THE DISCREPANCY OF GENERALIZED VAN-DER-CORPUT-HALTON SEQUENCES

MICHAEL DRMOTA

ABSTRACT. The purpose of this paper is to provide upper bounds for the discrepancy of generalized Van-der-Corput-Halton sequences that are built from Halton sequences and the Zeckendorf Van-der-Corput sequence.

1. Introduction

Suppose that $q \ge 2$ is a given integer and

$$n = \sum_{j\ge 0} \varepsilon_{q,j}(n) q^j, \qquad \varepsilon_{q,j}(n) \in \{0,1,\ldots,q-1\}$$

denotes the q-ary representation of a non-negative integer n. Then the q-ary Van-der-Corput $(\phi_q(n))_{n\ge 0}$ sequence is defined by

$$\phi_q(n) = \sum_{j\ge 0} \varepsilon_{q,j}(n) q^{-j-1}.$$

It is well known that the Van-der-Corput sequence is a low discrepancy sequence, that is, the (star-) discrepancy $D_N^*(\phi_q(n))$ satisfies

$$D_N^*(\phi_q(n)) \ll \frac{\log N}{N};$$

Similarly a d-dimensional Halton equences $\Phi_{\mathbf{q}}(n)$ that is defined by $\Phi_{\mathbf{q}}(n) = (\phi_{q_1}(n),\ldots,\phi_{q_s}(n))$ satisfies

$$D_N^*(\Phi_{\mathbf{q}}(n)) \ll \frac{(\log N)^s}{N},$$

if $\mathbf{q} = (q_1,\ldots,q_s)$ consists of pairwise coprime integers $q_j \ge 2$. In particular all these sequences are uniformly distributed modulo 1 (which just means that the discrepancy tends to 0 as $N \to \infty$). For more details on discrepancy theory and uniformly distributed sequences we refer to [2, 6]. We just note that the (star-) discrepancy $D_N^*(\mathbf{x}_n)$ of a s-dimensional real sequence $(\mathbf{x}_n)_{n\ge 0}$ is defined by

$$D_N(\mathbf{x}_n) = \sup_{0<\alpha_1,\ldots,\alpha_s\le 1} \left| \frac{1}{N} \sum_{n=0}^{N-1} \mathbf{1}_{[0,\alpha_1)\times\cdots\times[0,\alpha_s)}(\mathbf{x}_n \bmod 1) - \alpha_1\cdots\alpha_s \right|.$$

Recently Hofer, Iaco, and Tichy [4] considered generalized Van-der-Corput and Halton sequences of the following form. Suppose that $b \ge 1$ and $d \ge 2$ (or $b \ge 2$ if $d = 1$) are integers and $G = (G_n)_{n\ge 0}$ is an integer sequence defined by $G_0 = 1$, $G_k = b(G_{k-1} + \cdots + G_0) + 1$ for $1 \le k < d$ and

$$G_k = b(G_{k-1} + \cdots + G_{k-d})$$

for $k \ge d$. Then every non-negative integer n can be uniquely represented by

$$n = \sum_{j\ge 0} \varepsilon_{G,j}(n)\, G_j, \tag{1.1}$$

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where the digits $\varepsilon \in \{0,1,\ldots,\lfloor G_{j+1}/G_j\rfloor\}$ are (uniquely) chosen by the *greedy condition*

$$\sum_{j=0}^{J-1} \varepsilon_{G,j}(n) G_j < G_J.$$

We will call (1.1) the G-ary expansion of n. Of course, this generalizes q-ary expansions, where $b = q$, $d = 1$, and $G_k = q^k$.

Furthermore let $\beta > 1$ be the dominating root of the adjoint polynomial

$$x^d - bx^{d-1} - \cdots - b = 0.$$

Then the (so-called) β-Van-der-Corput sequence $(\phi_\beta(n))_{n\ge 0}$ is defined by

$$\phi_\beta(n) = \sum_{j\ge 0} \varepsilon_{G,j}(n)\beta^{-j-1}.$$

As in the case of the usual Van-der-Corput sequence it is well known that in this case the β-Van-der-Corput sequence is a low-discrepancy sequence, that is $D_N^*(\phi_\beta(n)) \ll (\log N)/N$, see [1, 8]. A very prominent example is the β-Van-der-Corput sequence corresponding to the golden mean $\varphi = (1+\sqrt{5})/2$, where the base sequence $G = (G_n)_{n\ge 0}$ is given by the Fibonacci numbers $G_n = F_{n+2}$. Here the digital expansion (1.1) is called Zeckendorf expansion. And the digits $\varepsilon_{G,j}(n) \in \{0,1\}$ just have to satisfy $\varepsilon_{G,j}(n)\varepsilon_{G,j+1}(n) = 0$, that is, there are no consecutive 1's. For convenience we will denote this sequence the *Zeckendorf Van-der-Corput sequence* $(\phi_Z(n))_{n\ge 0}$.

