Embedded Trees and the Support of the ISE*

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Abstract. Embedded trees are labelled rooted trees, where the root has zero label and where the labels of adjacent vertices differ by ± 1. Recently it was proved by Chassaing and Schaeffer, and Janson and Marckert that the distribution of the maximum and minimum label are closely related to the support of the density of the integrated superbrownian excursion (ISE). The purpose of this paper is make this probabilistic limiting relation more explicit by using a generating function approach due to Bouttier, Di Francesco, and Guitter that is based on properties of Jacobi's θ-functions. In particular we derive an integral representation of the joint distribution function of the supremum and infimum of the support of the ISE in terms of the Weierstrass $\wp$-function.

1 Introduction

A *planted plane tree* is a rooted ordered tree, which means that all successors of a node have a left-to-right order. It is a classical result that the number p_n of planted planted plane trees with n edges equals the Catalan number

$$p_n = \frac{1}{n+1}\binom{2n}{n}.$$

An *embedded tree* (with increments ± 1) is a planted plane tree, where the vertices are labelled by integers such that the root has label 0 and labels of adjacent vertices differ by ± 1 (see Figure 1). By construction the number q of different embedded trees (with increments ± 1) is given by

$$q_n = 2^n p_n = \frac{2^n}{n+1}\binom{2n}{n}.$$

In what follows we assume that every embedded tree (with n edges) is equally likely. Of course, in this random setting every parameter on embedded trees becomes a random variable.

Let $X_n(j)$ denote the number of vertices with label j in a (random) embedded tree of size n. The sequence $(X_n(j))_{j\in\mathbb{Z}}$ is then the *label profile*, and let $X_n(t)$,

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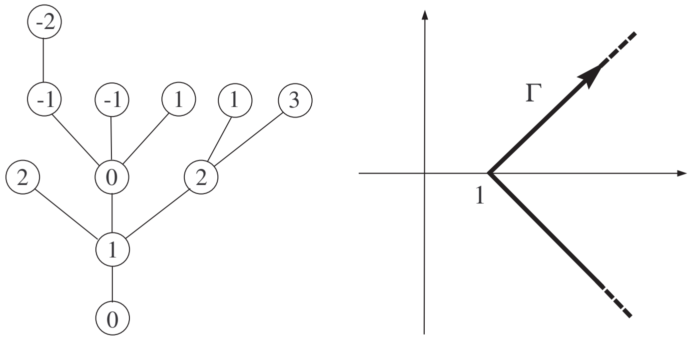

Fig. 1. Embedded tree and contour of integration Γ

$t \in \mathbb{R}$, be the the linearly interpolated (random) function. Recently, Bousquet-Mélou and Janson [3] proved that

$$\left(n^{-3/4} X_n(n^{1/4}t),\ -\infty < t < \infty\right) \xrightarrow{d} \left(f_{\mathrm{ISE}}(t),\ -\infty < t < \infty\right), \qquad (1)$$

where $\xrightarrow{d}$ denotes weak convergence in the space $\mathrm{C}_0(\mathbb{R})$ and the stochastic process $(f_{\mathrm{ISE}}(t),\ -\infty < t < \infty)$ is the densitiy of the integrated superbrownian excursion (ISE). Recall that the ISE is a random measure which can be seen – despite a scaling factor $\sqrt{2}$ – as the occupation measure of the head of the Brownian snake (see Chassaing and Schaeffer [6], Janson and Marckert [9], and Bousquet-Mélou and Janson [3]).

One interesting feature of the ISE is that the support of its density $[L_{\mathrm{ISE}}, R_{\mathrm{ISE}}]$ is (almost surely) a finite interval. By (1) it is clear that the largest label M_n and the smallest label m_n of a random embedded tree with n edges is related to R_{ISE} and L_{ISE}:

$$\frac{M_n}{n^{1/4}} \xrightarrow{d} R_{\mathrm{ISE}} \quad \text{and} \quad \frac{m_n}{n^{1/4}} \xrightarrow{d} L_{\mathrm{ISE}}.$$

We also have

$$\frac{M_n - m_n}{n^{1/4}} \xrightarrow{d} R_{\mathrm{ISE}} - L_{\mathrm{ISE}}.$$

Note that R_{ISE} and $-L_{\mathrm{ISE}}$ have the same distribution but they are not independent.

