SEMI-DISCRETE CONSTANT MEAN CURVATURE SURFACES

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Abstract. We study semi-discrete surfaces in three dimensional euclidean space which are defined on a parameter domain consisting of one smooth and one discrete parameter. More precisely, we consider only those surfaces which are glued together from individual developable surface strips. In particular we investigate minimal surfaces and constant mean curvature (cmc) surfaces with non vanishing mean curvature in the setting of Koenigs nets and Christoffel duality. We obtain incidence-geometric characterizations of the dualizability of Koenigs nets as well as for the Gauss image of cmc surfaces. We also consider isothermic semi-discrete cmc surfaces and a specific type of Cauchy problem in this regard.

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1. Introduction and Preliminaries

1.1. **Introduction.** We are investigating semi-discrete constant mean curvature (cmc) surfaces within the framework of discrete differential geometry. The mathematical research field of *discrete differential geometry*, which at a first glance seems to be self-contradictory, discretizes notions, objects and methods from classical or smooth differential geometry. A first systematical approach is [17,]. Another textbook with the focus on integrable structures was written by [2]. Discrete differential geometry considers all kinds of discrete objects such as polygons, polyhedral surfaces, non-polyhedral meshes, etc. and has a long range of applications in pure mathematics, physics, computer graphics, architecture etc. Also semi-discrete surfaces are included and can be looked at in different ways – on the one hand as a discrete [or smooth] evolution of a curve [or polygon] from the point of view of transformations of curves, and on the other hand as approximation or covering of a surface by a sequence of merged strips. [14] use the latter concept in applications that have manufacturing of real world objects as architectural freeform surfaces in mind. We will give a precise definition of semi-discrete surfaces in §1.2.

A special class of objects in differential geometry are surfaces of constant mean curvature. Different notions of a discrete mean curvature may lead to different cmc surfaces. There are approaches for triangle meshes, planar quadrilateral meshes and other polyhedral surfaces for example in [3, 6, 7, 10, 12,]). It would be desirable that different discretizations of notions, like the mean curvature in our case, converge to the smooth counterpart in some appropriate limit. However, convergence results are rare and the topic of current research.

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In our case we focus on semi-discrete cmc surfaces in the setting of Christoffel duality. Thus, we obtain semi-discrete cmc surfaces whose parametrization is a semi-discrete Koenigs net. The mean curvature is determined on the one hand by Christoffel duality and at the same time by Steiner's formula. Our main focus is on the characterization of the corresponding Gauss images of cmc surfaces. We consider two situations, the case where the mean curvature vanishes (Section 2) and the case where the mean curvature is constant but different from zero (Section 3). In particular we consider the circular case i.e., where the cmc surfaces are isothermic (Section 4). Finally, we investigate a semi-discrete Cauchy problem for Gauss images of cmc surfaces in the setting of isothermic surfaces (Section 5).

1.2. **Semi-discrete surfaces.** The present paper is about two-dimensional semi-discrete surfaces. A *semi-discrete surface* is a map of one discrete and one smooth parameter into $\mathbb{R}^3$ (see Figure 1 left):

$$\begin{aligned} f : \mathbb{Z} \times \mathbb{R} \supset U &\longrightarrow \mathbb{R}^3 \\ (k,t) &\longmapsto f(k,t). \end{aligned}$$

Thus, a semi-discrete surface consists of *surface strips* or just *strips* which we obtain by connecting corresponding points on successive curves by straight line segments:

$$(u,v) \longmapsto (1-v)f(k,u) + vf(k+1,u), \quad \text{where } v \in [0,1].$$

To make our formulas shorter we use the following abbreviations:

$$f = f(k,t),\ f_{-1} = f(k-1,t),\ f_1 = f(k+1,t),\ f_2 = f(k+2,t), \text{ etc.}$$

For all $k \in \mathbb{Z}$ we assume the one parameter function $f(k,\cdot)$ to be sufficiently smooth and denote its derivatives by

$$\partial f = \frac{\partial f(k,t)}{\partial t} \quad \text{and} \quad \partial f_1 = \frac{\partial f(k+1,t)}{\partial t} \quad \text{etc.}$$

The 'discrete derivatives', i.e., the first forward differences, are denoted by

$$\Delta f = f_1 - f = f(k+1,t) - f(k,t) \quad \text{etc.}$$

1.3. **Semi-discrete conjugate nets.** *Conjugate nets* $f : \mathbb{R}^2 \to \mathbb{R}^3$ are objects of projective differential geometry. They are characterized by the property that the mixed derivatives are contained in the linear span of the first derivatives, i.e., $f_{uv} \in \operatorname{span}(f_u, f_v)$. Discrete analogs, namely *discrete conjugate nets*, are studied in the context of pure mathematics (see e.g., [2, 17,]) as well as with respect to applications in computer graphics and architecture (see e.g., [9, 13,]). The common discretization is a polyhedral surface with planar quadrilateral faces.

A *semi-discrete conjugate net* can be seen as the limit of an appropriate refinement process of discrete conjugate nets where the refinement is applied to one coordinate direction of the net. Thus, opposite edges of the quadrilaterals in one direction become smaller and smaller whereas the edge lengths of the other direction stay away from zero. Therefore, the limit surface of each strip has the same tangent plane along the line segment connecting f and f_1 and is therefore a developable surface. Thus, a semi-discrete conjugate net is a semi-discrete surface where all strips are developable. It is

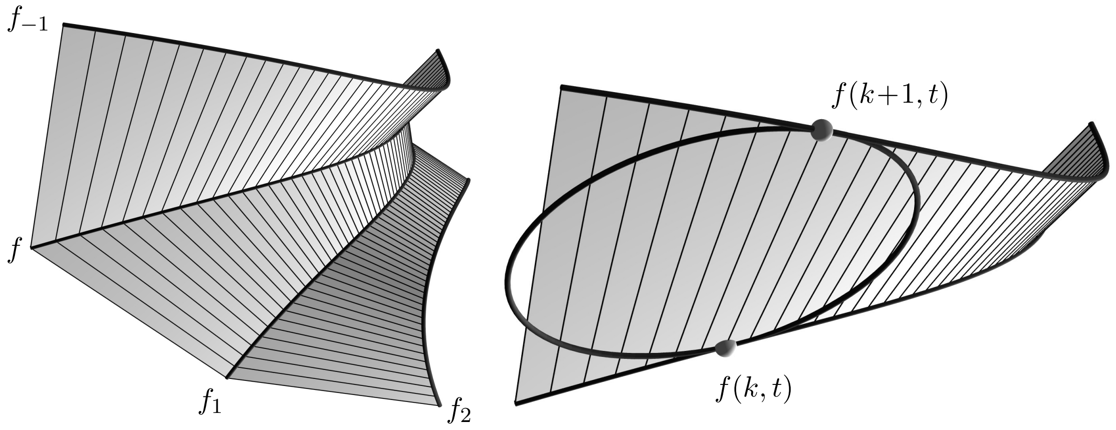


FIGURE 1. *Left:* A semi-discrete surface. A discrete family of curves $f_{-1},\dots,f_2$ is successively connected by straight line segments. *Right:* A circular strip. In each point there is a circle which makes tangential contact with the curves f and f_1 at corresponding points.

easy to see that developability of a strip in terms of semi-discrete parametrizations is equivalent to the linear dependence of $\{\partial f, \partial f_1, \Delta f\}$.

