Analysis of a Recurrence Related to Critical Nonhomogeneous Branching Processes

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Abstract

Some classes of controlled branching processes (with nonhomogeneous migration or with nonhomogeneous state-dependent immigration) lead in the critical case to a recurrence for the extinction probabilities. Under some additional conditions it was shown that this recurrence depends from some parameter $\beta$ and converges for $0 < \beta < 1$. Now we show that the recurrence does converge for all positive values of the parameter $\beta$ which leads to an extension of some limit theorems for the corresponding branching processes. We give also a generalization of the recurrence and an asymptotic analysis of its behaviour.

1 Introduction

Branching processes can be interpreted as mathematical models of population dynamics where the reproduction of the individuals follows some stochastic laws.

It is well known that the history of branching processes begins with the pioneer papers of Bienaymé [3] and Galton and Watson [7] analysing some demographic problems. However the terminology “branching processes” was introduced by A.N.Kolmogorov and among the first asymptotic results of the modern mathematical basis were obtained by him and his students, see Kolmogorov [13], [14], Kolmogorov and Dmitriev [15], Kolmogorov and Sevastyanov [16], Yaglom [22], Sevastyanov [19], Zolotorev [28]. More details of the further development can be obtained from the books of Harris [11], Sevastyanov [20], Athreya and Ney [2], Jagers [12], Asmussen and Hering [1] and Guttorp [10].

Remember that the classical Bienaymé-Galton-Watson branching process can be defined by the following recurrence:

$$Z_{n+1} = \sum_{j=1}^{Z_n} X_n(j), \quad n = 0, 1, 2, \ldots,$$

where $\{X_n(j)\}$ are (in both $n$ and $j$) i.i.d. non-negative integer-valued random variables.

The BGW process can be used to describe the growth of an isolated population with an independent reproduction of the individuals. However, in real situations there is always an interaction with the environment and/or between the individuals, which means that, in general, the evolutions of the individuals are not independent.

Controlled branching processes represent one direction in describing this more complicated situation.

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Sevastyanov and Zubkov [21] introduced the following generalization of (1):

\[ Z_{n+1} = \sum_{j=1}^{\varphi(Z_n)} X_n(j), \ n = 0, 1, 2, \ldots, \]  

(2)

where \( \varphi(k) \) is a deterministic and non-negative integer-valued control function. They obtained conditions for extinction or non-extinction in the case \( \varphi(n) \sim an, n \to \infty, a > 0 \).

Yanev [25] considered a generalization of (2) in the case of random control functions (RCF):

\[ Z_{n+1} = \sum_{j=1}^{\varphi_n(Z_n)} X_n(j), \ n = 0, 1, 2, \ldots, \]  

(3)

where \( \varphi_n = \{ \varphi_n(k), k = 0, 1, 2, \ldots \}, n \geq 0 \), are independent non-negative integer-valued random processes with identical one-dimensional distributions. Conditions of extinction or non-extinction where obtained by Yanev [25] [26] and Bruss [5] assuming a.s. linear growth of the RCF. Yanev [26] assumed also a random environment. Gonzales, Molina and Del Puerto [8] [9] investigated the model (3) when the means of the RCF have a linear growth.

The controlled branching processes with multi-type RCF can be defined as follows:

\[ Z_{n+1} = \sum_{i \in I} \sum_{j=1}^{\varphi_{i,n}(Z_n)} X_{i,n}(j), \ n = 0, 1, 2, \ldots, \]  

(4)

where \( I \) is an index set (finite or infinite) and the set \( X = \{ X_{i,n}(j) \} \) consists of i.i.d. for each fixed \( i \) random variables with p.g.f. \( F_i(s) = E(s^{X_{i,n}(j)}) \), \( i \in I \). On the other hand, the set \( X \) is independent of the set of the RCF \( \varphi = \{ \varphi_{i,n}(k) \} \), where \( \varphi_n = \{ \varphi_{i,n}(k), i \in I, k = 0, 1, 2, \ldots \} \) are independent random fields with identical one-dimensional distributions.

Zubkov [29] investigated the case of deterministic control functions, i.e. \( \varphi_{i,n}(k) \equiv \varphi_i(k) \) a.s. For more general controls, sufficient extinction criteria were obtained by Bruss [4]. G.Yanev and N.Yanev [23] [24] obtained conditions for extinction or non-extinction in the general model (4). In the last paper it is assumed also a random environment.

Note that the general model (4) describes a very large class of stochastic processes, in particular all Markov chains and the most popular considered models of branching processes in discrete time: BGW process, processes with immigration, branching processes with random migration components, population-size-dependent branching processes, branching processes in random environment and so on. However, in all these models it is always fulfilled the condition \( \sum_{i \in I} \varphi_{i,n}(k) \to \infty, a.s., n \to \infty \), which can be considered as a general branching property.

Of course, in the general situation (4) it is not possible to obtain detailed asymptotic results. That is why it is very interesting to consider every particular case where such asymptotic analysis is possible. On the other hand, it is also very important for applications.

Yanev and Mitov [27] considered the following model allowing nonhomogeneous migration: in the general definition (4) it is assumed that \( I = \{1, 2\} \) and

\[ \varphi_{1,n}(k) = \max\{\min(k, k + Y_n), 0\}, \varphi_{2,n}(k) = \max\{Y_n, 0\}, \]  

(5)

where the independent r.v. \( Y_n \) have the following distributions

\[ P(Y_n = -1) = p_n, P(Y_n = 0) = q_n, P(Y_n = 1) = r_n, p_n + q_n + r_n = 1, \ n = 0, 1, 2, \ldots \]  

(6)

Then by (4)-(6) it follows that \( \{Z_n\} \) is a nonhomogeneous Markov chain which can be represented as follows:

\[ Z_{n+1} = \sum_{j=1}^{Z_n} X_{1,n}(j) + M_n, \ n = 0, 1, 2, \ldots \]  

(7)
where
\[
M_n = -X_{1,n}(1)I\{Z_n = 0\} \text{ with probability } p_n, \\
= 0 \text{ with probability } q_n, \\
= X_{2,n}(1) \text{ with probability } r_n.
\]

One can assume without loss of generality that \(Z_0 = 0\) a.s.