By using the relation between M_n and R_{ISE} and asymptotics of generating functions Bousquet-Mélou [2] proved a remarkable integral representation of the tail distribution function $G(\lambda) = \mathbb{P}\{R_{\mathrm{ISE}} > \lambda\}$:

$$G(\lambda) = \frac{12}{i\sqrt{\pi}} \int_{\Gamma} \frac{v^5 e^{v^4}}{\sinh^2(\lambda v)} dv, \qquad (2)$$

where

$$\Gamma = \{1 - te^{-i\pi/4}, t \in (-\infty, 0]\} \cup \{1 + te^{i\pi/4}, t \in [0, \infty)\}, \qquad (3)$$

see Figure 1. In [7] one can also find a relation for the Laplace transform of the function $x^{-\frac{3}{2}}\mathbb{P}\{R_{\mathrm{ISE}} > x^{\frac{1}{4}}\}$ which is given by

$$\int_0^\infty x^{-\frac{3}{2}}\mathbb{P}\{R_{\mathrm{ISE}} > x^{\frac{1}{4}}\}\, e^{-sx}\, dx = \frac{6\sqrt{\pi s}}{\left(\sinh((s/2)^{1/4})\right)^2}$$

and representations for the moments

$$\mathbb{E}(R_{\mathrm{ISE}}^r) = \frac{24\sqrt{\pi}\,\Gamma(r+1)\zeta(r-1)}{2^r\Gamma((r-2)/4)},$$

for $\Re(r) > -4$, where the right hand side has to be analytically continued at the points $-3, -2, -1, 2$.

The purpose of this paper is to extend the result (2) by Bousquet-Mélou. We will provide integral representations for the joint distribution of L_{ISE} and R_{ISE} and also for the length $R_{\mathrm{ISE}} - L_{\mathrm{ISE}}$ of the support of the ISE. In the proof we use an explicit representation of the corresponding generating function of embedded trees in terms of θ-functions (see [5]) and use asymptotics of these generating functions, where Eisenstein series the Weierstrass $\wp$-function appear.

The structure of the paper is as follows. In Section 2 we give precise statements of our results. The proof is then divided into two major parts. First we discuss combinatorics on embedded trees (Section 3) and derive then the asymptotic results in Section 4.

2 Results

As above let M_n and m_n denote the maximum and minimum labels in embedded trees of size n, respectively. In order to formulate our main result we need the notion of the Weierstrass $\wp$-function

$$\wp(z;\tau) = \frac{1}{z^2} + \sum_{(m_1,m_2)\in\mathbb{Z}^2\setminus\{(0,0)\}} \left(\frac{1}{(z-m_1\tau-m_2)^2} - \frac{1}{(m_1\tau+m_2)^2}\right),$$

where τ and z are complex variables with $\Im(\tau) > 0$ and $z \notin \mathbb{Z} + \tau\mathbb{Z}$. The $\wp$-function – considered as a function in z – is an elliptic function that has periods 1 and τ. It is analytic in τ and meromorphic in z with double poles on the lattice points $\mathbb{Z} + \tau\mathbb{Z}$; for details we refer to [10].

Theorem 1. *The distribution function*

$$F(\lambda_1,\lambda_2) = \mathbb{P}\{R_{\mathrm{ISE}} \le \lambda_1,\ -L_{\mathrm{ISE}} \le \lambda_2\}$$

of the limit

$$\left(\frac{M_n}{n^{1/4}}, \frac{-m_n}{n^{1/4}}\right) \xrightarrow{d} (R_{\mathrm{ISE}}, -L_{\mathrm{ISE}})$$

is given by

$$F(\lambda_1,\lambda_2) = \frac{20}{3i\pi^{5/2}} \int_\Gamma \wp\left(-\frac{i\lambda_1}{\pi}v; -\frac{i(\lambda_1+\lambda_2)}{\pi}v\right) v^5\, e^{\frac{5}{9}v^4}\, dv.$$

Similarly we obtain an integral representation for the length of the support of the ISE.