Similarly to the usual Halton sequence the β-Halton sequence is defined by $\Phi_\beta(n) = (\phi_{\beta_1}(n),\ldots,\phi_{\beta_s}(n))$, where the entries of $\beta = (\beta_1,\ldots,\beta_s)$ correspond to G_i-adic expansions of the above kind, $1 \le i \le s$.

By the use of ergodic properties it was shown in [4] that $\Phi_\beta(n)$ is uniformly distributed modulo 1 provided that the integers b_i are pairwise coprime and the dominant roots β_i have the property that $\beta_i^k/\beta_j^l \notin \mathbb{Q}$ for all integers $k, l \ge 1$ and $i \ne j$. However, the discrepancy was not considered at all.

The purpose of the present paper is to provide a first quantitative discrepancy analysis of β-Halton sequences.

Theorem 1. *Suppose that $q_1,\ldots,q_s \ge 2$ are pairwise coprime integers. Then the discrepancy of the $(s+1)$-dimensional sequence*

$$\Phi(n) = (\phi_{q_1}(n),\ldots,\phi_{q_s}(x),\phi_Z(n))$$

satisfies

$$D_N^*(\Phi(n)) \ll N^{1-\varepsilon}$$

for every $\varepsilon > 0$.

It remains an open problem whether this kind of generalized Halton sequences are low discrepancy sequences. Nevertheless the upper bound given in Theorem 1 is close to optimality. We leave this as an open problem.

Problem. *Suppose that $\Phi_\beta(n) = (\phi_{\beta_1}(n),\ldots,\phi_{\beta_s}(n))$ is a s-dimensional β-Halton sequence that is uniformly distributed modulo 1. Is $\Phi_\beta(n)$ also a low-discrepancy sequence, too, that is,*

$$D_N^*(\Phi_\beta(n)) \ll \frac{(\log N)^s}{N} \quad ?$$

Actually it is not clear how far Theorem 1 can be generalized. It would be desirable to cover (at least) the kind of sequences that are discussed in [4]. However, it seems that the methods that are applied in the present paper are not sufficient to handle these cases. In the case of the Zeckendorf Van-der-Corput sequence the distribution can be reduced to distribution properties of the Weyl sequence $n\varphi$ mod 1 (see Lemma 5). This is due to the fact that the Zeckendorf expansion agrees with the Ostrowski expansion related to the golden mean φ. In the more general case it might be possible to replace this approach by a distribution analysis of a more-dimensional linear sequence (modulo 1) in Rauzy fractal type sets (see [7, 11]).

2. Discrepancy Bounds for Halton Sequences

The purpose of this section is to present a very basic approach to the discrepancy of Halton sequences $\Phi_{\mathbf{q}}(n)$. These results are by no means new but are helpful to prepare the proof of Theorem 1.

Actually all subsequent properties follow from the following observation (that is immediate from the definition).

Lemma 2. *Suppose that $q \ge 2$, $k \ge 0$, and $0 \le m < q^k$ are integers. Then we have*

$$\phi_q(n) \in \left[\frac{m}{q^k}, \frac{m+1}{q^k} \right)$$

if and only if

$$\sum_{\ell=0}^{k-1} \varepsilon_{q,k-1-\ell}(n) q^\ell = m,$$

that is, the digits $\varepsilon_{q,0}(n), \ldots, \varepsilon_{q,k-1}(n)$ are fixed or, equivalently, n is contained in a fixed residue class mod q^k.

Lemma 2 implies a discrepancy bound (2.1) for the Van-der-Corput sequence. Note that in the present case we trivially have $\delta_k \le 1$ so that the upper bound $q^{-L} + L/N \ll (\log N)/N$ follows immediately by choosing $L = \lfloor \log_q N \rfloor$. The reason for using the formulation (2.1) with explicit δ_k is that this kind of formula naturally generalizes to (generalized) Halton sequences.