The process (7) allows for the following interpretation. In the \((n+1)th\) generation \((n = 0, 1, 2, \ldots)\) the next three situations are possible: \((i)\) with probability \(p_n\), one family is eliminated and does not take part in a further evolution (emigration); \((ii)\) with probability \(q_n\), there is no migration (like in a BGW process); \((iii)\) with probability \(r_n\), there is an immigration of new individuals according to a p.g.f. \(G(s) = E(s^{X_{2,n}(1)})\).

Let \(F' = E(s^{X_{1,n}(1)})\) be the offspring p.g.f. and \(H_n(s) = E(s^{Z_n})\) be the p.g.f. of the process (7). Then by (7) one can obtain the equation (see Yanev and Mitov [27]):
\[
H_{n+1}(s) = U_n(n, s) + \sum_{k=0}^{n} p_{n-k}H_{n-k}(0)[1 - 1/F_{k+1}(s)]U_{k-1}(n, s),
\]
where \(U_{k-1}(n, s) = \prod_{i=0}^{k} \Delta_{n-i}(F_i(s)), \Delta_n(s) = p_n/F(s) + q_n + r_nG(s)\) and \(F_i(s)\) is the \(i\)th iterate of \(F(s)\), i.e. \(F_0(s) = s, F_{i+1}(s) = F(F_i(s)), i = 0, 1, 2, \ldots\).

Further on the critical case \(F'(1) = 1, 0 < F''(1) = 2\alpha < \infty\), with a balanced migration component \(r_nG'(1) \equiv p_n \sim K \log n\) is considered. In this case it follows by (8):
\[
H_{n+1}(0) = 1 - \sum_{k=0}^{n} (1 - F_k(0))p_{n-k}H_{n-k}(0) + O(1/\log n),
\]
where \(1 - F_n(0) \sim 1/(\alpha n), n \to \infty\). This corresponds to the famous asymptotic result of Kolmogorov [13] for the probability of non-extinction in a BGW process.

When \(\beta = K/\alpha < 1\) then it was shown in Yanev and Mitov [27] that \(\lim H_n(0) = 1/(1 + \beta)\), respectively \(\lim R_n = \beta/(1 + \beta)\), where \(R_n = 1 - H_n(0)\) is the probability of non-extinction.

The case \(\beta \geq 1\) remained open for a very long period.

In this paper, we will first consider recurrences of the above type:

**Main recurrence equation**
\[
a_{n+1} = T_n - \sum_{k=0}^{n} a_k P_k Q_{n-k}, (n \geq 0),
\]
where \(P_n, Q_n\) and \(T_n\) are given sequences. We establish general properties of the solutions \(a_n\) in Section 2. It will turn out that, under quite natural monotonicity conditions, all solutions of (10) are bounded and that there are solutions with \(0 \leq a_n \leq 1\).

Finally, in Section 3 we discuss the case \(P_n \sim \beta/\log n, Q_n \sim 1/n\) and \(T_n \to 1\) (that is, the original problem from Yanev and Mitov [27]) in more details and we show that \(a_n\) is convergent for any choice of \(\beta > 0\). This result allows us to generalize Theorem 1 of Yanev and Mitov [27]. Let \(\beta = K/\alpha\).

**Theorem 1.1** Let \(F'(1) = 1, 0 < F''(1) = 2\alpha < \infty\) and \(r_nG'(1) \equiv p_n \sim K/\log n, n \to \infty, K > 0\). Then for all \(\beta\) we have:
\[
\lim R_n = \beta/(1 + \beta), E(Z_n) \sim \beta \alpha n/\log n, E(Z_n^2) \sim \beta \alpha^2 n^2/\log n
\]
and
\[
\lim P(\log Z_n/\log n \leq x \mid Z_n > 0) = x, 0 \leq x \leq 1.
\]
Branching processes with decreasing state-dependent immigration represent another example where the same analytical problem appears.

Let us assume in (4) that $I = \{1,2\}$, $\varphi_{1,n}(k) \equiv k$ and $\varphi_{2,k}(k) \equiv \delta_{k,0}$, where as usual $\delta_{k,0} = 0$ for $k \geq 1$ and $\delta_{0,0} = 1$. Then

$$Z_{n+1} = \sum_{j=1}^{Z_n} X_{1,n}(j) + X_{2,n}(1) I\{Z_n = 0\}, \quad n = 0,1,2,\ldots, \tag{11}$$

where $F(s) = E(s^{X_{1,n}(1)})$ and $G_n(s) = E(s^{X_{2,n}(1)})$ are the offspring p.g.f. and the p.g.f. of the immigrants in state zero, respectively.

The critical case $F'(1) = 1, 0 < F''(1) = 2\alpha < \infty$ with $m_n = G'_n(1) \sim K/\log n$ is considered in Mitov and Yanov [18] and an equation similar to (9) is obtained. It is also proved that $\lim R_n = \beta/(1 + \beta)$ for $\beta = K/\alpha < 1$ and it is pointed out that the case $\beta \geq 1$ is an open problem (see Remark, p.30).

Therefore this problem is now solved and we are able to prove a similar limit theorem as Theorem 1.1, which generalizes the corresponding Theorem 1 from Mitov and Yanov [18].

Finally, the continuous time analogy of the process (1) is considered by Mitov, Vatutin and Yanov [17], where it is also pointed out that the case $\beta \geq 1$ is ”still an open problem” (Remark, p.707).

Section 4 concludes the paper.

2 Boundedness and Convergence

2.1 Convergence by “Contraction”

If one want to obtain convergent solutions $a_n$ of (10), then one way to attack this problem is to first assume that the sequences $T_n$ and

$$C_n := \sum_{k=0}^{n} P_k Q_{n-k} \tag{12}$$

are convergent, say $T = \lim_{n \to \infty} T_n$ and $C = \lim_{n \to \infty} C_n$. Then the expected limit $\alpha$ of the sequence $a_n$ is

$$\alpha = \frac{T}{1 + C}.$$

Our first result considers the case $C < 1$ that is also related to the case covered in [27]. The constant $C < 1$ serves (more or less) as a contraction factor that forces the sequence being convergent.