Theorem 2. *The distribution function*

$$H(\lambda) = \mathbb{P}\{R_{\mathrm{ISE}} - L_{\mathrm{ISE}} \le \lambda\}$$

of the limit

$$\frac{M_n - m_n}{n^{1/4}} \xrightarrow{d} R_{\mathrm{ISE}} - L_{\mathrm{ISE}}$$

is given by

$$H(\lambda) = -\frac{20}{3\pi^{7/3}} \int_0^\lambda \int_\Gamma \frac{\partial}{\partial \tau} \wp\left(-\frac{is}{\pi}v; -\frac{i\lambda}{\pi}v\right) v^6\, e^{\frac{5}{9}v^4}\, dv\, ds.$$

There is almost no literature on explicit results on the support $[L_{\mathrm{ISE}}, R_{\mathrm{ISE}}]$ of the ISE. Besides the aforementioned results on R_{ISE} the expected values

$$\mathbb{E}\left(-R_{\mathrm{ISE}} L_{\mathrm{ISE}}\right) = -3\sqrt{2\pi} + 2\sqrt{2\pi} \int_1^\infty \int_1^\infty \frac{(u+1)}{\sqrt{t^3-1}\sqrt{u^3-1}(u+\sqrt{u^2+u+1})}\, du\, dt$$

and

$$\mathbb{E}\left(\min\{R_{\mathrm{ISE}}, -L_{\mathrm{ISE}}\}\right) = 6\sqrt{2\pi}\left(1 - \frac{1}{8}\left(\int_1^\infty \frac{du}{\sqrt{u^3-1}}\right)^2\right)$$

have been computed by Delmas [7].

3 Combinatorics

Let $P(t)$ denote the generating function of planted plane trees, where the exponent of t counts the number of edges. Then by using the combinatorial decomposition – namely that all subtrees of the root are again planted plane trees, see Figure 2 – we obtain the relation

$$P(t) = 1 + tP(t) + t^2P(t)^2 + t^3P(t)^3 + \cdots = \frac{1}{1 - tP(t)}$$

and consequently

$$P(t) = \frac{1 - \sqrt{1-4t}}{2t} = \sum_{n \ge 0} \frac{1}{n+1}\binom{2n}{n} t^n.$$

It is also very easy to count embedded trees without any restriction with the help of generating functions. Let $R(t)$ denote the generating function of embedded trees, where the exponent of t counts the number of edges. Furthermore let $R_n(t)$, $n \in \mathbb{Z}$, be the generating function of embedded trees, where we assume

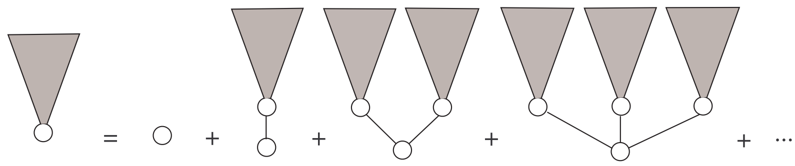

Fig. 2. Recursion for planted plane trees

that the root is labelled by n (and labels of adjacent vertices differ by ± 1). Then by using the same decomposition as above we have

$$R_n(t) = \frac{1}{1 - t(R_{n-1}(t) + R_{n+1}(t))}. \qquad (4)$$

Since there are no restrictions on the embedded trees we have $R_n(t) = R_0(t) = R(t)$ for all $n \in \mathbb{Z}$ leading to the relation

$$R(t) = \frac{1}{1 - 2tR(t)}$$

and to the explicit representation

$$R(t) = \frac{1 - \sqrt{1-8t}}{4t} = \sum_{n \ge 0} \frac{2^n}{n+1} \binom{2n}{n} t^n.$$

The situation becomes more interesting if we just consider embedded trees, where all labels are non-negative. Let $R_n^{[0]}(t)$ be the generating functions of those embedded trees, where the root has label n. By definition $R_n^{[0]}(t) = 0$ if $n < 0$. However, we have the same recurrence relation as above:

$$R_n^{[0]}(t) = \frac{1}{1 - t(R_{n-1}^{[0]}(t) + R_{n+1}^{[0]}(t))}, \qquad (n \ge 0). \qquad (5)$$

Interestigly, this system of equations has an explicit solution of the form

$$R_n^{[0]}(t) = R(t) \frac{u_n u_{n+4}}{u_{n+1} u_{n+3}},$$

where

$$u_n = u_n(t) = Z(t)^{\frac{n+1}{2}} - Z(t)^{-\frac{n+1}{2}}$$

and

$$Z(t) = \frac{1 - (1-8t)^{1/4}}{1 + (1-8t)^{1/4}}$$

is the solution of the equation

$$Z + \frac{1}{Z} + 2 = \frac{1}{tR(t)}$$

that is analytic at $t = 0$. This miraculous relation was observed by Bouttier, Di Francesco, and Guitter [4]. In fact this explicit solution was used by Bousquet-Mélou [2] to obtain the integral representation for (2).

