \\ b\not\equiv 0 \bmod q}} |F_{\delta'}(q^{\theta}b,\alpha)|^2 \min\left(|q|^{\delta'-\rho}, \tau\left(\frac{br'}{q^{\delta'-\theta}}\right)\right) \\
\le |q|^{\lambda-\delta'+\mu} \sum_{0\le\theta<\delta'} \sum_{\substack{b\in\mathcal{F}_{\delta'-\theta} \\ b\not\equiv 0 \bmod q}} |F_{\delta'-\theta}(b,\alpha)|^2 \min\left(|q|^{\delta'-\rho}, \tau\left(\frac{br'}{q^{\delta'-\theta}}\right)\right)
\end{aligned}$$

and

$$\hat{S}_2'' = \lambda(1+Q^{\nu-\lambda})Q^{(2-\tau_Q'(\alpha))\lambda}(\log Q)\cdot Q^{(1-2\eta_Q)\triangle}.$$

It remains to estimate $S_5(\delta')$ and to calculate its contribution to S_2. By Corollary 6.5 we have for some $0 \le \gamma_Q(\alpha) < 1/2$,

$$\begin{aligned} S_5(\delta) &\le |q|^{\lambda-\delta'+\mu} \sum_{0\le\theta<\delta'} \sum_{\substack{b\in\mathcal{F}_{\delta'-\theta}\\ b\not\equiv 0 \bmod q}} |F_{\delta'-\theta}(b,\alpha)|^2\, \tau\left(\frac{br'}{q^{\delta'-\theta}}\right) \\ &\ll Q^{\mu/2}|q|^{\lambda-\delta'} \sum_{0\le\theta<\delta'} Q^{\gamma_Q(\alpha)(\delta'-\theta)} \ll Q^{\mu/2}\cdot|q|^{\lambda-(1-2\gamma_Q(\alpha))(\delta-\triangle)}. \end{aligned}$$

Finally, we end up with

$$\begin{aligned} S_2 &\ll (1+Q^{\nu-\lambda})|q|^{\rho}Q^{\mu/2} \sum_{\triangle<\delta\le\lambda} Q^{\delta+2\eta_q(\lambda-\delta)}|q|^{\lambda-(1-2\gamma_Q(\alpha))(\delta-\triangle)} + \hat{S}_2'' \\ &\ll (1+Q^{\nu-\lambda})|q|^{\rho}Q^{\mu/2}\left(Q^{\lambda}\cdot|q|^{\lambda-(1-2\gamma_Q(\alpha))(\lambda-\triangle)} + Q^{\triangle+2\eta_Q(\lambda-\triangle)}|q|^{\lambda}\right) + \hat{S}_2''. \end{aligned} \tag{51}$$

It remains to estimate the three summands in (51). For the sake of simplicity we do not aim for the best possible estimate here, but look for a reasonable bound which makes the final calculation analogous to that of [16, section 7.2]. Obviously, by removing $\triangle$ from the summands, we have

$$\begin{aligned} S_2 &\ll \lambda(1+Q^{\nu-\lambda})\left(Q^{\left(2-\tau_Q'(\alpha)/2\right)\lambda} + |q|^{\rho}\cdot Q^{\left(5/4+\gamma_Q(\alpha)/2\right)\lambda+\mu/2}\right) \\ &\le \lambda(1+Q^{\nu-\lambda})Q^{\rho}\left(Q^{\left(2-\tau_Q'(\alpha)/2\right)\lambda} + Q^{\left(5/4+\gamma_Q(\alpha)/2\right)\lambda+\mu/2}\right) \\ &\le \lambda(1+Q^{\nu-\lambda})Q^{(2-\varepsilon_Q(\alpha))\lambda+\rho}, \end{aligned}$$

where

$$\varepsilon_Q(\alpha) := \min\left(\frac{\tau_Q'(\alpha)}{2},\ \frac{1}{4}-\frac{\gamma_Q(\alpha)}{2}\right) > 0.$$

In order to conclude, it is sufficient to ensure the validity of the following two inequalities (compare with [16]),

$$\begin{aligned} (2-\varepsilon_Q(\alpha))\,\mu + (5-2\varepsilon_Q(\alpha))\rho &< \mu+\nu-\rho \\ (1-\varepsilon_Q(\alpha))\,\mu + \nu + (3-2\varepsilon_Q(\alpha))\rho &< \mu+\nu-\rho, \end{aligned}$$

which hold true for $\rho = \xi(\mu+\nu)$ with

$$\xi = \xi_Q(\alpha) < \frac{\varepsilon_Q(\alpha)}{12-4\varepsilon_Q(\alpha)}.$$

Hence this choice yields $S_2 \ll Q^{\mu+\nu-\rho}$ and by Lemma 3.3 this finishes the treatment of the type II-sums in Lemma 3.1.