**Theorem 2.1** Suppose that $P_n, Q_n, T_n$ are non-negative sequences such that $T_n$ is convergent with limit $T$, $Q_n$ converges to 0, and $C_n$ convergent to $C < 1$. Then every sequence $a_n$ that satisfies (10) is convergent with limit

$$\lim_{n \to \infty} a_n = \frac{T}{C + 1}.$$

**Proof.** First of all, it is easy to show that the sequence $a_n$ is bounded. Set $\overline{T} = \sup_{n \geq 0} T_n$ and $\overline{C} = \sup_{n \geq 0} C_n$. First, suppose that $\overline{C} < 1$. Then it follows by induction that

$$|a_n| \leq \overline{C^n}|a_0| + (\overline{C}^{n-1} + \cdots + C + 1)\overline{T}$$

and, thus, $a_n$ is bounded. (If $\overline{C} \geq 1$ we just have to shift the argument to $n \geq n_0$, where $n_0$ is chosen in a way that $\sup_{n \geq n_0} C_n < 1$.)

Furthermore, since $C_n$ is bounded and $C_n \geq P_{n-t_0} Q_{t_0}$ (where $Q_{t_0} \neq 0$) it follows that $P_n$ is bounded, too.

Next set $b_n = a_n - \frac{T}{\overline{C} + 1}$. Then $b_n$ satisfies the recurrence

$$b_{n+1} = -\sum_{k=0}^{n-k} b_k P_k Q_{n-k} + \overline{T}_n,$$
where
\[ \bar{T}_n = T_n - (C_n + 1) \frac{T}{C + 1} \to 0. \]
We again suppose that \( \bar{C} = \sup_{n \geq 0} C_n < 1 \) (otherwise we have to shift the argument as above). We further set \( \bar{F} = \sup_{n \geq 1} |b_n| \) and \( \bar{P} = \sup_{n \geq 1} |P_n| \). Since \( Q_n \to 0 \) there exists an increasing integer sequence \( n_k \) (with \( n_0 = 0 \)) and a sequence \( \eta_k > 0 \) that is monotonically decreasing to 0 such that
\[ Q_n + Q_{n-1} + \cdots + Q_{n-n_k} \leq \eta_k \]
for all \( n \geq n_{k+1} \). Set
\[ B_k := \max_{n_k \leq n < n_{k+1}} |b_n| \]
and
\[ \tau_k := \bar{F} \eta_k + \sup_{n \geq n_{k+1}} |\bar{T}_n|. \]
By construction we have \( \tau_k \to 0 \). Finally choose \( \bar{B}_0 \geq B_0 \) in a way that \( \bar{C} \bar{B}_0 + \tau_0 \leq \bar{B}_0 \) and set
\[ \bar{B}_{k+1} = \bar{C} \bar{B}_k + \tau_k \]
for \( k \geq 0 \). Then it follows by induction that \( \bar{B}_{k+1} \leq \bar{B}_k \) and that
\[ \bar{B}_k = \bar{C}^k \bar{B}_0 + \sum_{\ell=0}^{k-1} \bar{C}^{k-1-\ell} \tau_\ell \to 0 \]
as \( k \to \infty \), that is, \( \bar{B}_k \) converges monotonically to 0.

In a final step we show by induction that \( B_k \leq \bar{B}_k \) for all \( k \geq 0 \). Of course this shows that \( b_n \to 0 \) and consequently \( a_n \to T/(C + 1) \).

By definition we have \( B_0 \leq \bar{B}_0 \). Now suppose that \( B_k \leq \bar{B}_k \) for some \( k \geq 0 \). Then, for \( n = n_{k+1} \) we have
\[
|b_{n_{k+1}}| \leq \sum_{k=0}^{n_k} |b_k| P_k Q_{n_{k+1} - k} + \sum_{k=n_k+1}^{n_{k+1}-1} |b_k| P_k Q_{n-1-k} + |\bar{T}_{n-1}|
\leq \bar{F} \eta_k + \bar{C} B_k + |\bar{T}_{n-1}|
\leq \bar{C} \bar{B}_k + \tau_k
\leq \bar{B}_{k+1}
\]
Note that we also have \( |b_{n_{k+1}}| \leq \bar{B}_k \). In the same way we get (inductively) for \( n_{k+1} \leq n < n_{k+1} \)
\[
|b_n| \leq \bar{C} \bar{B}_k + \tau_k = \bar{B}_{k+1}
\]
and, thus,
\[ B_{k+1} = \max_{n_{k+1} \leq n < n_{k+2}} |b_n| \leq \bar{B}_{k+1}. \]
This completes the proof of Theorem 2.1.

\[ \boxed{\text{Remark 2.2} \, \text{Note that with help of the same proof techniques it follows that} \, a_n \, \text{is bounded if} \]
\[ \limsup_{n \to \infty} T_n < \infty \quad \text{and} \quad \limsup_{n \to \infty} C_n < 1 \]
\[ \text{and that} \, a_n \to 0 \quad \text{if} \]
\[ \lim_{n \to \infty} T_n = 0 \quad \text{and} \quad \limsup_{n \to \infty} C_n < 1 \]
2.2 Boundedness Conditions

We are now interested in conditions on the sequences \( P_n, Q_n, \) and \( T_n \) that imply that all solutions \( a_n \) of (1) are bounded even if \( C \geq 1 \) so that we cannot use a contraction argument, compare with the preceding Remark.

**Theorem 2.3** Suppose that \( Q_n \) and \( C_n \) are monotone convergent sequences, that \( Q_0P_n \leq 1 \) for \( n \geq n_0 \) (for some \( n_0 \)) and that

\[
\sum_{n \geq 0} |T_{n+1} - T_n| < \infty.
\]

Then every sequence \( a_n \) that satisfies (10) is bounded.

**Proof.** Since \( C_n \) is bounded it follows as in the proof of Theorem 2.1 that \( P_n \) and \( Q_n \) have to bounded, too.