Lemma 3. *Suppose that $q \ge 2$ is an integer. Then we have for every integer $L \ge 1$*

$$D_N^*(\phi_q(n)) \ll \frac{1}{q^L} + \frac{1}{N} \sum_{1 \le k \le L} \delta_k, \tag{2.1}$$

where

$$\delta_k = \max_{0 \le u < q^k} \left| \#\{n < N : n \equiv u \bmod q^k\} - \frac{N}{q^k} \right|$$

Proof. Suppose that the interval $[0, \alpha) = I_1 \cup I_2 \cup \cdots \cup I_R$ is partitioned into R disjoint intervals I_r of lengths ℓ_r, $1 \le r \le R$. Then by the triangle inequality we have

$$\left| \sum_{n=0}^{N-1} \mathbf{1}_{[0,\alpha)}(x_n) - N\alpha \right| \le \sum_{r=1}^{R} \left| \sum_{n=0}^{N-1} \mathbf{1}_{I_r}(x_n) - N\ell_r \right|.$$

Now if $\alpha \in [vq^{-L}, (v+1)q^{-L})$ and v has the digital expansion $v = v_0 + v_1 q + \cdots + v_{L-1} q^{L-1}$ then we can partition the interval $[0, \alpha)$ into $v_0 + v_1 + \cdots + v_{L-1} + 1$ intervals: v_0 intervals of the form $[m_0 q^{-1}, (m_0+1)q^{-1})$, $0 \le m_0 < v_1$; v_1 intervals of the form $[v_0 q^{-1} + m_1 q^{-2}, v_0 q^{-1} + (m_1+1) q^{-2})$, $0 \le m_1 < v_1$ etc., and finally the interval $[vq^{-L}, \alpha)$.

By Lemma 2 it follows that for every interval of the form $I = [mq^{-k}, (m+1)q^{-k})$ we have

$$\left| \sum_{n=0}^{N-1} \mathbf{1}_I(\phi_q(n)) - Nq^{-k} \right| \le \delta_k.$$

Finally for the interval $[vq^{-L},\alpha)$ we set $J=[vq^{-L},(v+1)q^{-L})$ and obtain

$$\begin{aligned}\sum_{n=0}^{N-1}\mathbf{1}_{[vq^{-L},\alpha)}(\phi_q(n)) &\le \sum_{n=0}^{N-1}\mathbf{1}_J(\phi_q(n))\\ &\le \left|\sum_{n=0}^{N-1}\mathbf{1}_J(\phi_q(n)) - Nq^{-L}\right| + Nq^{-L}\\ &\le \delta_L + Nq^{-L}.\end{aligned}$$

and consequently

$$\left|\sum_{n=0}^{N-1}\mathbf{1}_{[vq^{-L},\alpha)}(\phi_q(n)) - (\alpha - vq^{-L})N\right| \le \delta_L + 2Nq^{-L}.$$

Of course this proves Lemma 3. □

By using precisely the same proof method as in the proof of Lemma 3 we obtain a direct generalization for Halton sequences. Note that the Lemma 2 together with the Chinese remainder theorem has to be used to obtain (2.2). Note again that $\delta_{k_1,\ldots,k_s}\le 1$ so that we derive from (2.2) the upper bound $D_N^*(\Phi_{\mathbf{q}}(n)) \ll (\log N)^s/N$ by choosing $L_j = \lfloor \log_{q_j} N\rfloor$.

Lemma 4. *Suppose that $q_1,q_2,\ldots,q_s\ge 2$ are pairwise coprime integers and $\mathbf{q}=(q_1,\ldots,q_s)$. Then we have for all integers $L_1,\ldots,L_s\ge 1$*

$$D_N^*(\Phi_{\mathbf{q}}(n)) \ll \sum_{j=1}^{s}\frac{1}{q_j^{L_j}} + \frac{1}{N}\sum_{1\le k_1\le L_1}\cdots\sum_{1\le k_s\le L_s}\delta_{k_1,\ldots k_s} \tag{2.2}$$

with

$$\delta_{k_1,\ldots,k_s} = \max_{0\le u<q_1^{k_1}\cdots q_s^{k_s}}\left|\#\{n<N : n\equiv u \bmod q_1^{k_1}\cdots q_s^{k_s}\} - \frac{N}{q_1^{k_1}\cdots q_s^{k_s}}\right|$$