1.4. **Integrability of conjugate nets.** It is sometimes important to generate a semi-discrete surface f which is *parallel* to a given semi-discrete surface s. That is, we are looking for f such that both $\{\Delta f, \Delta s\}$ and $\{\partial f, \partial s\}$ are linearly dependent for all parameter values. The question of the existence of f can be restated as follows. Is there a semi-discrete surface f such that

$$\partial f = \alpha \partial s \quad \text{and} \quad \Delta f = \beta \Delta s, \tag{1}$$

where α and β are real valued functions? The existence of f is equivalent to the semi-discrete integrability condition $\Delta(\partial f) = \partial(\Delta f)$, which using α and β turns into the following lemma.

Lemma 1. *There is a semi-discrete surface f parallel to s such that* (1) *holds if and only if α and β fulfill*

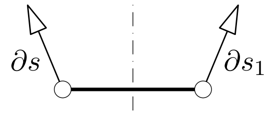
$$\alpha_1 \partial s_1 - \alpha \partial s - (\partial \beta)\Delta s - \beta \Delta \partial s = 0. \tag{2}$$

Let n denote a unit normal vector of the strip f, f_1. After multiplying Equation (2) by $(\Delta s) \times n$, which is a vector in the tangent plane orthogonal to Δs, we immediately obtain a necessary but not sufficient local condition for a pair of parallel semi-discrete surfaces f and s

$$\det(\Delta(\alpha \partial s), \Delta s, n) - \det(\Delta \partial s, \beta \Delta s, n) = 0. \tag{3}$$

A different derivation of Equation (3) (via the so called infinitesimal quadrilaterals) also appears in [8, Lem. 1.8].

An example for Equation (3) being not sufficient as integrability condition for the existence of f is the following. Let $\alpha = \alpha_1 = 1$, $\beta = 2$, and $\partial s, \partial s_1$ symmetric with respect to the perpendicular bisector of s, s_1 as in the image at right. Then (3) is fulfilled but (2) obviously not.



1.5. **Semi-discrete Koenigs nets and duality.** *Semi-discrete Koenigs nets* $f : \mathbb{Z} \times \mathbb{R} \supset U \to \mathbb{R}^3$ constitute a subclass of semi-discrete conjugate nets namely those which permit dualization. f is *dualizable* if there exists another semi-discrete surface $f^* : \mathbb{Z} \times \mathbb{R} \supset U \to \mathbb{R}^3$ and a positive semi-discrete function $\nu : \mathbb{Z} \times \mathbb{R} \supset U \to \mathbb{R}^+$ such that

$$\partial f^* = -\frac{1}{\nu^2}\partial f \quad \text{and} \quad \Delta f^* = \frac{1}{\nu\nu_1}\Delta f.$$

In that case f^* is called *dual* to f. It is easy to see that f is the dual of f^* up to translation and scaling. It is important to note here that putting the minus sign in the smooth equation was done arbitrarily and could as well be put in the discrete equation instead, with the consequence $f^* \to -f^*$. Dualizability of f is equivalent to the existence of a positive function ν such that the difference-differential equation (see also [11,])

$$\Delta\partial f = \frac{\nu_1}{\nu}\,\partial f - \frac{\nu}{\nu_1}\,\partial f_1 + \Big(\frac{\partial\nu}{\nu} + \frac{\partial\nu_1}{\nu_1}\Big)\Delta f \tag{4}$$

holds, which we easily obtain by applying Lemma 1. Further, dualizability of f is equivalent to the existence of a semi-discrete surface f^* which is parallel to f such that

$$\det(\partial f + \partial f_1, \Delta f^*, n) + \det(\partial f^* + \partial f_1^*, \Delta f, n) = 0 \tag{5}$$

holds, where n is a unit normal vector of the strip f, f_1.

If there is a family of circles for each strip of a semi-discrete surface such that there is a circle which makes tangential contact with f and f_1 for each t we call the semi-discrete surface *circular* (Figure 1 right). It turns out that a circular semi-discrete surface is dualizable if and only if it is isothermic in the sense of Definition 2. The proof of that is given by [11, Thm. 4.3]. For more details on the motivation and discussion of the definition of semi-discrete isothermic surfaces see [11,].

Definition 2. *A circular semi-discrete surface f is* isothermic *if there are positive scalar functions $\nu, \sigma, \tau : \mathbb{R} \to \mathbb{R}^+$ with*

$$\|\Delta f\|^2 = \sigma\nu\nu_1, \quad \|\partial f\|^2 = \tau\nu^2, \quad \textit{where } \partial\sigma = 0,\ \Delta\tau = 0$$

(i.e., σ and τ depend on the discrete and the continuous variable only, respectively). A complex valued isothermic semi-discrete function

$$f : \mathbb{Z} \times \mathbb{R} \supset U \to \mathbb{R}^2 \cong \mathbb{C}$$

is called holomorphic.

A Weierstrass type representation for semi-discrete isothermic parametrizations of minimal surfaces was recently formulated in [16,] using this definition of semi-discrete holomorphic functions. We will return to that in §2.2 where we will give an example.

2. Semi-discrete minimal surfaces

In this section we focus on semi-discrete surfaces which discretize minimal surfaces. We revisit the semi-discrete version [11,] of the well known construction of minimal surfaces by [4]. In the smooth setting it starts with an isothermic parametrization of the sphere. Thus, in our case we need a semi-discrete Koenigs net f which approximates the unit sphere. Here, approximation just means that f is 'close' to the unit sphere. There are two common possibilities. First, the curves $f(k, \cdot)$ are contained in the sphere

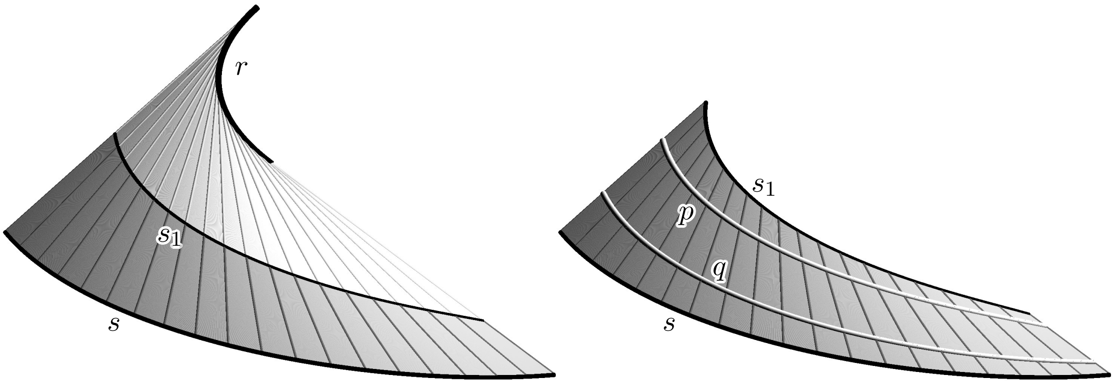


FIGURE 2. *Left:* A developable strip s, s_1. The rulings envelope the curve of regression r. These are the singular points of the developable surface. It is allowed for singular points to vanish to points at infinity. *Right:* For each strip s, s_1 we introduce two curves p and q on the surface such that $p - s = s_1 - q = (1-\beta)\Delta s$ for some function β and for all parameter values t.