First, consider the case that \( Q_n \) is increasing (to a finite limit \( Q \)). Since \( C_n \) is convergent, it follows that

\[
\sum_{n \geq 0} P_n < \infty.
\]

Now, by subtracting (10) for \( a_{n+2} \) and \( a_{n+1} \) it directly follows that \( a_n \) satisfies the recurrence

\[
a_{n+2} = a_{n+1}(1 - P_{n+1}Q_0) + \sum_{k=0}^{n} a_k P_k(Q_{n-k} - Q_{n-k+1}) + T_{n+1} - T_n,
\]

too. Set \( A_n = \max_{0 \leq k \leq n} |a_k| \). Then we have

\[
|a_{n+2}| \leq A_{n+1} \left( 1 - P_{n+1}Q_0 - \sum_{k=0}^{n} a_k P_k(Q_{n-k} - Q_{n-k+1}) \right) + |T_{n+1} - T_n|
\]

\[
= A_{n+1} (1 + (C_{n+1} - C_n) - 2P_{n+1}Q_0) + |T_{n+1} - T_n|
\]

If \( Q_n \) is increasing we surely have

\[
C_{n+1} - C_n = \sum_{k=0}^{n} a_k P_k(Q_{n+1-k} - Q_{n-k}) + P_{n+1}Q_0 \geq 0,
\]

(that is, it is not necessary to assume that \( C_n \) is monotone in that case). Thus, the series

\[
\sum_{n \geq 0} (C_{n+1} - C_n - 2P_{n+1}Q_0)
\]

is absolutely convergent and from

\[
A_{n+2} \leq A_{n+1} (1 + (C_{n+1} - C_n) + 2P_{n+1}Q_0) + |T_{n+1} - T_n|
\]

it follows that \( A_n \) is a bounded sequence. Thus, \( a_n \) is bounded, too.

If \( Q_n \) is decreasing we get (as above)

\[
|a_{n+2}| \leq A_{n+1} \left( 1 - P_{n+1}Q_0 + \sum_{k=0}^{n} a_k P_k(Q_{n-k} - Q_{n-k+1}) \right) + |T_{n+1} - T_n|
\]

\[
= A_{n+1} (1 - (C_{n+1} - C_n)) + |T_{n+1} - T_n|.
\]

Hence, if \( C_n \) is increasing then we have

\[
A_{n+2} \leq A_{n+1} + |T_{n+1} - T_n|
\]
and if $C_n$ is decreasing

$$A_{n+2} \leq A_{n+1} (1 + (C_n - C_{n+1})) + |T_{n+1} - T_n|.$$  

In both cases, it follows that $A_n$ is bounded.  

Let us consider the example $P_n = \beta/\log(n)$ (for $n \geq 2$, $P_0 = P_1 = 0$) and $Q_n = 1/(n + 1)$. Then

$$C_n = \frac{\beta}{\log(n)} \sum_{k=2}^{n} \frac{1}{(n-k+1) \log k}$$

$$= \beta \left( 1 + \frac{\gamma}{\log n} + O \left( \frac{1}{(\log n)^2} \right) \right)$$

$$\rightarrow \beta,$$

where $\gamma$ denotes Euler’s constant. Furthermore, after some algebra one obtains

$$C_n = \beta \left\{ \frac{1}{\log(n)} [\ln(n) + \gamma] + \frac{\pi^2}{6\log^2(n)} \right\} + O \left( \frac{1}{\log^3(n)} \right)$$

$$C_n - C_{n+1} = \beta \frac{\gamma}{n \log(n)^2} + O \left( \frac{1}{n \log(n)^3} \right) > 0$$

(for sufficiently large $n$, compare also with Figure 1). Thus, we can apply Theorem 2.3 and obtain that every solution $a_n$ is bounded.

Next we present a second condition for boundedness that does not rely on monotonicity or convergence properties of $C_n$. It also shows that there exist solutions with $0 \leq a_n \leq 1$, that is, $a_n$ may be interpreted as probabilities.

**Theorem 2.4** Suppose that $P_n$ and $Q_n$ are non-negative and strictly monotonely decreasing sequences, that $T_n = 1$, and that $P_3Q_0 \leq 1$. Then there exists a solution $a_n$ of (10) with $0 \leq a_n \leq 1$ for all $n \geq 0$. Furthermore, all solutions of (10) are bounded.
Proof. The idea of the proof is to show that there exist two different solutions \( a_n^{(1)}, a_n^{(2)} \) with \( 0 \leq a_n^{(1)}, a_n^{(2)} \leq 1 \). Then any solution is given by

\[
a_n = a_n^{(1)} + \frac{a_0 - a_0^{(1)}}{a_0^{(2)} - a_0^{(1)}} (a_n^{(2)} - a_n^{(1)})
\]

and is, thus, bounded, too.

We consider several cases. First, suppose that \( P_1Q_0 \leq 1 \) and consider the system of equations

\[
\begin{align*}
    a_0P_0Q_0 + a_1 &= 1, \\
    a_0P_0Q_1 + a_1P_1Q_0 + a_2 &= 1.
\end{align*}
\]

Since \( P_0Q_0 < P_0Q_1 \) and \( 1 \geq P_1Q_0 \) it follows that for every \( a_2 \) with \( 0 \leq a_2 \leq 1 - Q_1/Q_0 \) there is a unique solution for \( a_0 \) and \( a_1 \) with \( 0 \leq a_0, a_1 \leq 1 \). Next set recursively

\[
a_{n+1} = 1 - \sum_{k=0}^{n} a_k P_k Q_{n-k} \quad (n \geq 2).
\]

We induction we show that we always have \( 0 \leq a_{n+1} \leq 1 \). Note that \( P_1Q_0 \leq 1 \) also implies \( P_nQ_0 \leq 1 \) for all \( n \geq 1 \). Assume that \( 0 \leq a_0, a_1, \ldots, a_n \leq 1 \) for some \( n \geq 2 \). Then

\[
0 \leq \sum_{k=0}^{n} a_k P_k Q_{n-k} \leq \sum_{k=0}^{n-1} a_k P_k Q_{n-1-k} + a_n = 1.
\]

and consequently \( 0 \leq a_{n+1} \leq 1 \). Thus, we have \( 0 \leq a_n \leq 1 \) for all \( n \geq 0 \).