Finally we mention that it is easy to we extend Lemma 4 to subsequences of Halton sequences. Suppose (again) that $q_1,q_2,\ldots,q_s\ge 2$ are pairwise coprime integers. If $c(n)$ a sequence of non-negative integers then we obtain

$$D_N^*(\Phi_{\mathbf{q}}(c(n))) \ll \sum_{j=1}^{s}\frac{1}{q_j^{L_j}} + \frac{1}{N}\sum_{1\le k_1\le L_1}\cdots\sum_{1\le k_s\le L_s}\delta_{k_1,\ldots k_s}$$

with

$$\delta_{k_1,\ldots,k_s} = \max_{0\le u<q_1^{k_1}\cdots q_s^{k_s}}\left|\#\{n<N : c(n)\equiv u \bmod q_1^{k_1}\cdots q_s^{k_s}\} - \frac{N}{q_1^{k_1}\cdots q_s^{k_s}}\right|$$

Similar observations have been already made in [5]. And actually we can reprove them with the above estimates. We just recall the obervation from [5] that $(\Phi_{\mathbf{q}}(c(n)))_{n\ge 0}$ is uniformly distributed modulo 1 if and only if $c(n)$ is uniformly distributed in the residue classes modulo $(q_1\cdots q_s)^k$ for all $k\ge 0$. (Of course if $c(n)$ is uniformly distributed in the residue classes modulo $(q_1\cdots q_s)^k$ then it is also uniformly distributed in the residue classes modulo $q_1^{k_1}\cdots q_s^{k_s}$ for all $k_j\le k$.)

It is also of interest to start with a sequence $r(n)$ of non-negative real numbers and to consider the sequence $(\Phi_{\mathbf{q}}(\lfloor r(n)\rfloor)_{n\ge 0}$. Here it follows that if the sequence $r(n)/(q_1^{k_1}\cdots q_s^{k_s})$ is uniformly distributed modulo 1 for all integers $k_1,\ldots k_s\ge 0$ then $(\Phi_{\mathbf{q}}(\lfloor r(n)\rfloor)_{n\ge 0}$ is uniformly distributed modulo 1, too. Furthermore

$$D_N^*(\Phi_{\mathbf{q}}(\lfloor r(n)\rfloor) \ll \sum_{j=1}^{s}\frac{1}{q_j^{L_j}} + \sum_{1\le k_1\le L_1}\cdots\sum_{1\le k_s\le L_s} D_N(r(n)/(q_1^{k_1}\cdots q_s^{k_s})).$$

3. Discrepancy Bounds for the Zeckendorf Van-der-Corput Sequence

Next we provide a quantitive appraoch to the Zeckendorf Van-der-Corput Sequence $(\phi_Z(n))_{n\ge 0}$ that has strong similarities to the q-ary case from the previous section.

We recall that the Fibonacci number $(F_n)_{n\ge 0}$ are given by $F_0 = 0$, $F_1 = 1$, and $F_k = F_{k-1} + F_{k-2}$ (for $k \ge 2$) and that every non-negative integer $n \ge 0$ has a unique representation

$$n = \sum_{j\ge 2} \varepsilon_{Z,j}(n) F_j,$$

where $\varepsilon_{Z,j}(n) \in \{0,1\}$ and $\varepsilon_{Z,j}(n)\varepsilon_{Z,j+1}(n) = 0$. The Zeckendorf Van-der-Corput Sequence $\phi_Z(n)$ is then given by

$$\phi_Z(n) = \sum_{j\ge 2} \varepsilon_{Z,j}(n) \varphi^{-j+1},$$

where $\varphi = (1+\sqrt{5})/2$ is the golden mean.

There is an interesting analogue to Lemma 2. Actually we get slightly more information than in the q-adic case.