In another paper Bouttier, Di Francesco, and Guitter [5] considered the class of embedded trees, where all labels are bounded between 0 and L, where L is a non-negative integer. Let $R_n^{[0,L]}(t)$ be the generating functions of those embedded trees, where the root has label n. By definition $R_n^{[0,L]}(t) = 0$ if $n < 0$ of $n > L$. We have the same recurrence relation as above:

$$R_n^{[0,L]}(t) = \frac{1}{1 - t(R_{n-1}^{[0,L]}(t) + R_{n+1}^{[0,L]}(t))}, \qquad (0 \le n \le L). \tag{6}$$

Interestingly there is an explicit solution of this system of equation in terms of the Jacobi theta function

$$\theta_1(u;q) = 2i\sin(\pi u)\prod_{j\ge 1}\left(1 - 2q^j\cos(2\pi u) + q^{2j}\right). \tag{7}$$

First let $q = q(t)$ be determined by the equation

$$t = \frac{\theta_1\left(\frac{1}{L+6}, q\right)^4 \theta_1\left(\frac{4}{L+6}, q\right)}{\theta_1\left(\frac{2}{L+6}, q\right)^5} \tag{8}$$

Then we have (see [5])

$$R_n^{[0,L]}(t) = \frac{\theta_1\left(\frac{2}{L+6}, q\right)^3}{\theta_1\left(\frac{1}{L+6}, q\right)^2 \theta_1\left(\frac{4}{L+6}, q\right)} \frac{\theta_1\left(\frac{n+1}{L+6}, q\right)\theta_1\left(\frac{n+5}{L+6}, q\right)}{\theta_1\left(\frac{n+2}{L+6}, q\right)\theta_1\left(\frac{n+4}{L+6}, q\right)}. \tag{9}$$

4 Asymptotic Analysis

In [5] the generating functions $R_n^{[0,L]}(t)$ have been analyzed by considering so-called scaling limits which can be interpreted in terms of potentials and characteristic lengths etc. For our purpose we have to be more precise, since we are interested in asymptotics of the coefficients. Nevertheless, we use – more or less – the same of scaling as in [5].

By shifting labels from 0 to j it follows that

$$\mathbb{P}\{M_n \le k,\, m_n \ge -j\} = \frac{[t^n]\, R_j^{[0,j+k]}(t)}{\frac{2^n}{n+1}\binom{2n}{n}}. \tag{10}$$

Thus, in order to prove Theorem 1 we need asymptotics of the coefficient $[t^n]\, R_j^{[0,j+k]}(t)$. Note that it is not necessary to prove asymptotics in the full range of parameters. In particular, we will set $j \sim \lambda_1 n^{1/4}$ and $k \sim \lambda_2 n^{1/4}$ for positive real numbers λ_1, λ_2.

We use Cauchy's formula

$$[t^n]\, R_j^{[0,j+k]}(t) = \frac{1}{2\pi i}\int_\gamma R_j^{[0,j+k]}(t) t^{-n-1}\, dt,$$

where γ is a certain contour of winding number $+1$ around the origin, contained in the analyticity region of $R_j(t)^{[0,j+k]}$. In this case we will use a path of integration γ of the form $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$, where $x_0 = \frac{1}{8}$, $c > 0$,

$$\begin{aligned}
\gamma_1 &= \left\{ x = x_0 \left(1 - c\frac{i + n^{1/4} - s}{n}\right) : 0 \le s \le n^{1/4} \right\},\\
\gamma_2 &= \left\{ x = x_0 \left(1 - c\frac{1}{n}e^{-i\varphi}\right) : -\frac{\pi}{2} \le \varphi \le \frac{\pi}{2} \right\},\\
\gamma_3 &= \left\{ x = x_0 \left(1 + c\frac{i + s}{n} :\right) 0 \le s \le n^{1/4} \right\},
\end{aligned}$$

and γ_4 is a circular arc centred at the origin and making γ a closed curve (see also Figure 3). Note that $\gamma_1 \cup \gamma_2 \cup \gamma_3$ constitutes a so-called Hankel contour that appears in Hankel's integral representation of $1/\Gamma(s)$.