Now we use this construction for \( a_2^{(1)} = 0 \) and for \( a_2^{(2)} = 1 - Q_1/Q_0 > 0 \). Consequently we obtain two different solutions \( a_n^{(1)}, a_n^{(2)} \) with \( 0 \leq a_n^{(1)}, a_n^{(2)} \leq 1 \) and the result follows.

Second, suppose that \( P_1Q_0 > 1 \) and \( P_2Q_0 \leq 1 \). Here we consider the system of equations

\[
\begin{align*}
    a_0P_0Q_0 + a_1 &= 1, \\
    a_0P_0Q_1 + a_1P_1Q_0 + a_2 &= 1, \\
    a_0P_0Q_2 + a_1P_1Q_1 + a_2P_2Q_0 + a_3 &= 1. 
\end{align*}
\]

After eliminating \( a_0 \) we get

\[
\begin{align*}
a_1 & \frac{P_1Q_0 - \frac{Q_1}{Q_0}}{1 - \frac{Q_1}{Q_0}} + a_2 \frac{1}{1 - \frac{Q_1}{Q_0}} = 1, \\
a_1 & \frac{P_1Q_1 - \frac{Q_2}{Q_0}}{1 - \frac{Q_2}{Q_0}} + a_2 \frac{P_2Q_0}{1 - \frac{Q_2}{Q_0}} + a_3 \frac{1}{1 - \frac{Q_2}{Q_0}} = 1.
\end{align*}
\]

Note that the coefficients of \( a_1 \) satisfy

\[
\frac{P_1Q_0 - \frac{Q_1}{Q_0}}{1 - \frac{Q_1}{Q_0}} > \frac{P_1Q_1 - \frac{Q_2}{Q_0}}{1 - \frac{Q_2}{Q_0}} > 0.
\]

If

\[
\frac{1}{1 - \frac{Q_1}{Q_0}} \geq \frac{P_2Q_0}{1 - \frac{Q_2}{Q_0}}
\]

then for arbitrary \( a_1, a_2 \geq 0 \) that satisfy (14) we obtain

\[
0 \leq a_1 \frac{P_1Q_1 - \frac{Q_2}{Q_0}}{1 - \frac{Q_2}{Q_0}} + a_2 \frac{P_2Q_0}{1 - \frac{Q_2}{Q_0}} \leq a_1 \frac{P_1Q_0 - \frac{Q_1}{Q_0}}{1 - \frac{Q_1}{Q_0}} + a_2 \frac{1}{1 - \frac{Q_1}{Q_0}} = 1
\]
and consequently (from (15))

$$0 \leq a_3 \leq 1 - \frac{Q_2}{Q_0}.$$  

Note that $a_1, a_2 \geq 0$ that satisfy (14) are bounded by

$$a_1 \leq \frac{1 - \frac{Q_1}{Q_0}}{P_1 Q_0 - \frac{Q_1}{Q_0}} \leq 1 \quad \text{and} \quad a_2 \leq 1 - \frac{Q_1}{Q_0} \leq 1.$$  

Next set

$$a_0 = \frac{1 - a_1}{1 - P_0 Q_0}.$$  

Thus, we have found solutions $0 \leq a_0, a_1, a_2 \leq 1$ of the system (13). Since $P_n Q_0 \leq 1$ for all $n \geq 2$ we can proceed as above and obtain $0 \leq a_n \leq 1$ for all $n \geq 0$. Since $a_1, a_2 \geq 0$ can be chosen in an arbitrary way we again get two different solutions.

Next, consider the case, where (16) is not satisfied, that is,

$$\frac{1}{1 - \frac{Q_1}{Q_0}} < \frac{P_2 Q_0}{1 - \frac{Q_2}{Q_0}}.$$  

Then for every $a_3$ with

$$0 \leq a_3 \leq \left(1 - \frac{P_1 Q_1 - Q_2/Q_0}{1 - \frac{Q_2}{Q_0}}\right) \left(1 - \frac{Q_2}{Q_0}\right)$$

there (uniquely) exist $a_1, a_2 \geq 0$ that satisfy (14) and (15). Setting $a_0 = (1 - a_1)/(1 - P_0 Q_0)$ we get a solution of (13) with $0 \leq a_0, a_1, a_2 \leq 1$ and proceed as in the previous case.

Finally, let us consider the case $P_2 Q_0 > 1$ and $P_3 Q_0 \leq 1$. Here we consider again the system (14), (15) and observe that

$$\frac{P_1 Q_0 - \frac{Q_1}{Q_0}}{1 - \frac{Q_1}{Q_0}} > \frac{P_1 Q_1 - \frac{Q_2}{Q_0}}{1 - \frac{Q_2}{Q_0}} > 0$$

and

$$\frac{1}{1 - \frac{Q_1}{Q_0}} \geq \frac{P_2 Q_0}{1 - \frac{Q_2}{Q_0}}.$$  

Thus, we can proceed as the case where $P_1 Q_0 > 1$, $P_2 Q_0 \leq 1$, and (16) are satisfied. This completes the proof of Theorem 2.4.

It is not clear whether the above arguments (recursive elimination etc.) can be extended to all cases where $P_n$ and $Q_n$ are monotonically decreasing. The condition $P_0 Q_0 \leq 1$ seems to be artificial. We leave this as an open problem.

### 2.3 Convergence Conditions

If $C = \lim C_n \geq 1$ then one would not expect convergence. However, there is a simple trick to reduce the original problem to the contraction case that has been considered in Theorem 2.1.