(*inscribed*) as in Figure 5 left and second, the strips are tangent to the sphere (*circumscribed*). Other ways of approximation might be considered as well but one might expect more geometric properties in the more special situations. For specialities of the 'inscribed' case see Section 4.

E. Christoffel's construction of minimal surfaces can be translated into our semi-discrete setting as follows. Let f be a semi-discrete surface approximating the unit sphere. Then the dual f^* can be consistently defined to be a semi-discrete minimal surface. This construction is called the *Christoffel dual construction.* For details on semi-discrete minimal surfaces see [11,]. It turns out that a semi-discrete minimal surface in the setting of Christoffel duality has vanishing mean curvature everywhere for the semi-discrete mean curvature given later by Definition 8. Finally, we give a novel example of semi-discrete minimal surfaces namely an isothermically parametrized helicoid based on a semi-discrete version of $z \mapsto \exp((i+1)z)$.

2.1. **Geometric characterization of dualizability.** We aim to characterize dualizability of a semi-discrete surface by means of incidence geometry. We therefore study some special curves on the strips. Since we are in the setting of semi-discrete conjugate surfaces all strips are developable surfaces. It is well known that all ruled surfaces which are developable have a *curve of regression* (see e.g. [15,] and Figure 2 left) which consists of the singular points of the surface. These singular points are either in $\mathbb{R}^3$ or possibly points at infinity. It is well known [15,] that the curve of regression can be computed as stated in the following lemma.

Lemma 3. *Let f be a semi-discrete conjugate net. Then the curve of regression r of the strip f, f_1 is determined as*

$$r = f + v\Delta f, \quad \textit{where } v = -\frac{\partial f \times \Delta f}{\Delta(\partial f) \times \Delta f} = \Big(1 - \frac{\partial f_1 \times \Delta f}{\partial f \times \Delta f}\Big)^{-1}. \tag{6}$$

Note that v is the quotient of two parallel vectors.

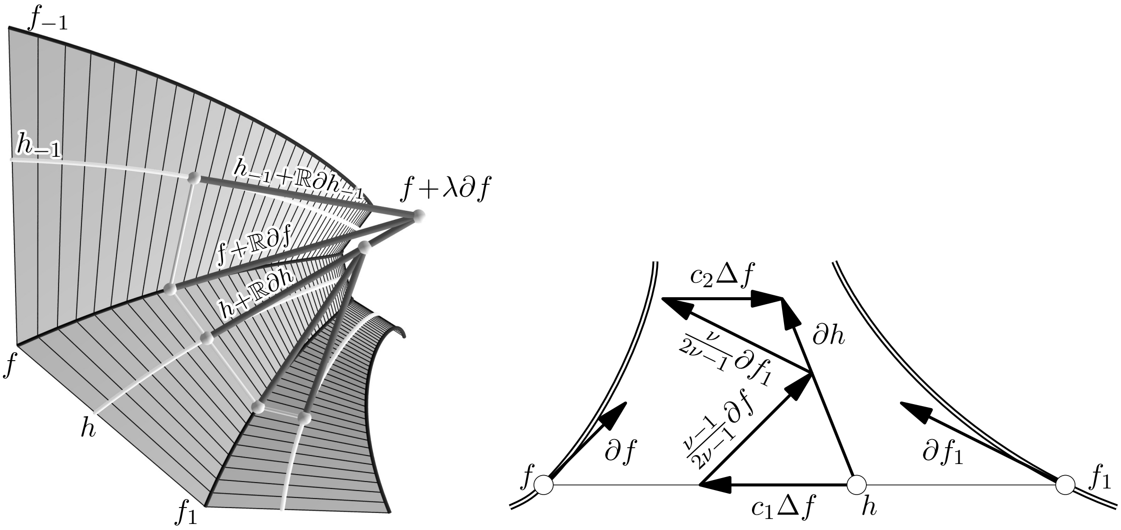


FIGURE 3. *Left:* Illustration of the incidence-geometric property $(*)$from Definition 5. A semi-discrete surface has property $(*)$, if the three tangents $f+\mathbb{R}\partial f$, $h+\mathbb{R}\partial h$, and $h_{-1}+\mathbb{R}\partial h_{-1}$ intersect each other in a point $f+\lambda\partial f$ or are parallel. Here, h is defined via the cross-ratio condition $cr(f, f_1, h, r) = -1$, where r is the curve of regression of the strip f, f_1. *Right:* Illustration of Equation (10). We decompose ∂h into the sum of two vectors $\left(\frac{v-1}{2v-1}\partial f + c_1\Delta f\right)$ and $\left(\frac{v}{2v-1}\partial f_1 + c_2\Delta f\right)$ which are both parallel to ∂h.

We also need the well known notion of the *cross-ratio* of four collinear points a, b, c, d, which is defined by

$$\mathrm{cr}(a, b, c, d) = \frac{a-c}{b-c} : \frac{a-d}{b-d} \in \mathbb{R} \cup \{\infty\}.$$

Lemma 4. *For four collinear points p, q, g, r with $r = p + \lambda(q-p)$ and $g = p + \mu(q-p)$ we have*

$$\mathrm{cr}(p, q, g, r) = -1 \quad \Longleftrightarrow \quad \mu = \lambda(2\lambda - 1)^{-1}.$$

Proof. This is easy to verify as $\mathrm{cr}(p, q, g, r) = \frac{p-g}{q-g} : \frac{p-r}{q-r} = \frac{\mu}{1-\mu} : \frac{\lambda}{1-\lambda}$. $\square$

Next, we introduce a curve h on each strip such that for all parameter values t the four points f, f_1, h, r are harmonic conjugate (cf. also [11, Def. 6.1.]), meaning that their cross-ratio equals -1, i.e., we have

$$\mathrm{cr}(f, f_1, h, r) = -1. \tag{7}$$

Definition 5 (Property $(*)$)**.** *We say that a conjugate semi-discrete surface f has* property $(*)$ *if and only if the three tangents*

$$f + \mathbb{R}\partial f, \quad h + \mathbb{R}\partial h, \quad h_{-1} + \mathbb{R}\partial h_{-1},$$

are intersecting in a common point or are parallel. For an illustration see Figure 3 (left).