Lemma 5. *Suppose that $k \ge 3$ and that the first digits $\varepsilon_{Z,2}(n), \ldots, \varepsilon_{Z,k-1}(n)$ are fixed. Then we equivalently have*

$$\phi_Z(n) \in \begin{cases} \left[\frac{\mu}{\varphi^{k-2}}, \frac{\mu+1}{\varphi^{k-2}}\right) & \text{if } \varepsilon_{Z,k-1}(n) = 0, \\ \left[\frac{\mu}{\varphi^{k-2}}, \frac{\mu+\varphi}{\varphi^{k-2}}\right) & \text{if } \varepsilon_{Z,k-1}(n) = 1, \end{cases} \tag{3.1}$$

where

$$\mu = \sum_{\ell=0}^{k-3} \varepsilon_{Z,k-1-\ell}(n) \varphi^{\ell}.$$

Furthermore we have

$$(-1)^k n\varphi \in (-1)^k u\varphi + \begin{cases} A_k^{(0)} + \mathbb{Z} & \text{if } \varepsilon_{Z,k-1}(n) = 0, \\ A_k^{(1)} + \mathbb{Z} & \text{if } \varepsilon_{Z,k-1}(n) = 1, \end{cases} \tag{3.2}$$

where

$$u = \sum_{j=2}^{k-1} \varepsilon_{Z,j}(n) F_\ell$$

and

$$A_k^{(0)} = \left[-\frac{1}{\varphi^{k-1}}, \frac{1}{\varphi^k}\right) \quad \text{and} \quad A_k^{(1)} = \left[-\frac{1}{\varphi^{k+1}}, \frac{1}{\varphi^k}\right).$$

Proof. Both properties, (3.1) and (3.2), are completely elementary. We just mention that (3.2) is a general property related to the Ostrowski expansion, compare with [3, Section 3.2]. (For a proof in the present case we refer to [10].) □

We first note that the intervals $\left[\frac{\mu}{\varphi^{k-2}}, \frac{\mu+1}{\varphi^{k-2}}\right)$ (or $\left[\frac{\mu}{\varphi^{k-2}}, \frac{\mu+\varphi}{\varphi^{k-2}}\right)$, respectively) partition the unit interval $[0,1)$ if the digits $\varepsilon_{Z,2}(n), \ldots, \varepsilon_{Z,k-1}(n)$ vary over all valid 0-1-sequences. Similarly the sets $(-1)u\varphi + A_k^{(0)} \bmod 1$ (or $(-1)u\varphi + A_k^{(1)} \bmod 1$, respectively) partition the unit interval. This implies that the distribution of the Zeckendorf Van-der-Corput sequence $\phi_Z(n)$ can be directly related to the distibution of the sequence $n\varphi \bmod 1$. This leads us to a corresponding variant of Lemma 3.

Lemma 6. *For every integer $L \ge 1$ we have*

$$D_N^*(\phi_Z(n)) \ll \frac{1}{\varphi^L} + \frac{1}{N} \sum_{1\le k\le L} \delta_k,$$

where

$$\delta_k = \sup_{0\le\beta<1} \left| \sum_{n=0}^{N-1} \mathbf{1}_{[\beta,\beta+\varphi^{-k})}(\varphi n \bmod 1) - \frac{N}{\varphi^k} \right|.$$

Proof. The proof is very close to the proof of Lemma 3. First, for every $\alpha \in (0,1]$ there exists μ of the form

$$\mu = \sum_{\ell=0}^{L-3} \varepsilon_{L-1-\ell}\varphi^{\ell}.$$

with $\varepsilon_2,\ldots,\varepsilon_{L-1} \in \{0,1\}$ and $\varepsilon_j\varepsilon_{j+1} = 0$ (that is, there are no consecutive 1's) such that $\mu\varphi^{-L+2} \le \alpha < (\mu+1)\varphi^{-L+2}$ if $\varepsilon_{L-1} = 0$ or $\mu\varphi^{-L+2} \le \alpha < (\mu+\varphi)\varphi^{-L+2}$ if $\varepsilon_{L-1} = 1$. For notational convenience we only take into account non-zero digits and write

$$\mu = \sum_{j=1}^{L'} \varphi^{L-2-\ell_j},$$

where $L' \le L/2$ and $1 \le \ell_1 < \ell_2 < \cdots < \ell_{L'} \le L-2$. Now we partition the interval $[0,\alpha)$ into $L'+1$ intervals of the form

$$\left[0,\frac{1}{\varphi^{\ell_1}}\right) \cup \left[\frac{1}{\varphi^{\ell_1}},\frac{1}{\varphi^{\ell_1}}+\frac{1}{\varphi^{\ell_2}}\right) \cup\cdots\cup \left[\sum_{j=1}^{L'-1}\frac{1}{\varphi^{\ell_j}},\sum_{j=1}^{L'}\frac{1}{\varphi^{\ell_j}}\right) \cup \left[\sum_{j=1}^{L'}\frac{1}{\varphi^{\ell_j}},\alpha\right)$$