8 The type I-sums

The estimate of the "type I-sums" basically follows the lines of [16, section 8]. We show that for $\alpha\in\mathbb{R}$, $(a^2+2a+2)\alpha\notin\mathbb{Z}$, $a\ge 2$ there exists $\kappa_Q(\alpha)$ such that for all $0<\kappa<\kappa_Q(\alpha)$,

$M \le x^{1/10}$, it holds

$$\sum_{\frac{M}{Q}<|m|^2\le M} \max_{\frac{x}{Q|m|^2}<t\le\frac{x}{|m|^2}} \left| \sum_{\frac{x}{Q|m|^2}<|n|^2\le t} \mathrm{e}(\alpha s_q(mn)) \right| \ll_\kappa x^{1-\kappa}. \tag{52}$$

Note that this already completes the proof of (2), since by our estimates in Section 7 we are able to use the type II-estimate in Lemma 3.1 for the larger interval $x^{1/10} \le M \le x^{\beta_2}$.

In order to get rid of the dependency of m and $t(m)$, we show more generally, by enlarging the interval of t and including $t = \frac{x}{QM}$, that

$$\max_{\frac{x}{QM}\le t\le \frac{xQ}{M}} \sum_{\frac{M}{Q}<|m|^2\le M} \left| \sum_{|n|^2\le t} \mathrm{e}(\alpha s_q(mn)) \right| \ll_{Q,\kappa} x^{1-\kappa}.$$

Denote by R_m a complete residue system mod m, whose elements all have modulus $\le \frac{\sqrt{2}}{2}|m|$. By the orthogonality relation $\sum_{k\in R_m} \mathrm{e}\left(\frac{1}{2}\operatorname{tr}\frac{kl}{m}\right) = |m|^2$, if $m \mid l$, and 0 otherwise, we have

$$\left|\sum_{|n|^2\le t} \mathrm{e}(\alpha s_q(mn))\right| = \left| \frac{1}{|m|^2} \sum_{k\in R_m} \sum_{0\le|l|^2\le|m|^2 t} \mathrm{e}\left(\alpha s_q(l) + \frac{1}{2}\operatorname{tr}\frac{kl}{m}\right)\right|. \tag{53}$$

A crude upper bound of the right hand side of (53) suffices our purposes. We tessellate the disc $|l|^2 \le |m|^2\, t \le x$ by translates of $\mathcal{F}_\lambda$. Observe that for each translate $\mathcal{F}'_\lambda$ there are constants $\alpha, \beta \in \mathbb{Z}$ such that there is a bijection between $l \in \mathcal{F}_\lambda$ and $l + \alpha \mathrm{i} q^\lambda + \beta q^\lambda \in \mathcal{F}'_\lambda$. Thus, by point counting, we obtain for any $\lambda \ge 0$,

$$\left| \sum_{0\le|l|^2\le|m|^2 t} \mathrm{e}\left(\alpha s_q(l) + \frac{1}{2}\operatorname{tr}\frac{kl}{m}\right)\right| \ll \frac{x}{Q^\lambda} \left| \sum_{l\in\mathcal{F}_\lambda} \mathrm{e}\left(\alpha s_q(l) + \frac{1}{2}\operatorname{tr}\frac{kl}{m}\right)\right| + O(x^{1/2}Q^\lambda).$$

We may choose λ later on in some suitable way depending on x and Q. As for the next step, we need an estimate of the exponential sum now evolving from (53).