Note that we called property $(*)$ differently in [11,], namely *property H*. We changed the notion to avoid confusions with the notion of H-surfaces which can be used as an alternative for cmc surfaces.

In [11,] we showed that every dualizable conjugate semi-discrete surface has property $(*)$. Further, we showed the reverse direction with the restriction of circularity: Any *circular* surface with property $(*)$ can be re-parametrized to become dualizable (i.e., isothermic). We now improve the latter result by dropping the property of circularity.

Theorem 6. *A conjugate semi-discrete surface f is a Koenigs net (i.e., dualizable) if and only if f has property $(*)$.*

Proof. The implication which yields property $(*)$ from dualizability is covered by [11, Th. 6.2]. Thus it remains to show the other implication by verifying Equation (4) for some function ν. The proof is rather technical and involves some identities which we have to derive first. We start by considering the strip f, f_1. The curve of regression is given by Lemma 3 and reads

$$r = f + v\Delta f, \quad \text{where } v = \Big(1 - \frac{\partial f_1 \times \Delta f}{\partial f \times \Delta f}\Big)^{-1}.$$

Thus, using Lemma 4, the curve h with $\mathrm{cr}(f, f_1, h, r) = -1$ is determined by

$$h = f + \frac{v}{2v-1}\Delta f \quad \text{or equivalently by} \quad h = f_1 + \frac{1-v}{2v-1}\Delta f. \tag{8}$$

Further, property $(*)$ yields the existence of functions λ for each curve f such that

$$f + \lambda\partial f = (h + \mathbb{R}\partial h) \cap (h_{-1} + \mathbb{R}\partial h_{-1}).$$

For an illustration see Figure 3 (left). We write down the proof only for the case $\lambda < \infty$. The reader can easily verify that the proof in the case of $\lambda = \infty$ (parallelity of ∂h and ∂f) works just as well after some slight modifications mainly concerning the notation (e.g., $1/\infty = 0$ or $c_1 = 0$ in Equation (10)). We compute

$$\partial h = \frac{v-1}{2v-1}\partial f + \frac{v}{2v-1}\partial f_1 - \frac{\partial v}{(2v-1)^2}\Delta f. \tag{9}$$

We can decompose ∂h into a sum of two vectors which are both parallel to ∂h in the following way (see Figure 3 right)

$$\partial h = \Big(\frac{v-1}{2v-1}\partial f + c_1\Delta f\Big) + \Big(\frac{v}{2v-1}\partial f_1 + c_2\Delta f\Big), \tag{10}$$

with $c_1 + c_2 = -\frac{\partial v}{(2v-1)^2}$. Since both $\big(\frac{v-1}{2v-1}\partial f + c_1\Delta f\big)$ and $\lambda\partial f + f - h$, are parallel to ∂h by construction, respectively, we obtain the following equality of ratios of parallel vectors

$$\Big(\frac{v-1}{2v-1}\partial f\Big) : (\lambda\partial f) = (c_1\Delta f) : (f - h) \quad \text{and thus} \quad \frac{1}{\lambda} = -c_1\frac{(2v-1)^2}{v(v-1)}, \tag{11}$$

using (8). Analogously, parallelity of $\big(\frac{v}{2v-1}\partial f + c_2\Delta f\big)$ and $\lambda_1\partial f_1 + f_1 - h$ yields

$$\Big(\frac{v}{2v-1}\partial f_1\Big) : (\lambda_1\partial f_1) = (c_2\Delta f) : (f_1 - h) \quad \text{and thus} \quad \frac{1}{\lambda_1} = c_2\frac{(2v-1)^2}{v(v-1)}.$$

We immediately obtain

$$\frac{1}{\lambda} - \frac{1}{\lambda_1} = (-c_1 - c_2)\frac{(2v-1)^2}{v(v-1)} = \frac{\partial v}{v(v-1)} = \frac{\partial v}{v-1} - \frac{\partial v}{v} = \partial \log\frac{v-1}{v}. \tag{12}$$

We set $\nu := \exp(-\int 1/\lambda)$ meaning $\nu(k,t) := \exp(-\int_{t_0}^{t} 1/\lambda(s)\,ds)$ and conclude

$$\frac{\nu}{\nu_1} = \frac{\exp(-\int 1/\lambda)}{\exp(-\int 1/\lambda_1)} = \exp(-\int(1/\lambda - 1/\lambda_1)) \overset{(12)}{=} \exp(-\int \partial \log \tfrac{v-1}{v}) = \frac{v}{v-1}. \tag{13}$$

Further, we have

$$\frac{\partial\nu}{\nu} = \partial \log \nu = \partial \log \exp(-\int 1/\lambda) = -\frac{1}{\lambda}. \tag{14}$$

We express ∂f_1 in the basis $\{\partial f, \Delta f\}$ and get

$$\partial f_1 = \frac{\partial f_1 \times \Delta f}{\partial f \times \Delta f}\partial f + \frac{\partial f \times \partial f_1}{\partial f \times \Delta f}\Delta f.$$

Note that the coefficients are ratios of parallel vectors. Then we insert ∂f_1 into Equation (9) which then turns into

$$\partial h = \Big(\frac{v-1}{2v-1} + \frac{v}{2v-1}\frac{\partial f_1 \times \Delta f}{\partial f \times \Delta f}\Big)\partial f + \Big(\frac{-\partial v}{(2v-1)^2} + \frac{v}{2v-1}\frac{\partial f \times \partial f_1}{\partial f \times \Delta f}\Big)\Delta f.$$

The value of v in (6) implies $(v-1)/v = (\partial f_1 \times \Delta f) : (\partial f \times \Delta f)$ and thus

$$\partial h = 2\frac{v-1}{2v-1}\partial f + \Big(\frac{-\partial v}{(2v-1)^2} + \frac{v}{2v-1}\frac{\partial f \times \partial f_1}{\partial f \times \Delta f}\Big)\Delta f. \tag{15}$$

Note that the last equation together with the choice of c_1 above in Equation (10) implies that the coefficient of Δf here equals $2c_1$. Finally, we show Equation (4), i.e., we need to show that the following expression equals 0. We get

$$\begin{aligned}
&\Big(\frac{\nu_1}{\nu} + 1\Big)\partial f - \Big(\frac{\nu}{\nu_1} + 1\Big)\partial f_1 + \Big(\frac{\partial\nu}{\nu} + \frac{\partial\nu_1}{\nu_1}\Big)\Delta f \\
&\overset{(14)}{\underset{(13)}{=}} \Big(\frac{v-1}{v} + 1\Big)\partial f - \Big(\frac{v}{v-1} + 1\Big)\partial f_1 - \Big(\frac{1}{\lambda} + \frac{1}{\lambda_1}\Big)\Delta f \\
&\overset{(9)}{=} 2\frac{2v-1}{v}\partial f - \frac{(2v-1)^2}{v(v-1)}\partial h - \Big(\frac{\partial v}{v(v-1)} + \frac{1}{\lambda} + \frac{1}{\lambda_1}\Big)\Delta f \\
&\overset{(12)}{=} 2\frac{2v-1}{v}\partial f - \frac{(2v-1)^2}{v(v-1)}\partial h - \frac{2}{\lambda}\Delta f \\
&\overset{(11)}{=} 2\frac{2v-1}{v}\partial f - \frac{(2v-1)^2}{v(v-1)}\partial h + 2c_1\frac{(2v-1)^2}{v(v-1)}\Delta f \\
&= \frac{(2v-1)^2}{v(v-1)}\Big(-\partial h + 2\frac{v-1}{2v-1}\partial f + 2c_1\Delta f\Big) \overset{(15)}{=} 0.
\end{aligned}$$