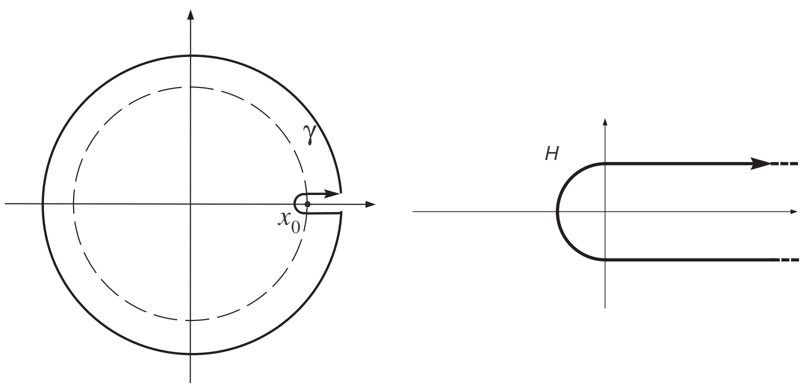

Fig. 3. Contour of integration and Hankel contour

By the relation (8), t and q are related. We will first study this relation for $t \in \gamma_1 \cup \gamma_2 \cup \gamma_3$. For this purpose we use the following asymptotic property.

Lemma 1. *Suppose that $q = e^{2\pi i \tau}$ and L satisfy $|1-q| \ge c/L$ for some constant $c > 0$. Then*

$$\frac{\theta_1\left(\frac{1}{L+6}, q\right)^4 \theta_1\left(\frac{4}{L+6}, q\right)}{\theta_1\left(\frac{2}{L+6}, q\right)^5} = \frac{1}{8}\left(1 - \frac{25}{(L+6)^4}G_4(\tau) + O\left(\frac{1}{L^6|1-q|^6}\right)\right), \quad (11)$$

where $G_4(\tau)$ denotes the Eisenstein series

$$G_4(\tau) = \sum_{(m_1,m_2)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(m_1+m_2\tau)^4}.$$

Proof. By using the Taylor series expansions for $\sin(\pi u)$ and $\cos(2\pi u)$ we can represent $\theta_1(u,q)$ as

$$\theta_1(u,q) = 2\pi i\, u \exp\left(\left(-\frac{\pi^2}{6} + 4\pi^2 \sum_{j\geq 1} \frac{q^j}{(1-q^j)^2}\right)u^2 + \left(-\frac{\pi^4}{180} - \frac{4\pi^4}{3}\sum_{j\geq 1}\frac{q^j}{(1-q^j)^2} - 8\pi^4 \sum_{j\geq 1}\frac{q^{2j}}{(1-q^j)^4}\right)u^4 + O\left(\frac{u^6}{|1-q|^6}\right)\right).$$

This gives

$$\frac{\theta_1\left(\frac{1}{L+6},q\right)^4 \theta_1\left(\frac{4}{L+6},q\right)}{\theta_1\left(\frac{2}{L+6},q\right)^5} = \frac{1}{8}\left(1 - \frac{100\pi^4}{(L+6)^4}\left(\frac{1}{180} + \frac{4}{3}S_1 + 8S_2\right) + O\left(\frac{1}{L^6|1-q|^6}\right)\right),$$

where S_1 and S_2 abbreviate

$$S_1 = \sum_{j\geq 1}\frac{q^j}{(1-q^j)^2} \quad \text{and} \quad S_2 = \sum_{j\geq 1}\frac{q^{2j}}{(1-q^j)^4}$$

By using the notation $\sigma_\ell(n) = \sum_{d|n} d^\ell$ we have

$$\frac{4}{3}S_1 + 8S_2 = \frac{4}{3}\sum_{j,k} k^3 q^{jk} = \frac{4}{3}\sum_{n\geq 1}\sigma_3(n)q^n.$$

Since

$$G_4(\tau) = \sum_{(m_1,m_2)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(m_1+m_2\tau)^4} = \frac{\pi^4}{45} + \frac{16\pi^4}{3}\sum_{n\geq 1}\sigma_3(n)e^{2\pi i n\tau}.$$

it follows that

$$\frac{\pi^4}{180} + \frac{4}{3}\pi^4 S_1 + 8\pi^4 S_2 = \frac{1}{4}G_4(\tau). \tag{12}$$

This proves (11).

Alternatively to the previous proof we could have used the relation

$$G_4(\tau) = \frac{5}{8}\left(\frac{\theta_1'''(0,q)}{\theta_1'(0,q)}\right)^2 + \frac{3}{8}\frac{\theta_1^{(5)}(0,q)}{\theta_1'(0,q)},$$

where $'$ denotes the derivative with respect to the first variable u.

Next we discuss the behaviour of $G_4(\tau)$ if τ is close to 0.