Lemma 8.1. *In the setting of (52) we have*

$$\sum_{\frac{M}{Q}<|m|^2\le M} \sum_{k\in R_m} \left| \sum_{l\in\mathcal{F}_\lambda} \mathrm{e}\left(\alpha s_q(l) + \frac{1}{2}\operatorname{tr}\frac{kl}{m}\right)\right| \ll_Q M^2 Q^{\lambda/2} + Q^{\gamma_Q(\alpha)\lambda} M^{3-2\gamma_Q(\alpha)}. \tag{54}$$

For the proof we use a two-dimensional large sieve based on the Sobolev-Gallagher inequality [10, Lemma 9.3]. Let $\mathcal{T} = \{t_i\}$ denote a collection of points in $\mathbb{C}$ with $|t_i| \le C$, which are well-spaced, i.e., there exists $\delta > 0$ with $|t_i - t_j| \ge 2\delta$ for $i \ne j$.

Proposition 8.2. *Let $H(z)$ be a smooth function (up to a set of measure zero) on $\mathcal{S} := \{z \in \mathbb{C} :\ |z| \le C + \delta\}$ and write $z = t^{(1)} + \mathrm{i}t^{(2)}$ with $t^{(1)}, t^{(2)} \in \mathbb{R}$. Then we have*

$$\begin{aligned}\sum_{t_i\in\mathcal{T}} |H(t_i)| \le \frac{1}{\delta^2}\int_{\mathcal{S}} |H(t)|\, \mathrm{d}t^{(1)}\, \mathrm{d}t^{(2)} + \frac{1}{\delta}\int_{\mathcal{S}} \left|\frac{\partial H}{\partial t^{(1)}}(t)\right| \mathrm{d}t^{(1)} \\ + \frac{1}{\delta}\int_{\mathcal{S}} \left|\frac{\partial H}{\partial t^{(2)}}(t)\right| \mathrm{d}t^{(2)} + \int_{\mathcal{S}} \left|\frac{\partial^2 H}{\partial t^{(1)}\partial t^{(2)}}(t)\right| \mathrm{d}t^{(1)}\, \mathrm{d}t^{(2)}.\end{aligned}$$

Proof. The statement follows by using the idea of the one-dimensional Sobolev-Gallagher inequality twice (for $t^{(1)}$ and $t^{(2)}$, respectively), together with the fact that two different points t_i and t_j have distance at least 2δ (see also [5, Proof of Lemma 3.10]). $\square$

Proof of Lemma 8.1. Define a function

$$\Phi_\lambda(t) = Q^\lambda \cdot |F_\lambda(-tq^\lambda, \alpha)| = \prod_{j=0}^{\lambda-1} \varphi_Q\left(\alpha + \frac{1}{2}\operatorname{tr}\,(q^j t)\right).$$

With this notion, the left hand side of (54) equals

$$\sum_{|d|^2 \le \frac{M}{2}} \sum_{\frac{M}{Q} < |m|^2 \le M} \sum_{\substack{k \in R_m \\ (k,m)=d}} \Phi_\lambda\left(\frac{k}{m}\right).$$

By our previous considerations in Section 6, we have that for any $\lambda_1 \in \mathbb{Z}$ with $0 \le \lambda_1 \le \lambda$ there is $0 \le \gamma_Q'(\alpha) < 1$ such that

$$\Phi_\lambda(t) \le \Phi_{\lambda_1}(t) \cdot Q^{\gamma_Q'(\alpha)(\lambda-\lambda_1)}. \tag{55}$$

We will again choose λ_1 in some suitable way depending on d, M and λ. The points k/m satisfy $-\sqrt{2}/2 \le |k/m| \le \sqrt{2}/2$ and are well-spaced with distance $|d|/M^{1/2}$. By Proposition 8.2 we therefore obtain

$$\begin{aligned}\sum_{\frac{M}{Q} < |m|^2 \le M} \sum_{\substack{k \in R_m \\ (k,m)=d}} \Phi_{\lambda_1}\left(\frac{k}{m}\right) \le \frac{1}{\delta^2}\int_{\mathcal{S}} |\Phi_{\lambda_1}(t)|\,\mathrm{d}t^{(1)}\,\mathrm{d}t^{(2)} + \frac{1}{\delta}\int_{\mathcal{S}}\left|\frac{\partial \Phi_{\lambda_1}}{\partial t^{(1)}}(t)\right|\mathrm{d}t^{(1)} \\ + \frac{1}{\delta}\int_{\mathcal{S}}\left|\frac{\partial \Phi_{\lambda_1}}{\partial t^{(2)}}(t)\right|\mathrm{d}t^{(2)} + \int_{\mathcal{S}}\left|\frac{\partial^2 \Phi_{\lambda_1}}{\partial t^{(1)}\partial t^{(2)}}(t)\right|\mathrm{d}t^{(1)}\,\mathrm{d}t^{(2)},\end{aligned} \tag{56}$$