Therefore, with our choice of ν, Equation (4) holds which yields dualizability of f. $\square$

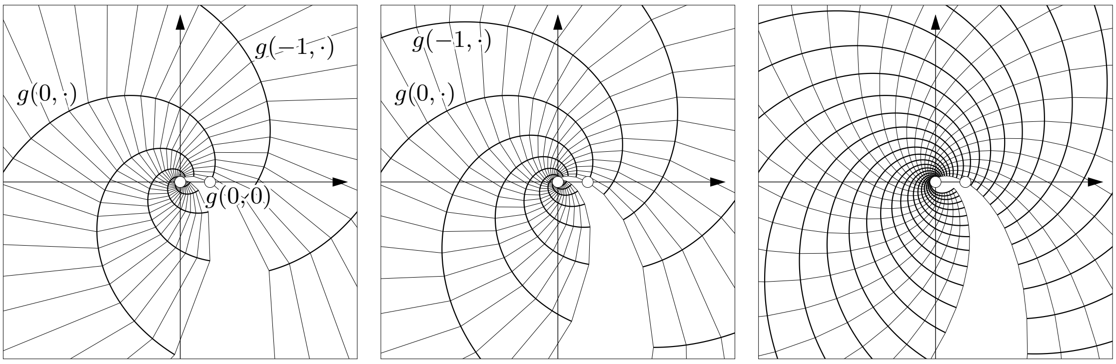


FIGURE 4. Illustration of the semi-discrete holomorphic function $g(k,t)$ of Theorem 7. It is a semi-discrete analogue of the smooth exponential function $z \mapsto \exp(az)$ with $a \in \mathbb{C} \setminus \{0\}$. The parameters $\alpha = \pi/3$, $r = \sqrt{2}$ are equal in all three images whereas φ equals $\pi/8$, $\pi/12$, and $\pi/32$ from left to right. We can say that φ regulates the density of the strips.

2.2. **Example: A semi-discrete helicoid.** The semi-discrete helicoid is an example of a minimal surface. One of the many remarkable insights in the theory of smooth minimal surfaces is the classical Weierstrass representation formula. It establishes a correspondence between holomorphic functions and minimal surfaces. For further details see e.g., [5,]. This formula has recently been formulated for semi-discrete surfaces [16, Theorem 2]. It says that for a semi-discrete holomorphic function $g : \mathbb{Z} \times \mathbb{R} \to \mathbb{C}$ (see Definition 2) we obtain the corresponding semi-discrete minimal surface f by solving

$$\partial f = -\frac{\tau}{2}\mathrm{Re}\Big[\frac{1}{\partial g}(1-g^2, i(1+g^2), 2g)\Big], \quad \Delta f = \frac{\sigma}{2}\mathrm{Re}\Big[\frac{1}{\Delta g}(1-gg_1, i(1+gg_1), g+g_1)\Big].$$

In classical differential geometry the holomorphic functions $g, h : \mathbb{C} \to \mathbb{C}$ with $g(z) = \exp(az)$, $h(z) = 1/a$, and $a \in \mathbb{C} \setminus \{0\}$ correspond to the isothermic parametrization of the helical surfaces in the Weierstrass representation [5, Th. 8.5.1]. For $a = 1$ we obtain the catenoid and for $a = 1 + i$ the helicoid. Consequently, to obtain semi-discrete helicoids we need to find a semi-discrete analogue of the map $z \mapsto \exp(az)$. In the following theorem we define a semi-discrete holomorphic function which has the desired property as we will discuss after the proof. Finally, we will get the semi-discrete helicoid (Figure 5 right) as the Christoffel dual of the stereographic projection of g (Figure 5 left). Since g is holomorphic (i.e., isothermic in $\mathbb{C}$; c.f. Definition 2) its stereographic projection is isothermic. Also the Christoffel dual construction keeps the property of beeing isothermic. Consequently, this particular parametrization of the helicoid is actually semi-discrete isothermic.

So far not many semi-discrete holomorphic functions are known. The following version of $z \mapsto \exp((i + 1)z)$ seems to be new.

Theorem 7. *Let $r, \alpha, \varphi \in \mathbb{R}$ with $\varphi \neq 0$, $r > 0$ such that $\lambda = \frac{\cos(\alpha+\varphi/2)}{\cos(\alpha-\varphi/2)} > 0$. Then*

$$g(k,t) = \exp(r\exp(i\alpha)t + (i\varphi + \log\lambda)k)$$

is a semi-discrete holomorphic function. For an illustration see Figure 4.

Proof. First, we show that g is circular. That is, we need to show that the normalized tangent vectors are symmetric with respect to $(\Delta g)^{\perp}$ (cf. also [11, Equation (8)]). Therefore, we need to show that

$$V := \Big(\frac{\partial g}{|\partial g|} + \frac{\partial g_1}{|\partial g_1|}\Big)(g_1 - g)^{-1} \in i\mathbb{R}$$

for all k. Setting $z = \cos\alpha + i\sin\alpha$, we obtain

$$\begin{aligned} V = &\Big(\big[\lambda^k rz\exp(rtz + ik\varphi)\big]\cdot\big[\lambda^k r\exp(rt\cos\alpha)\big]^{-1} \\ &+\big[\lambda^{k+1} rz\exp(rtz + i(k{+}1)\varphi)\big]\cdot\big[\lambda^{k+1} r\exp(t\cos\alpha)\big]^{-1}\Big) \\ &\Big(\lambda^{k+1}\exp(rtz + i(k+1)\varphi) - \lambda^k\exp(rtz + ik\varphi)\Big)^{-1} \\ = &\, c\cdot\exp(i\alpha)(\exp(i\varphi)+1)(\lambda\exp(i\varphi)-1)^{-1}, \end{aligned}$$

with some $c \in \mathbb{R}$. After a tedious but straightforward computation we obtain

$$\exp(i\alpha)(\exp(i\varphi)+1)(\lambda\exp(i\varphi)-1)^{-1} = -i\frac{\cos(\alpha-\varphi/2)}{\cos(\varphi/2)} \in i\mathbb{R}$$

and conclude that g is circular. To show that g is holomorphic we define three functions

$$\nu(k,t) = \lambda^k r^2\exp(rt\cos\alpha), \quad \sigma(k,t) = \frac{2\sin^2\varphi}{\cos 2\alpha + \cos\varphi}, \quad \text{and} \quad \tau(k,t) = r^{-2}.$$

We immediately see that σ and τ are independent of t and k, respectively, and therefore $\partial\sigma = 0$ and $\Delta\tau = 0$. It is further easy to see that

$$|\partial g|^2 = \tau\nu^2 \quad \text{and} \quad |\Delta g|^2 = \sigma\nu\nu_1.$$

Hence, the requirements of Definition 2 are fulfilled and consequently, g is a semi-discrete holomorphic function. ☐

Note that the parameter φ regulates the density of smooth curves in the semi-discrete holomorphic map as illustrated by Figure 4.