**Theorem 2.5** Suppose that $P_n, Q_n, T_n$ are non-negative sequences such that $P_n$ and $Q_n$ converges to 0, $T_n$ is convergent (with limit T), and $C_n$ convergent to $C \geq 1$. Let $Q_n^{(r)}$ be recursively defined by $Q_n^{(1)} = Q_{n-k}$ and

$$Q_{n,k}^{(r+1)} := \sum_{\ell=k}^{n-r} P_{\ell+1} Q_{\ell-k} Q_{n,\ell+1}^{(r)} \quad (r \geq 1).$$
Suppose that there exists \( r \geq 2 \) and positive real numbers \( c_1, c_2, \ldots, c_r \) with \( c_1 + c_2 + \cdots + c_r = 1 \) such that
\[
\limsup_{n \to \infty} \sum_{k=0}^{n-r+1} P_k \left( \sum_{j=1}^{r} (-1)^j c_j Q_{n,k}^{(j)} \right) < 1
\]
then every sequence \( a_n \) that satisfies (10) is convergent with limit
\[
\lim_{n \to \infty} a_n = \frac{T}{C+1}.
\]

**Proof.** Set \( b_n = a_n - T/(C + 1) \). Then it follows by induction that for every \( r \geq 0 \) the sequence \( b_n \) satisfies the recurrence
\[
b_{n+1} = (-1)^r \sum_{k=0}^{n-r+1} b_k P_k Q_{n,k}^{(r)} + b_0 P_0 \sum_{j=1}^{r-1} (-1)^j Q_{n,0}^{(j)} + \bar{T}_n^{(r)}
\]
for some sequence \( \bar{T}_n^{(r)} \) that tends to 0. Furthermore \( Q_{n,0}^{(r)} \to 0 \) as \( n \to \infty \). Thus, it follows that \( b_n \) satisfies the recurrence
\[
b_{n+1} = \sum_{k=0}^{n-r+1} b_k P_k \sum_{j=1}^{r} (-1)^j c_j Q_{n,k}^{(j)} + o(1),
\]
too. Hence, we are in a similar situation as in Theorem 2.1, and it follows by completely the same arguments that \( b_n \to 0 \). \( \blacksquare \)

In Section 3 we will use this method for proving convergence for the case \( P_n \sim \beta/\log n \) and \( Q_n \sim 1/n \) for any \( \beta \geq 1 \).

**3  The Original Case**

**3.1  Convergence**

In this section we will discuss the solution \( a_n \) of (10), where \( P_n \sim \beta/\log n \) and \( Q_n \sim 1/n \) in more detail.

**Theorem 3.1** Let \( \beta > 0 \) be given and suppose that the sequences \( P_n, Q_n, T_n \) are non-negative with \( P_n \sim \beta/\log n \), \( Q_n \sim 1/n \), and \( T_n \sim 1 \) as \( n \to \infty \). Then every solution \( a_n \) of (10) converges to the limit
\[
\lim_{n \to \infty} a_n = \frac{1}{\beta + 1}.
\]

If \( P_n = \beta/\log n \) (for \( n \geq 2 \)), \( Q_n = 1/(n+1) \), and \( T_n = 1 \) we can be even more precise.

**Theorem 3.2** Let \( \beta > 0 \). Then every solution of the recurrence
\[
a_{n+1} = 1 - \beta \sum_{k=2}^{n} \frac{a_k}{(n-k+1) \log k} \quad (n \geq 2)
\]
is convergent with limit
\[
\lim_{n \to \infty} a_n = \frac{1}{1 + \beta}.
\]
Furthermore there exists an asymptotic series expansion of the form
\[
a_n \sim \frac{1}{1 + \beta} + \sum_{j \geq 0} c_j (\log n)^{-j}
\]
for certain real numbers $c_j$ (which are independent of the initial value $\sigma_2$). In particular
\[
a_n = \frac{1}{1 + \beta} - \frac{\beta \gamma}{(1 + \beta)^2 \log(n)} - \frac{1}{6} \frac{\beta [ -6 \beta \gamma^2 + \pi^2 + \beta \pi^2 ]}{(1 + \beta)^3 \log^2(n)} + O \left( \frac{1}{\log^3(n)} \right)
\]

We start with some easy properties.

**Lemma 3.3** Suppose that $P_n \sim \beta / \log n$ and $Q_n \sim 1/n$ as $n \to \infty$. Then for every $r \geq 1$ the sequence $Q_{n,k}^{(r)}$ satisfies
\[
Q_{n,k}^{(r)} \sim r \beta^r \frac{(\log(n-k))^{r-1}}{(n-k+1) (\log(n))^{r-1}}
\]
uniformly for $n/2 \leq k \leq n - r$. Furthermore we have
\[
\sum_{0 \leq k \leq n/2} P_k Q_{n,k}^{(r)} = O \left( \frac{1}{\log n} \right)
\]
and uniformly for $\varepsilon \leq \rho \leq 1 - \varepsilon$
\[
\sum_{n - n^\varepsilon \leq k \leq n - r + 1} P_k Q_{n,k}^{(r)} = \beta^r \rho^r + o(1).
\]
as $n \to \infty$, where $\varepsilon > 0$.

**Proof.** The proof is easy. We leave the detail to the reader.

**Lemma 3.4** Suppose that
\[
T(x) = d_1 x - d_2 x^2 + d_3 x^3 + \cdots + (-1)^{r-1} d_r x^r
\]
is a polynomial with (absolute) coefficients $d_j > 0$ and set
\[
c_j = \frac{d_j \beta^r}{d_1 \beta + d_2 \beta^2 + \cdots + d_r \beta^r} = -\frac{d_j}{\beta^j \cdot T \left( -\frac{1}{\beta} \right)}.
\]
Then we have
\[
\lim_{n \to \infty} \sum_{k=0}^{n-r+1} P_k \left| \sum_{j=1}^{r} (-1)^j c_j Q_{n,k}^{(j)} \right| = \frac{1}{2} \int_0^1 |T''(x)| \, dx.
\]

**Proof.** The proof follows immediately from Lemma 3.3.

The final step of the proof of Theorem 3.1 is to find proper polynomials $T(x)$ with small integral $\int_0^1 |T'(x)| \, dx$ and large $|T(-1/\beta)|$. One possibility is to use Legendre polynomials
\[
P_m(x) = \frac{1}{2^m m!} \frac{d^m}{dx^m} (x^2 - 1)^m.
\]
that have all zeros in the interval $[-1, 1]$. They satisfy
\[
\int_{-1}^1 P_m(x)^2 = \frac{2}{2m + 1}
\]
and their generating function is given by
\[
\sum_{n=0}^{\infty} P_m(x) t^n = \frac{1}{\sqrt{1 - 2xt + t^2}}.
\]