where $2\delta = |d|/M^{1/2}$ and $\mathcal{S} = \{z \in \mathbb{C} : \; -\sqrt{2}/2 \le \Re(z), \Im(z) \le \sqrt{2}/2\}$. For the first derivative ($\nu = 1, 2$) we have

$$\begin{aligned}\left|\frac{\partial \Phi_{\lambda_1}}{\partial t^{(\nu)}}(t)\right| &\le \sum_{0 \le i < \lambda_1} Q^i \prod_{0 \le j < \lambda_1} \varphi_Q\left(\alpha + \frac{1}{2}\operatorname{tr}\,(q^j t)\right) \\ &\le \sum_{0 \le i < \lambda_1} Q^i Q^{\lambda_1 - i} \prod_{0 \le j < i} \varphi_Q\left(\alpha + \frac{1}{2}\operatorname{tr}\,(q^j t)\right) = Q^{\lambda_1} \sum_{0 \le i < \lambda_1} \Phi_i(t).\end{aligned}$$

On the other hand, by Cauchy-Schwarz inequality and Parseval's identity, we have

$$\int_{-\sqrt{2}/2}^{\sqrt{2}/2}\int_{-\sqrt{2}/2}^{\sqrt{2}/2}\left|\Phi_{\lambda_1}\left(\alpha + \frac{1}{2}\operatorname{tr}\,(q^i t)\right)\right|\mathrm{d}t^{(1)}\,\mathrm{d}t^{(2)} \ll Q^{\lambda_1/2}.$$

Therefore, by choosing $\lambda_1 = \min\left(\lambda, \left\lfloor \log_Q\left(\frac{M}{|d|^2}\right)\right\rfloor\right)$ we have that (56) can be bounded by

$$\frac{1}{\delta^2}Q^{\lambda_1/2} + \frac{1}{\delta}Q^{3\lambda_1/2} + Q^{5\lambda_1/2} \ll \frac{M^2}{|d|^4}Q^{\lambda_1/2} + \frac{M}{|d|^2}Q^{3\lambda_1/2} + Q^{5\lambda_1/2},$$

and taking into account (55), the left hand side of (54) is

$$\ll Q^{\gamma_Q'(\alpha)(\lambda-\lambda_1)} \cdot \frac{M^2}{|d|^4}Q^{\lambda_1/2} \ll \frac{M^2}{|d|^4}Q^{\lambda/2} + \frac{M^{3-2\gamma_Q'(\alpha)}}{|d|^{6-4\gamma_Q'(\alpha)}}Q^{\gamma_Q'(\alpha)\lambda + \gamma_Q'(\alpha)}.$$

Summation over the divisors d gives (54). $\square$

We now return to the proof of (52). By Lemma 8.1 we have

$$\begin{aligned}
&\sum_{\frac{M}{Q}<|m|^2\le M}\left|\sum_{|n|^2\le t}\mathrm{e}(\alpha s_q(mn))\right| \\
&\qquad\ll \frac{1}{M}\sum_{\frac{M}{Q}<|m|^2\le M}\sum_{k\in R_m}\left(\frac{x}{Q^\lambda}\left|\sum_{l\in\mathcal{F}_\lambda}\mathrm{e}\left(\alpha s_q(l)+\frac{1}{2}\operatorname{tr}\frac{kl}{m}\right)\right|+O\left(x^{1/2}Q^\lambda\right)\right)\\
&\qquad\ll \frac{x}{MQ^\lambda}\left(M^2Q^{\lambda/2}+Q^{\gamma_Q'(\alpha)\lambda}M^{3-2\gamma_Q'(\alpha)}\right)+Mx^{1/2}Q^\lambda\\
&\qquad= \frac{xM}{Q^{\lambda/2}}+xM^{2-2\gamma_Q'(\alpha)}Q^{(\gamma_Q'(\alpha)-1)\lambda}+Mx^{1/2}Q^\lambda. \qquad (57)
\end{aligned}$$