The semi-discrete holomorphic function of Theorem 7 can be seen as an analogue of the smooth holomorphic function $\exp(az)$ in addition to the similarity of its representation for the following reason. Let us consider the two families of parameter lines of $\exp(az)$. Each parameter line is a logarithmic spiral. That is, each parameter line is characterized by the fact that the tangent vector in each point is obtained from its position vector after applying a uniquely determined similarity. In our case the two tangents of the two parameter lines through $\exp(az)$ are $\overline{a}\exp(az)$ and $i\overline{a}\exp(az)$.

Analogous properties are true in our semi-discrete case. The smooth curves $g(k,\cdot)$ are logarithmic spirals too. The tangent vector at $g(k,t)$ is obtained from its position vector after applying the similarity $z \mapsto r\exp(i\alpha)z$. The discrete curves (polygons) $g(\cdot,t)$ are discrete logarithmic spirals in the sense that we obtain the discrete derivative $\Delta g(k,t)$ from its position vector $g(k,t)$ after applying the similarity $z \mapsto (\lambda\exp(i\phi)-1)$.

These analogies between the smooth and the semi-discrete setting justify calling g a *semi-discrete exponential map*.

Comparing now the complex numbers defining the just mentioned similarities from the smooth and the semi-discrete settings, then a either corresponds to $r\exp(-i\alpha)$ or

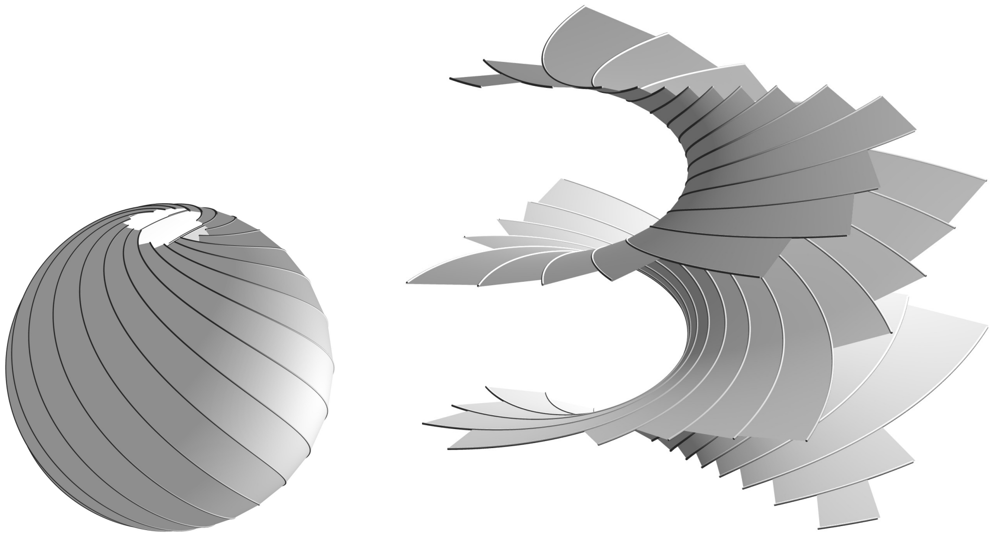


FIGURE 5. Illustration of a semi-discrete isothermic parametrization of a helicoid (right) together with its Gauss image (left). The helicoid is generated as a Christoffel dual of the Gauss image and thus is a semi-discrete minimal surface. Each strip is a developable surface. The corresponding semi-discrete holomorphic function represented by Theorem 7 takes $r = \sqrt{2}$, $\alpha = \pi/4$, and $\varphi = \pi/16$ as parameters.

to $i(\lambda \exp(-i\varphi) - 1)$. For our illustration in Figure 5 we chose the first one, i.e., $a = 1 + i = r\exp(-i\alpha)$. Thus, $r = \sqrt{2}$ and $\alpha = \pi/4$.

3. Semi-discrete cmc surfaces

In contrast to Section 2 here we focus on semi-discrete surfaces with constant but non-vanishing mean curvature. After the definition of mean curvature we focus on characterizations and properties of the Gauss image of a cmc surface.

3.1. **Mean curvature from Steiner's formula.**

For conjugate semi-discrete surfaces in [11,] we introduced the notion of mean curvature with respect to a certain Gauss image. We are given a pair of parallel semi-discrete surfaces f and s (i.e., $\Delta f \parallel \Delta s$ and $\partial f \parallel \partial s$). Here, s plays the role of the Gauss image. In classical differential geometry the Gauss image is a map to the sphere. To mimic this in our semi-discrete setting the semi-discrete Gauss image s should be approximating the sphere in an appropriate way. Like as mentioned in Section 2, there are two common possibilities: Either the curves $s(k, \cdot)$ are contained in the sphere (*inscribed*) as in Figure 5 left, or, the strips are tangent to the sphere (*circumscribed*). The following definition of the mean curvature works for either type of semi-discrete Gauss image.

Definition 8. *Let f and s be a pair of parallel, conjugate semi-discrete surfaces. Then we call*

$$H = -\frac{\det(\partial f + \partial f_1, \Delta s, n) + \det(\partial s + \partial s_1, \Delta f, n)}{2\det(\partial f + \partial f_1, \Delta f, n)} \tag{16}$$

semi-discrete mean curvature of f with respect to s, *where n is the unit normal vector of the parallel tangent planes of the two strips.*

This definition can be seen as the limit of an appropriate refinement process in one direction of the definition for planar quadrilateral meshes in [1,]. The combinatorial geometries of this definition imply that the mean curvature $H(k,t)$ is not associated with the vertex $f(k,t)$ but rather with the *segments* $f(k,t)f(k+1,t)$.

3.2. **CMC surfaces parametrized by semi-discrete Koenigs nets.** Here we would like to recall a connection between classical Christoffel duality and cmc surfaces which also holds in our semi-discrete setting. In doing so we follow [2, Theorem 4.49] and [11,].