Lemma 2. *We have uniformly for $\tau \to 0$ with $\varepsilon \le \arg(\tau) \le \pi - \varepsilon$ (for any $\varepsilon > 0$)*

$$G_4(\tau) = \frac{\pi^4}{45}\,\tau^{-4} + O\left(\tau^{-3}\right). \tag{13}$$

Proof. We just study the behaviour of S_2, since it constitutes the asymptotic leading term in the representation (12). Set $q = e^{-x}$. Then by using the representation

$$S_2 = S_2(x) = \frac{1}{6}\sum_{n\ge 1}\left(\sigma_3(n) - \sigma_1(n)\right)e^{-nx}$$

we obtain that the Mellin transform of $S_2(x)$ (see [8]) is given by

$$F(s) = \int_0^\infty S_2(x)x^{s-1}\,dx = \frac{\Gamma(s)}{6}\zeta(s)\left(\zeta(s-3) - \zeta(s-1)\right).$$

for complex s with $\Re(s) > 4$. By taking the inverse Mellin transform (and shifting the line of integration to the left and taking into account the residue at $s = 4$) one gets directly

$$S_2(x) = \zeta(4)\,x^{-4} + O(x^{-3}) = \frac{\pi^4}{90}\,x^{-4} + O(x^{-3})$$

which is uniform for $x \to 0$ when $|\arg(x)| \le \pi - \varepsilon$ for any $\varepsilon > 0$ (see again [8]). Finally by setting $\tau = -x/(2\pi i)$ and by using the relation $G_4(\tau) \sim 32\pi^4 S_2$ we obtain (13).

We now assume that $L \sim j + k = (\lambda_1 + \lambda_2)n^{1/4}$ for some positive constants λ_1 and λ_2. Furthermore it is convenient to introduce a new variable

$$w = \frac{1}{2\pi}(\log(1/q))^{-1} = \frac{i}{\tau}.$$

Now suppose that t varies in $\gamma_1 \cup \gamma_2 \cup \gamma_3$ (with $x_0 = \frac{1}{8}$). If we write $t = \frac{1}{8}\left(1 - \frac{w'}{n}\right)$ then w' varies in $-H'$, where H' is a Hankel contour cut at real part $n^{1/4}$. For simplicity we neglect this cut for a moment. With the help of the asymptotic relations of Lemmas 1 and 2 we have

$$w' = \frac{5\pi^4}{9(\lambda_1 + \lambda_2)^4}\,w^4 + O(w^2).$$

Hence w varies on a contour coming from $+e^{i\pi/4}\infty$, cutting the real axis at some positive value and leaving to $+e^{-i\pi/4}\infty$ (compare with Figure 4). Hence, without loss of generality we can assume that w varies on $\hat{\Gamma}$, where Γ is defined in (3) and $\hat{}$ denotes the time reversed contour.

The next goal is to determine the asymptotic behaviour of $R_j^{[0,j+k]}(t)$ for $t \in \gamma_1 \cup \gamma_2 \cup \gamma_3$. For this purpose we will use the following property.

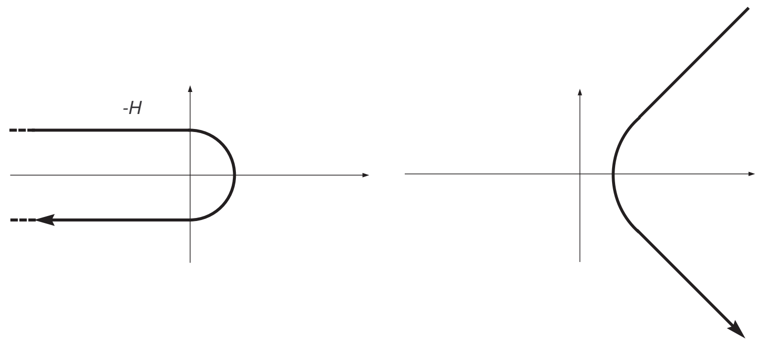

Fig. 4. Negative Hankel contour and contour of integration of w

Lemma 3. *Suppose that $q = e^{2\pi i\tau}$ and L satisfy $|1-q| \geq c/L$ for some constant $c > 0$. Then*

$$R_j^{[0,L]}(t) = 2\left(1 - \frac{3}{(L+6)^2}\,\wp\left(\frac{j+1}{L+6};\tau\right) + O\left(\frac{1}{L^4|1-q|^4}\right)\right) \tag{14}$$

uniformly for $\varepsilon \leq j/L \leq 1-\varepsilon$ (for any $\varepsilon > 0$), where $\wp(z;\tau)$ denotes the Weierstrass $\wp$-function

$$\wp(z;\tau) = \frac{1}{z^2} + \sum_{(m_1,m_2)\in\mathbb{Z}^2\setminus\{(0,0)\}}\left(\frac{1}{(z-m_1\tau-m_2)^2} - \frac{1}{(m_1\tau+m_2)^2}\right)$$