Let $M=x^{\beta_1}$ with $\beta_1>0$, and set λ such that $Q^\lambda=x^{3\beta_1}$. By (57) we have to guarantee the validity of the inequality

$$\max\left(1-\frac{1}{2}\beta_1,\ \ 1+\beta_1(2-2\gamma_Q'(\alpha))+3\beta_1(\gamma_Q'(\alpha)-1),\ \ 4\beta_1+\frac{1}{2}\right)<1-\kappa_Q(\alpha). \qquad (58)$$

Take $\beta_1=\frac{1}{10}$, then (58) can obviously be satisfied with

$$\kappa_Q(\alpha)<\min\left(\frac{1}{20},\frac{1}{10}-\frac{\gamma_Q'(\alpha)}{10}\right).$$

This finishes the proof of (52) and completes the proof of Theorem 2.1.

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References

[1] E. Fouvry, C. Mauduit, *Méthodes de crible et fonctions sommes des chiffres*, Acta Arith. **77** (1996), no. 4, 339–351. MR1414514 (97j:11046)

[2] A. O. Gelfond, *Sur les nombres qui ont des propriétés additives et multiplicatives données*, Acta Arith. **13** (1968), 259–265. MR0220693 (36 #3745)

[3] P. J. Grabner, P. Kirschenhofer and H. Prodinger, *The sum-of-digits function for complex bases*, J. London Math. Soc. (2) **57** (1998), no. 1, 20–40. MR1624777 (99e:11010)

[4] P. J. Grabner and P. Liardet, *Harmonic properties of the sum-of-digits function for complex bases*, Acta Arith. **91** (1999), no. 4, 329–349. MR1736016 (2001f:11126)

[5] D. R. Heath-Brown, *Primes represented by $x^3 + 2y^3$*, Acta Math. **186** (2001), 1–84.

[6] M. N. Huxley, *Area, lattice points, and exponential sums* London Mathematical Society Monographs, Oxford University Press, New York, 1996.

[7] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Fourth edition, Oxford University Press, Oxford 1975.

[8] E. Hlawka, J. Schoißengeier and R. Taschner, *Lehrbuch: Analytische Zahlentheorie*, Wien 1986.

[9] H. Iwaniec, *Almost-primes represented by quadratic polynomials*, Invent. Math. **47** (1978), no. 2, 171–188. MR0485740 (58 #5553)

[10] H. Iwaniec and E. Kowalski, *Analytic number theory*, American Mathematical Society Colloquium Publications, Providence, RI, 2004. xii+615 pp. MR2061214 (2005h:11005)

[11] I. Kátai and B. Kovács, *Canonical number systems in imaginary quadratic fields*, Acta Math. Acad. Sci. Hungar. **37** (1981), no. 1-3, 159–164. MR0616887 (83a:12005)

[12] I. Kátai and J. Szabó, *Canonical number systems for complex integers*, Acta Sci. Math. (Szeged) **37** (1975), no. 3-4, 255–260. MR0389759 (52 #10590)

[13] L. Kuipers and H. Niederreiter, *Uniform distribution of sequences*, Pure and Applied Mathematics, Wiley-Interscience, New York-London-Sydney, 1974. MR0419394 (54 #7415)

[14] E. Landau, *Über Gitterpunkte in mehrdimensionalen Ellipsoiden*, Math. Zeit. **21** (1924), 126–132. MR1544690

[15] E. Landau, *Über Gitterpunkte in mehrdimensionalen Ellipsoiden*, Math. Zeit. **24** (1926), 299–310. MR1544766

[16] Chr. Mauduit and J. Rivat, *Sur un probléme de Gelfond: La somme des chiffres des nombres premiers*, preprint; electronically available at `http://iml.univ-mrs.fr/~rivat/publications.html`.

[17] W. Narkiewicz, *Elementary and analytic theory of algebraic numbers*, 3rd edition, Springer Monographs in Mathematics, Berlin, 2004.

[18] W. Sierpinski, *Elementary Number Theory*, Polska Academia Nauk, Warszawa, 1964.