Theorem 9. *Let f and s be a pair of parallel, conjugate semi-discrete surfaces. Then f is a Koenigs net which has constant mean curvature 1 with respect to the Gauss image s if and only if there is a function ν such that*

$$\partial s = -\frac{\nu^2+1}{\nu^2}\partial f \quad \textit{and} \quad \Delta s = \frac{1-\nu\nu_1}{\nu\nu_1}\Delta f.$$

Proof. We use the definition of the mean curvature (16) and obtain

$$H = 1 \Longleftrightarrow \det(\partial f + \partial f_1, \Delta f + \Delta s, n) + \det(\partial f + \partial s + \partial f_1 + \partial s_1, \Delta f, n) = 0.$$

Then Equation (5) implies that f is dualizable which yields the existence of a dual f^* and a semi-discrete function ν such that

$$\partial f + \partial s = \partial f^* = -\frac{1}{\nu^2}\partial f \quad \text{and} \quad \Delta f + \Delta s = \Delta f^* = \frac{1}{\nu\nu_1}\Delta f.$$

This concludes the proof. □

3.3. **Semi-discrete Gauss images of cmc surfaces (algebraic).** We aim at characterizations of pairs of parallel semi-discrete surfaces f and s, where f is a semi-discrete cmc surface with respect to its Gauss image surface s. The following theorem characterizes semi-discrete surfaces s which are Gauss images of cmc surfaces.

Theorem 10. *Let s be a conjugate semi-discrete surface. Then each of the following statements is equivalent to the others.*

(a) *s is a Gauss image of a cmc surface.*
(b) *There exists a positive function $\nu : \mathbb{Z}\times\mathbb{R} \supset U \to \mathbb{R}^+$ such that the system of difference-differential equations*

$$\partial f = -\frac{\nu^2}{\nu^2+1}\partial s \quad \textit{and} \quad \Delta f = \frac{\nu\nu_1}{1-\nu\nu_1}\Delta s$$

is integrable. (f is a Koenigs net and cmc with respect to the Gauss image s.)
(c) *There exists a positive function $\nu : \mathbb{Z}\times\mathbb{R} \supset U \to \mathbb{R}^+$ such that*

$$\frac{\partial \log(\nu\nu_1 - 1)}{\nu+\nu_1}\Delta s + \frac{\nu}{\nu^2+1}\partial s - \frac{\nu_1}{\nu_1^2+1}\partial s_1 = 0. \tag{17}$$

Proof. (a) $\iff$ (b) is clear by Theorem 9.

(b) $\iff$ (c): The difference-differential equations of (b) are integrable if and only if the integrability condition $\Delta\partial f = \partial\Delta f$ holds. That is, f exists if and only if

$$\Delta\Big(\frac{\nu^2}{\nu^2+1}\partial s\Big) = \partial\Big(\frac{\nu\nu_1}{\nu\nu_1-1}\Delta s\Big),$$

which after rearranging turns into Equation (17). $\square$

3.4. **Semi-discrete Gauss images of cmc surfaces (geometric).** In this section we give a necessary geometric characterization for a semi-discrete surface s being a Gauss image of a cmc surface. We will end up with a similar property just like 'property ($*$)' in Theorem 13 but with different curves involved. For the sake of clarity we split up the proof into the following technical lemmas. In the following ν always denotes a function which is needed in Theorem 10 such that f is a Koenigs net.

Lemma 11. *Let s be a semi-discrete Gauss image of a cmc surface f. Further let ν be a positive function which makes f a Koenigs net (cf. Th. 10). Then we have the following quotient of parallel vectors*

$$\frac{\partial s_1 \times \Delta s}{\partial s \times \Delta s} = \frac{\nu(\nu_1^2+1)}{\nu_1(\nu^2+1)}.$$

Proof. We obtain the equation after a vector product of (17) with Δs. $\square$

We construct two curves p and q on each strip at the same but opposite oriented distance $\beta\|\Delta s\|$ from s_1 and s, respectively, (see Figure 2 right)

$$p = s + (1-\beta)\Delta s \qquad q = s + \beta\Delta s. \tag{18}$$

Then we easily obtain

$$s + v\Delta s = p + \lambda(q-p) \quad\iff\quad \lambda = \frac{v-1+\beta}{2\beta-1}. \tag{19}$$

Then, we generate a curve g on each strip such that the four points p, q, g, r are in harmonic position, i.e., $\mathrm{cr}(p,q,g,r) = -1$.

Lemma 12. *Let s be a semi-discrete Gauss image of a cmc surface f and let ν be a corresponding positive function which makes f a Koenigs net. Let r be the curve of regression of the strip s, s_1 and let $\beta = \nu\nu_1/(\nu\nu_1-1)$ and p, q be as in* (18). *Further, let g be a curve on the strip s, s_1 such that* $\mathrm{cr}(p,q,g,r) = -1$. *Then we have*

$$g = s + \frac{\nu_1(\nu^2-1)}{(\nu+\nu_1)(\nu\nu_1-1)}\Delta s.$$

Proof. Clearly, $g = s + \theta\Delta s = p + \mu(q-p)$ for some θ and μ. Equation (19) and Lemma 4 imply

$$\mu = \frac{\theta-1+\beta}{2\beta-1} \quad\text{and}\quad \mu = \frac{\lambda}{2\lambda-1},$$

where λ determines the curve of regression $r = p + \lambda(q-p)$. Again, as $r = s + v\Delta s = p + \lambda(q-p)$ we obtain with (19)