With help of these properties we can easily prove the following estimates.
Lemma 3.5 Let \( T(x) \) be defined by \( T(0) = 0 \) and \( T(x) = P_m(2x - 1) \), where \( P_m(x) \) is the \( m \)-th Legendre polynomial. Then we have
\[
\int_0^1 |T'(x)| \, dx \leq \frac{1}{\sqrt{2m+1}}
\]
and for every (fixed) \( \eta > 0 \)
\[
|T(-\eta)| \sim \frac{\sqrt{\pi} (\eta(1 + \eta))^{1/4}}{2m^{3/2}} (1 - 2\sqrt{\eta(1 + \eta)} + 2\eta)^{-m-1/2}
\]
as \( m \to \infty \).

**Proof.** The first relation (23) follows from (21) and Cauchy’s inequality. For the proof of the second relation (24) we use the formula
\[
\int_{-1}^x P_m(t) \, dt = \frac{(1 - x^2) P_m'(x)}{m(m + 1)}
\]
and obtain
\[
T(x) = \frac{2x(1 - x) P_m'(2x - 1)}{m(m + 1)}.
\]
From (22) it follows that the generating function of \( P_m'(-2\eta - 1) \) is given by
\[
\sum_{n=0}^{\infty} P_m'(-2\eta - 1) t^n = \frac{t}{(1 + 2(2\eta + 1)t + t^2)^{3/2}}.
\]
If \( \eta > 0 \) then the dominating singularity is given by
\[
t_0 = 1 - 2\sqrt{\eta(1 + \eta)} + 2\eta < 1
\]
and the dominant behaviour around the singularity is of the form
\[
\frac{1}{8t_0^{1/2}(\eta(1 + \eta))^{3/4}} \left(1 - \frac{t}{t_0}\right)^{-3/2}.
\]
Hence we obtain
\[
P_m'(-2\eta - 1) \sim \frac{\sqrt{m \pi}}{4t_0^{1/2}(\eta(1 + \eta))^{3/4}} t_0^{-m}
\]
and consequently (24). □

The proof of Theorem 3.1 is now immediate. By Lemma 3.5 for every \( \beta \geq 1 \) there exists \( m \) such that with \( T'(x) = P_m(2x - 1) \) we have
\[
\frac{\int_0^1 |T'(x)| \, dx}{\left| T\left( -\frac{1}{\beta} \right) \right|} < 1.
\]
(Note that \( 1 - 2\sqrt{\eta(1 + \eta)} + 2\eta < 1 \) for all \( \eta > 0 \).) Hence, by combining Lemma 3.4 and Theorem 2.5 it follows that \( a_n \) is convergent.

The proof of Theorem 3.2 runs along the same lines. The only difference is that we can be a little bit more precise concerning the error terms that are all of order \( O(1/\log n) \). Thus it also follows that \( b_n = O(1/\log n) \) and consequently \( a_n = T/(1 + C) + O(1/\log n) \). Finally, we can use this estimate and a simple bootstrapping procedure (via the basic recurrence (18)) to derive the asymptotic series expansion (20).
3.2 The Homogeneous Solution

In the proof of Theorem 2.4 we have used the property that every solution $a_n$ is given by

$$a_n = a_n^{(1)} + \frac{a_2 - a_2^{(1)}}{a_2^{(2)} - a_2^{(1)}} (a_n^{(2)} - a_n^{(1)}) \quad (n \geq 2),$$

where $a_n^{(1)}$, $a_n^{(2)}$ are two different solutions. We can also say that

$$a_n = a_n^{(p)} + \frac{a_2 - a_2^{(p)}}{a_2^{(h)} - a_2^{(h)}} a_n^{(h)} \quad (n \geq 2),$$

where $a_n^{(p)}$ is a particular solution and $a_n^{(h)}$ the homogeneous solution of (10), that is, the solution for $T_n = 0$:

$$a_2^{(h)} = 1, \quad a_{n+1}^{(h)} = -\beta \sum_{k=2}^{n} \frac{a_k^{(h)}}{(n - k + 1) \log k} \quad (n \geq 2). \quad (25)$$

By Theorem 2.4 we can expect that there always exists a particular solution $a_n^{(p)}$ with $0 \leq a_n^{(p)} \leq 1$. Thus, the shape of the homogeneous solution $a_n^{(h)}$ describes the structure of all other solutions.

Since we know from Theorem 3.1 that every solution converges to the same limit it follows that the homogeneous solution satisfies

$$\lim_{n \to \infty} a_n^{(h)} = 0.$$

Nevertheless it is of some interest to analyze this sequence for small $n$. It turns out that it is oscillating and quite large for $n \ll 2^{\beta}$ (even for moderate $\beta > 1$) and gets small for $n \gg 2^{\beta}$.

In fact, after a few experiments, we find that we must analyse 3 regions: $R1 : n = O(1)$, $R2 : n = O(2^\beta)$, $R3 : n \gg O(2^\beta)$.

3.2.1 Region 1: $n = O(1)$

As $\beta > 1$, (25) leads to an increasing, alternating sequence, shown in Figure 2. We have chosen $\beta = 5$. $|a_n^{(h)}|$ increases first exponentially. In fact, we have for every fixed $n$ and $\beta \to \infty$

$$a_n^{(h)} \sim (-1)^n \frac{\beta^{n-2} a_2^{(h)}}{\prod_{j=2}^{n-1} \log j}.$$

3.2.2 Region 2: $n = O(2^\beta)$

We observe the alternating behaviour shown in Figure 3 (again $\beta = 5$, as in all next figures).