Proof. By proceeding as in the proof of Lemma 1 we obtain

$$\frac{\theta_1\left(\frac{2}{L+6},q\right)^3}{\theta_1\left(\frac{1}{L+6},q\right)^2\theta_1\left(\frac{4}{L+6},q\right)} = 2\left(1 - \frac{1}{(L+6)^2}\frac{\theta_1'''(0,q)}{\theta_1'(0,q)} + O\left(\frac{1}{L^4|1-q|^4}\right)\right).$$

Furthermore we have for $u = (j+1)/(L+6)$ (and uniformly for $\varepsilon \leq u \leq 1-\varepsilon$)

$$\frac{\theta_1\left(\frac{j+1}{L+6},q\right)\theta_1\left(\frac{j+5}{L+6},q\right)}{\theta_1\left(\frac{j+2}{L+6},q\right)\theta_1\left(\frac{j+4}{L+6},q\right)} = 1 + \frac{3}{(L+6)^2}\left(\frac{\theta_1''(u,q)}{\theta_1(u,q)} - \left(\frac{\theta_1'(u,q)}{\theta_1(u,q)}\right)^2\right) + O\left(\frac{1}{L^4\,|1-q|}\right)$$

Finally, by using the relation (see [1])

$$\frac{\theta_1'''(0,q)}{3\,\theta_1'(0,q)} - \frac{\theta_1''(u,q)}{\theta_1(u,q)} + \left(\frac{\theta_1'(u,q)}{\theta_1(u,q)}\right)^2 = \wp(u;\tau)$$

we obtain the asymptotic relation (14).

We are now ready to prove Theorem 1. We set $j+3 = \lambda_1 n^{1/4}$, $k+3 = \lambda_2 n^{1/4}$ and $L+6 = (j+3)+(k+3) = (\lambda_1+\lambda_2)n^{1/4}$. As mentioned above we use Cauchy's formula. For technical reasons we apply it to $R_j^{[0,j+k]}(t)-2$ instead of $R_j^{[0,j+k]}(t)$. Of course, if $n>0$ we have

$$[t^n]\,R_j^{[0,j+k]}(t) = [t^n]\,\left(R_j^{[0,j+k]}(t)-2\right) = \frac{1}{2\pi i}\int_\gamma \left(R_j^{[0,j+k]}(t)-2\right) t^{-n-1}\,dt$$
$$= \frac{1}{2\pi i}\int_{\gamma_1\cup\gamma_2\cup\gamma_3} \left(R_j^{[0,j+k]}(t)-2\right) t^{-n-1}\,dt + \frac{1}{2\pi i}\int_{\gamma_4} \left(R_j^{[0,j+k]}(t)-2\right) t^{-n-1}\,dt$$

We will focus on the contribution coming from the contour $\gamma_1\cup\gamma_2\cup\gamma_3$. Namely if $t\in\gamma_4$ then $|t|\sim\frac{1}{8}\left(1+cn^{-3/4}\right)$ (for some $c>0$) whereas $R_j^{[0,j+k]}(t)$ stays bounded (note that Lemma 3 still applies). Hence

$$\int_{\gamma_4}\left(R_j^{[0,j+k]}(t)-2\right) t^{-n-1}\,dt = O\left(8^n\,e^{-c\,n^{1/4}}\right)$$

which is negligible compared to the normalization $\frac{2^n}{n+1}\binom{2n}{n}\sim 8^n n^{-3/2}/\sqrt{\pi}$.

For $t\in\gamma_1\cup\gamma_2\cup\gamma_3$ we use the substitution

$$t=\frac{1}{8}\left(1-\frac{5}{9}\frac{\pi^4 w^4}{L^4}\right),$$

where $w=i/\tau$ now varies on a contour that we can deform (due to analyticity) to $\hat{\Gamma}$. Note, however, that we have to cut $\hat{\Gamma}$ to a finite contour $\hat{\Gamma}'$, since $t\in\gamma_1\cup\gamma_2\cup\gamma_3$ implies that $w=O(n^{1/16})$. In this range we use the approximations