The first L' intervals can be seen as intervals of the form given in (3.1). Let I denote one of these intervals. We now apply Lemma 5 and by (3.2) there is another interval J (mod 1) of the same length $\varphi^{-\ell_j}$ such that

$$\sum_{n=0}^{N-1} \mathbf{1}_I(\phi_Z(n)) = \sum_{n=0}^{N-1} \mathbf{1}_J(\varphi n \bmod 1).$$

Thus the local discrepancy of the sequence $\phi_Z(n)$ with respect to the interval I can be replaced by the local discrepancy of the sequence φn mod 1 with respect to the interval J. And the second one can be estimated by δ_{ℓ_j}.

Finally the remaining interval can be handled in the same was in the proof of Lemma 3. Summing up this leads to proposed discrepancy bound. □

We should add that the sets $[\beta,\beta+\varphi^{-k})$ are bounded remainder sets for the sequence φn mod 1 since $\varphi^{-k} \in \mathbb{Z}+\varphi\mathbb{Z}$ and we also have $\delta_k = O(1)$. This leads to another proof of the upper bound

$$D_N^*(\phi_Z(n)) \ll \frac{\log N}{N}.$$

We also want to mention that Lemma 5 extends to subsequences. Suppose that $c(n)$ is a sequence of non-negative integers. Then we have for every integer $L \ge 1$

$$D_N^*(\phi_Z(c(n))) \ll \frac{1}{\varphi^L} + \frac{1}{N}\sum_{1\le k\le L}\delta_k,$$

where

$$\delta_k = \sup_{0\le\beta<1} \left| \sum_{n=0}^{N-1} \mathbf{1}_{[\beta,\beta+\varphi^{-k})}(\varphi c(n) \bmod 1) - \frac{N}{\varphi^k} \right|.$$

In particular we get

$$D_N^*(\phi_Z(c(n))) \ll \log N\, D_N^*(\varphi c(n)).$$

For example it follows from Lemma 6 that $\phi_Z(p(n))$ or $\phi_Z(p_n)$ is uniformly distributed mod 1, where $p(n)$ is a non-negative integer valued polynomial and p_n is the sequence of primes.

Finally the method of Lemma 5 can be used to describe the joint distribution of a Halton sequence and the Zeckendorf Van-der-Corput sequence.

Lemma 7. *Suppose that* $q_1, q_2, \ldots, q_s \ge 2$ *are pairwise coprime integers and* $\mathbf{q} = (q_1, \ldots, q_s)$*. Then we have for all integers* $L, L_1, \ldots, L_s \ge 1$

$$D_N^*(\Phi_{\mathbf{q}}(n), \phi_Z(n)) \ll \frac{1}{\varphi^L} + \sum_{j=1}^{s} \frac{1}{q_j^{L_j}} + \frac{1}{N} \sum_{1\le k\le L} \sum_{1\le k_1\le L_1} \cdots \sum_{1\le k_s\le L_s} \delta_{k,k_1,\ldots k_s}$$

with

$$\delta_{k,k_1,\ldots,k_s} = \sup_{0\le\beta<1} \left| \sum_{n=0}^{N/(q_1^{k_1}\cdots q_s^{k_s})-1} \mathbf{1}_{[\beta,\beta+\varphi^{-k})}(\varphi q_1^{k_1}\cdots q_s^{k_s} n \bmod 1) - \frac{N}{\varphi^k q_1^{k_1}\cdots q_s^{k_s}} \right|.$$

Proof. Suppose that $\varepsilon_{Z,k-1} = 0$. Then

$$(\Phi_{\mathbf{q}}(n), \phi_Z(n)) \in \left[\frac{m_1}{q_1^{k_1}}, \frac{m_1+1}{q_1^{k_1}}\right) \times \cdots \times \left[\frac{m_1}{q_1^{k_1}}, \frac{m_1+1}{q_1^{k_1}}\right) \times \left[\frac{\mu}{\varphi^{k-2}}, \frac{\mu+1}{\varphi^{k-2}}\right)$$

if any only if n contained in a residue class modulo $q_1^{k_1}\cdots q_s^{k_s}$ and $\varphi n \mod 1$ is contained in an interval of length φ^{-k+2}. Thus, the number of $n < N$ with this property minus the expected number $N/(\varphi^k q_1^{k_1}\cdots q_s^{k_s})$ is bounded by $\delta_{k-2,k_1,\ldots,k_s}$. (Similarly if $\varepsilon_{Z,k-1} = 1$ then the upper bound is $\delta_{k-1,k_1,\ldots,k_s}$.) Hence, by using a decomposition of a $(d+1)$-dimensional interval $[0,\alpha_1) \times \cdots \times [0,\alpha_s) \times [0,\alpha)$ is in the proofs of Lemma 3 and Lemma 6 we immediately obtain the result. □