So we set $\bar{a}_{2i} = a_{2i}^{(h)}$, $\bar{a}_{2i+1} = -a_{2i+1}^{(h)}$. This leads to

$$\begin{align*}
\bar{a}_{2i} &= -\beta \sum_{\ell=1}^{i-1} \frac{\bar{a}_{2\ell}}{(2\ell - 1 - 2\ell + 1) \log (2\ell)} + \beta \sum_{\ell=1}^{i-1} \frac{\bar{a}_{2\ell+1}}{(2\ell - 1 - (2\ell + 1) + 1) \log (2\ell + 1)}, \\
\bar{a}_{2i+1} &= \beta \sum_{\ell=1}^{i} \frac{\bar{a}_{2\ell}}{(2\ell - 2\ell + 1) \log (2\ell)} - \beta \sum_{\ell=1}^{i-1} \frac{\bar{a}_{2\ell+1}}{(2\ell - (2\ell + 1) + 1) \log (2\ell + 1)},
\end{align*} \quad (26)$$

and $\bar{a}_{2} = 1$. The behaviour, shown in Figure 4, is now very regular.

In what follows we will present some heuristics to analyze (26) and (27). We use the fact that

$$\sum_{0}^{(n-1)/2} \frac{1}{2\ell + 1} = \frac{1}{2} \log(n) + \frac{\gamma}{2} + \frac{1}{2} \log(2) + O(1/n),$$

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Figure 2: $a_n^{(k)}$ ($\beta = 5$).

Figure 3: $a_n^{(h)}$. 
\[
\sum_{i=1}^{n/2} \frac{1}{2i} = \frac{1}{2} \log(n) - \frac{1}{2} \log(2) + \frac{\gamma}{2} + O(1/n),
\]
\[
\frac{1}{\log(n)} - \frac{1}{\log(n-i)} = \frac{\log(1-i/n)}{\log(n)[\log(n) + \log(1-i/n)]} \approx \frac{-i}{n \log^2(n)} - \frac{i^2 \log(n) + 2}{n^2 \log^3(n)} + O \left( \frac{i^3}{n^3 \log^2(n)} \right).
\]

So we try to approximate (26) or (27) by the following first order ordinary differential equation (ODE)
\[
\frac{\partial \tilde{a}(s)}{\partial s} = \left[ \frac{D_1}{\log(s)} - 1 \right] \tilde{a}(s),
\]
with \(D_1 = \beta \ln(2)\) and \(\tilde{a}(3) = \bar{a}_3\). The solution of (28) is given by
\[
\tilde{a}(s) = \bar{a}_3 \exp[-(s-3) + D_1[E_i(\log(s)) - E_i(\log(3))]],
\]
and displayed in Figure 5.

The behaviour is surprisingly similar to Figure 4, taking all our approximations into account and the sensitivity to \(\bar{a}_3\). The maximum of (29) occurs at \(s = 2^\beta\) (same value as for \(\bar{a}_3\)). A comparison is given in Figure 6.

We have observed that the two functions are more and more similar as \(\beta\) increases.

A better approximation can be derived as follows, using (27) and Euler-McLaurin’s formula.

Set \(f(i) := \frac{\bar{a}_3 - \tilde{a}(\frac{s-i}{\log(s)})}{\log(s-i)}\). We obtain, for even \(s\),
\[
\tilde{a}(s + 1) \approx \beta \left[ \sum_{i=2, \text{by } 2}^{s-2} f(i) - \sum_{i=1, \text{by } 2}^{s-3} f(i) \right] + \beta \frac{\bar{a}(s)}{\log(s)} \left[ \sum_{i=0, \text{by } 2}^{s-2} \frac{1}{i+1} - \sum_{i=1, \text{by } 2}^{s-3} \frac{1}{i+1} \right]
\]
\[
\approx \beta \left\{ \frac{1}{2} [f(2) - f(1)] + \frac{1}{2} [f(s-2) - f(s-3)] - \frac{1}{2} \int_1^2 f(i) \, di \right. \\
+ \left. \frac{1}{2} \int_{s-3}^{s-2} f(i) \, di + \frac{B_2}{2} [f(s-2) - f'(s-3) - f'(2)] + f'(1) \right\}
\]
\[
+ \frac{B_4}{4!} \cdot 2^3 \left[ f^{3/2}(s-2) - f^{3/2}(s-3) - f^{3/2}(2) + f^{3/2}(1) \right]
\]
\[
+ \beta \frac{\bar{a}(s)}{\log(s)} \left[ \log(2) + \frac{1}{2s} + \frac{1}{4s^2} \right].
\]
Figure 5: $\bar{a}(s)$.

Figure 6: $\bar{a}(s)$ (first order ODE) and $\bar{a}[s]$. 
We now replace, in the LHS, $\tilde{a}(s + 1)$ by $\tilde{a}(s) + \tilde{a}'(s) + \tilde{a}''(s)/2$ and, in the RHS, $\tilde{a}(s - i)$ by $\tilde{a}(s) - i\tilde{a}'(s) + i^2/2\tilde{a}''(s)$ in the neighbourhood of $s$. This leads to a $2^{nd}$ order ODE, which is easily (numerically) solved by MAPLE. Figure 7 displays $\hat{a}(s)$ and $\hat{a}$. The fit is now quite good.

3.2.3 Region 3: $n \gg 2^{\beta}$

Here we observe that $a_n^{(h)}$ gets small and tends to 0 (as predicted). This is shown in Figure 8.

3.2.4 Remark

Unfortunately these approximations for the homogeneous solution are not rigorous. However, they indicate that these kind of recurrences are quite interesting if one is interested for approximations that hold for all $n$. We have to split the the positive integers into several parts and approximate $a_n^{(h)}$ by different methods. Of course this kind of approach is not new, for example compare with [6].

4 Conclusion

The investigation of some probabilistic structures (concerning branching stochastic processes) leads to a new analytical recurrence. This phenomenon was considered in a more general situation. Using tools from analysis, we have proved the convergence of the recurrence for all positive values of the parameter $\beta$ and the asymptotic analysis of its behaviour have been done. This results have allowed us to generalize some theorems for branching processes. Finally, some heuristic approximations, based on differential equations, have also been provided.

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Figure 8: $a_n^{(h)}$ for $n \gg 2^\beta$


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