$$R_j^{[0,j+k]}(t)-2 = -\frac{6}{(\lambda_1+\lambda_2)^2\sqrt{n}}\,\wp\left(\frac{\lambda_1}{\lambda_1+\lambda_2};\frac{i}{w}\right)+O\left(\frac{w^4}{n}\right),$$
$$t^{-n-1} = 8^{n+1}\,\exp\left(\frac{5}{9}\frac{\pi^4}{(\lambda_1+\lambda_2)^4}w^4+O\left(\frac{w^6}{\sqrt{n}}\right)\right),$$

and the substitution

$$dt = -\frac{20}{8\cdot 9}\frac{\pi^4}{(\lambda_1+\lambda_2)^4 n}\,w^3\,dw$$

that lead to the integral

$$\frac{1}{2\pi i}\int_{\gamma_1\cup\gamma_2\cup\gamma_3}\left(R_j^{[0,j+k]}(t)-2\right)t^{-n-1}\,dt = \frac{\pi^4}{2\pi i}\frac{8^n}{n^{3/2}}\frac{40}{3(\lambda_1+\lambda_2)^6}$$
$$\times\int_{\hat{\Gamma}'}\left(\wp\left(\frac{\lambda_1}{\lambda_1+\lambda_2};\frac{i}{w}\right)+O\left(\frac{w^4+w^6}{\sqrt{n}}\right)\right)\exp\left(\frac{5}{9}\frac{\pi^4}{(\lambda_1+\lambda_2)^4}w^4\right)w^3\,dw$$

At this point we can neglect the error terms and extend the cutted path of integration $\hat{\Gamma}'$ to the infinite path $\hat{\Gamma}$. Furthermore, we substitute $v=\pi w/(\lambda_1+\lambda_2)$, use the relation $\wp(z;-1/\tau)=\tau^2\wp(z\tau;\tau)$ and obtain (after reversing $\hat{\Gamma}$ to Γ and deforming $(\pi/(\lambda_1+\lambda_2))\Gamma$ to Γ)

$$[t^n]\,R_j^{[0,j+k]}(t)\sim\frac{1}{\pi^3 i}\frac{8^n}{n^{3/2}}\frac{20}{3}\int_\Gamma\wp\left(-\frac{i\lambda_1}{\pi}v;-\frac{i(\lambda_1+\lambda_2)}{\pi}v\right)\,v^5\,e^{\frac{5}{9}v^4}\,dv.$$

Since $\frac{2^n}{n+1}\binom{2n}{n} \sim 8^n n^{-3/2}/\sqrt{\pi}$ we finally derive the proposed result of Theorem 1.

Theorem 2 can be deduced from Theorem 1 and by applying the following property. Suppose that $F(\lambda_1, \lambda_2)$ is the distribution function of a non-negative random vector (X_1, X_2) which has density $\frac{\partial^2}{\partial\lambda_1\partial\lambda_2}F(\lambda_1, \lambda_2)$. Then the distribution function $H(\lambda)$ of the sum $Y = X_1 + X_2$ is given by

$$H(\lambda) = \int_0^\lambda \int_0^{\lambda-\lambda_1} \frac{\partial^2}{\partial\lambda_1\partial\lambda_2}F(\lambda_1, \lambda_2)\,d\lambda_2\,d\lambda_1 = \int_0^\lambda \frac{\partial}{\partial\lambda_1}F(\lambda_1, \lambda-\lambda_1)\,d\lambda_1$$

This directly implies

$$H(\lambda) =$$
$$-\frac{20}{3\pi^{7/3}}\int_0^\lambda \int_\Gamma \left(\frac{\partial}{\partial\tau}\wp\left(-\frac{i\lambda_1}{\pi}v; -\frac{i\lambda}{\pi}v\right) + \frac{\partial}{\partial z}\wp\left(-\frac{i\lambda_1}{\pi}v; -\frac{i\lambda}{\pi}v\right)\right) v^6\, e^{\frac{5}{9}v^4}\,dv\,d\lambda_1.$$

Finally, since $\wp(\tau - z;\tau) = \wp(z;\tau)$ we have $\frac{\partial}{\partial\tau}\wp(\tau - z;\tau) + \frac{\partial}{\partial z}\wp(\tau - z;\tau) = \frac{\partial}{\partial\tau}\wp(z;\tau)$ and consequently (by setting $s = \lambda - \lambda_1$) we derive the proposed representation for $H(\lambda)$ that is given in Theorem 2. Note that it is not possible to interchange the integrals, since the $\wp$-function is singular at $z = 0$ and $z = \tau$.

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