4. Proof of Theorem 1

The proof of Theorem 1 is based on Lemma 7. So we have to estimate $\delta_{k,k_1,\ldots,k_s}$. Clearly we have

$$\delta_{k,k_1,\ldots,k_s} \le 2\,(N/q)\,D_{N/q}^*(\varphi q n),$$

where $q = q_1^{k_1}\cdots q_s^{k_s}$. It is well known that the discrepancy of a sequence x_n can be estimated by the inequality of Erdős-Turán (see [2, 6]) saying that for every integer $H \ge 1$

$$D_M^*(x_n) \ll \frac{1}{H} + \sum_{h=1}^{H} \frac{1}{h} \left| \frac{1}{M} \sum_{n=0}^{M-1} e^{2\pi i h x_n} \right|.$$

If $x_n = \alpha n$ for some irrational number α we have

$$\left| \sum_{n=0}^{M-1} e^{2\pi i h\alpha n} \right| \ll \frac{1}{\|h\alpha\|},$$

where $\|x\| = \min_{k\in\mathbb{Z}} |x-k|$ denotes the distance to the integers. Consequently we get (for every integer $H \ge 1$)

$$\delta_{k,k_1,\ldots,k_s} \ll \frac{N}{qH} + \sum_{h=1}^{H} \frac{1}{h} \frac{1}{\|hq\varphi\|}. \tag{4.1}$$

In order to handle this kind of sums we make use of Ridout's p-adic version of the Thue-Siegel-Roth theorem [9] which implies the following lemma.

Lemma 8. *Suppose that* $q_1, q_2, \ldots, q_s \ge 2$ *are pairwise coprime integers. Then for every* $\varepsilon > 0$ *there exists a constant* $C > 0$ *such that for all integers* $k_1, \ldots, k_s \ge 0$ *and* $h \ge 1$

$$\|q_1^{k_1}\cdots q_s^{k_s} h\varphi\| \ge \frac{C}{h^{1+\varepsilon}(q_1^{k_1}\cdots q_s^{k_s})^{\varepsilon}}$$

Proof. Set $q = q_1^{k_1} \cdots q_s^{k_s}$ and let D denote the set of primes that appear in the prime decomposition of $q_1, \ldots, q_s$. By Ridoux's theorem for every $\varepsilon > 0$ there exist a constant $C > 0$ with

$$\|qh\varphi\| \prod_{\ell \in D} |qh|_\ell \ge \frac{C}{(qh)^{1+\varepsilon}}$$

for all integers $q, h \ge 1$. Since

$$\prod_{\ell \in D} |qh|_\ell \le \prod_{\ell \in D} |q|_\ell = \frac{1}{q}$$

we obtain

$$\|qh\varphi\| \ge \frac{C}{h^{1+\varepsilon} q^\varepsilon}$$

as proposed. □

Lemma 9. *Suppose that $q_1, q_2, \ldots, q_s \ge 2$ are pairwise coprime integers. Then for every $\varepsilon > 0$ there exists a constant $C > 0$ such that for all integers $k_1, \ldots, k_s \ge 0$ and $H \ge 1$*

$$\sum_{h=1}^{H} \frac{1}{h} \frac{1}{\|q_1^{k_1} \cdots q_s^{k_s} h\varphi\|} \le C(q_1^{k_1} \cdots q_s^{k_s} H)^\varepsilon.$$

Proof. Let $Q_k = Q_k(\alpha)$ denote the denominators of the convergents P_k/Q_k of an irrational number α and suppose that $Q_{k-1} < H \le Q_k$. Then it follows from the approximation

$$\alpha = \frac{P_k}{Q_k} + \frac{\theta}{Q_K Q_{K+1}}, \qquad |\theta| \le 1,$$

that

$$\left| h\alpha - \frac{hP_k}{Q_k} \right| < \frac{1}{Q_k}, \qquad 1 \le h \le Q_k.$$

Since P_k and Q_k are coprime the numbers hP_k, $1 \le h \le Q_k$, run through all residue classes modulo Q_k. Thus, the numbers $\|h\alpha\|$, $1 \le h \le Q_k$, can be well approximated by $\|\ell/Q_k\|$ with $\ell \in \{0, 1, \ldots, Q_k - 1\}$ but with at most three exceptions that are related to $\ell \in \{0, 1, Q_k - 1\}$. For these exceptional values we can just say that $\|\alpha h\| \ge \min_{1 \le h \le H} \|h\alpha\|$ whereas for the other values we have $\|\alpha h\| \ge \|\ell/Q_k\| - 1/Q_k$. Consequently we obtain

$$\begin{aligned} \sum_{h=1}^{H} \frac{1}{\|h\alpha\|} &\le \frac{3}{\min\limits_{1 \le h \le H} \|h\alpha\|} + \sum_{\ell=2}^{Q_k - 2} \frac{1}{\|\ell/Q_k\| - 1/Q_k} \\ &\ll \frac{1}{\min\limits_{1 \le h \le H} \|h\alpha\|} + Q_k \log Q_k \end{aligned}$$

If we apply this procedure for $\alpha = q\varphi$ (with $q = q_1^{k_1} \cdots q_s^{k_s}$) then we can use the estimate from Lemma 8 to obtain

$$\frac{1}{\min\limits_{1 \le h \le H} \|hq\varphi\|} \ll H^{1+\varepsilon} q^\varepsilon.$$

Furthermore, since $\|Q_{k-1}\alpha\| \le 1/Q_k$ we obtain (again from Lemma 8)

$$\frac{1}{Q_k} \ge \|Q_{k-1} q\varphi\| \ge \frac{C}{Q_{k-1}^{1+\varepsilon} q^\varepsilon} \gg \frac{1}{H^{1+\varepsilon} q^\varepsilon}$$

and consequently

$$\sum_{h=1}^{H} \frac{1}{\|hq\varphi\|} \ll H^{1+\varepsilon} q^\varepsilon + H^{1+\varepsilon} q^\varepsilon \log(H^{1+\varepsilon} q^\varepsilon) \ll H^{1+2\varepsilon} q^{2\varepsilon}.$$

Finally by partial summation we obtain

$$\begin{aligned}\sum_{h=1}^{H}\frac{1}{h}\frac{1}{\|hq\varphi\|} &\le \frac{1}{H}\sum_{h=1}^{H}\frac{1}{\|hq\varphi\|}+\sum_{h=1}^{H}\frac{1}{h^2}\sum_{\ell=1}^{h}\frac{1}{\|\ell q\varphi\|}\\ &\ll (qH)^{2\varepsilon}.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary we can replace 2ε by ε and we are done. □

Now it is easy to complete the proof of Theorem 1.

Proof. We use Lemma 7, where we choose $L = \lfloor \log_\varphi N \rfloor$ and $L_j = \lfloor \log_{q_j} N \rfloor$. As above we abbreviate $q_1^{k_1}\cdots q_s^{k_s}$ by q.

We distinguish two cases. First suppose that $q > N$. Then we trivially have

$$\delta_{k,k_1,\ldots,k_s} \le 2.$$

In the other case we set $H = \lfloor N/q \rfloor \ge 1$ and apply (4.1) to obtain

$$\begin{aligned}\delta_{k,k_1,\ldots,k_s} &\ll \frac{N}{qH}+\sum_{h=1}^{H}\frac{1}{h}\frac{1}{\|hq\varphi\|}\\ &\ll 1+N^{\varepsilon}\\ &\ll N^{\varepsilon}.\end{aligned}$$

Summing up this implies

$$D_N^*(\Phi(n)) \ll \frac{1}{N}+\frac{1}{N}N^{\varepsilon}(\log N)^{s+1} \ll \frac{1}{N^{1-2\varepsilon}}.$$

Again, since $\varepsilon > 0$ is arbitrary we can replace 2ε by ε which completes the proof of Theorem 1. □

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E-mail address: michael.drmota@tuwien.ac.at

Institut für Diskrete Mathematik und Geometrie TU Wien, Wiedner Hauptstr. 8–10, 1040 Wien, Austria