$$\lambda = \frac{v-1+\beta}{2\beta-1},$$

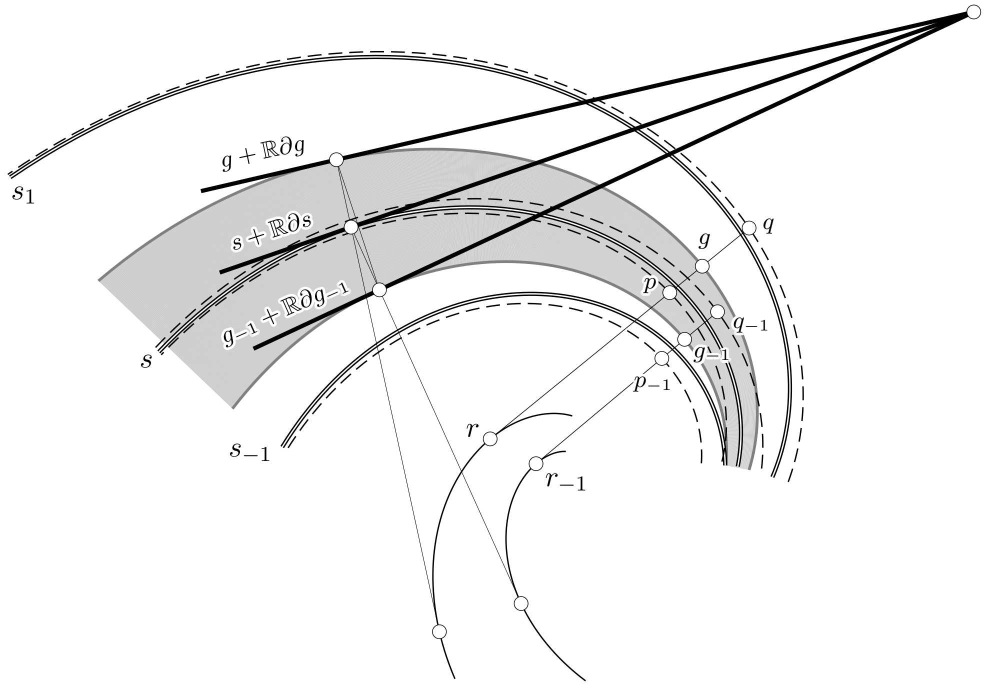


FIGURE 6. Illustration of Theorem 13. The curves s_{-1}, s, s_1 (double lines) illustrate the border of two adjacent strips of the Gauss image of a semi-discrete cmc surface. Each strip has a curve of regression r and r_{-1}. For each strip, say s, s_1, we have a pair of curves p and q (dashed lines) which have the same but opposite oriented distance from the curves s and s_1. The curve g consists of all points which are harmonic conjugate to r with respect to p and q, i.e., the cross-ratio $\operatorname{cr}(p, q, g, r)$ equals -1. Theorem 13 implies for Gauss images of cmc surfaces that the three tangents $s + \mathbb{R}\partial s$, $g + \mathbb{R}\partial g$, and $g_{-1} + \mathbb{R}\partial g_{-1}$ are intersecting in a common point or are parallel.

where Lemma 3 and Lemma 11 imply

$$v = \Big(1 - \frac{\nu(\nu_1^2 + 1)}{\nu_1(\nu^2 + 1)}\Big)^{-1}.$$

A tedious but straightforward computation by inserting backwards one equation into the other yields

$$\theta = \frac{\nu_1(\nu^2 - 1)}{(\nu + \nu_1)(\nu\nu_1 - 1)},$$

which concludes our proof. □

Finally we can state a necessary geometric condition for the Gauss image of semi-discrete cmc surfaces.

Theorem 13. *Let s be a Gauss image of a semi-discrete cmc surface f and let g and g_{-1} be the curves on the strips s, s_1 and s_{-1}, s as defined in Lemma 12. Then the three tangents*

$$s + \mathbb{R}\partial s, \quad g + \mathbb{R}\partial g, \quad g_{-1} + \mathbb{R}\partial g_{-1},$$

are intersecting in a common point or are parallel. For an illustration see Figure 6.

Proof. We set

$$\theta = \frac{\nu_1(\nu^2-1)}{(\nu+\nu_1)(\nu\nu_1-1)}$$

and obtain

$$\partial\theta = \frac{\nu_1(\nu^2+1)(\nu_1^2-1)\partial\nu - \nu(\nu^2-1)(\nu_1^2+1)\partial\nu_1}{(\nu+\nu_1)^2(\nu\nu_1-1)^2}.$$

Next we compute ∂g

$$\begin{aligned}\partial g &= \partial s + (\partial\theta)\Delta s + \theta\Delta(\partial s)\\ &= (\partial\theta)\Delta s + (1-\theta)\partial s + \theta\partial s_1\\ &= (\partial\theta)\Delta s + (1-\theta)\partial s + \frac{(\nu^2-1)(\nu_1^2+1)}{(\nu+\nu_1)(\nu\nu_1-1)}\Big(\frac{\nu_1}{\nu_1^2+1}\partial s_1\Big).\end{aligned}$$

We use Equation (17) which reads

$$\frac{\nu_1}{\nu_1^2+1}\partial s_1 = \frac{\partial\log(\nu\nu_1-1)}{\nu+\nu_1}\Delta s + \frac{\nu}{\nu^2+1}\partial s$$

and replace the corresponding "∂s_1 part" in the expression above to get

$$\partial g = \frac{2\nu_1(\nu^2\nu_1^2-1)\partial\nu}{(\nu+\nu_1)^2(\nu\nu_1-1)^2}\Delta s + \frac{2\nu(\nu\nu_1+1)}{(\nu+\nu_1)(\nu^2+1)}\partial s.$$

Now we split the proof into two cases. In the first case $\partial\nu = 0$. This corresponds to the parallelity of ∂g and ∂s. We make the following change of parameters: $\tilde{s}(k,t) = s(-k,t)$, i.e., we walk through the sequence of curves in the reverse direction. This immediately implies

$$\partial g_{-1} = \frac{2\nu_{-1}(\nu^2\nu_{-1}^2-1)\partial\nu}{(\nu+\nu_{-1})^2(\nu\nu_{-1}-1)^2}\Delta s_{-1} + \frac{2\nu(\nu\nu_{-1}+1)}{(\nu+\nu_{-1})(\nu^2+1)}\partial s,$$

since we only need to replace the index 1 in the last equation by -1. Thus, in our case where $\partial\nu$ equals 0 we obtain parallelity of ∂g_{-1} and ∂s as well. Consequently, for the case $\partial\nu = 0$ the theorem is proven.

In the second case we have $\partial\nu \neq 0$. Therefore, we can set

$$c_g = \frac{(1-\nu^2)(\nu+\nu_1)}{2(\nu\nu_1+1)\partial\nu} \quad \text{and} \quad c_s = \frac{(1-\nu^2)\nu}{(\nu^2+1)\partial\nu}.$$

Then it is easy to verify that

$$g + c_g\partial g = s + c_s\partial s,$$

which implies that the two tangents of s and g intersect at the common point

$$(s+\mathbb{R}\partial s)\cap(g+\mathbb{R}\partial g) = s + \frac{(1-\nu^2)\nu}{(\nu^2+1)\partial\nu}\partial s.$$

Furthermore, c_s only depends on ν and $\partial\nu$, i.e., c_s is independent from the curve s_1 and the function ν_1. Consequently, the change of parameters $\tilde{s}(k,t) = s(-k,t)$ implies that the tangents of $\tilde{s}$ and $\tilde{g}$ which correspond to the tangents of s and g_{-1} intersect at the same point. Thus

$$(s+\mathbb{R}\partial s)\cap(g_{-1}+\mathbb{R}\partial g_{-1})=(s+\mathbb{R}\partial s)\cap(g+\mathbb{R}\partial g)=s+\frac{(1-\nu^2)\nu}{(\nu^2+1)\partial\nu}\partial s,$$

which concludes our proof. □

Remark 14. *Counting degrees of freedom suggests that the other implication of Theorem 13 is also true, i.e., it should be possible to conclude that s is the Gauss image of a cmc surface assuming that the incidence geometric condition holds. So far a proof is missing.*

4. Isothermic cmc surfaces

The class of surfaces we are considering here are semi-discrete isothermic surfaces as given by Definition 2. One of the properties of a semi-discrete surface f being isothermic is circularity. It is a consequence [14,] of circularity of f that there exists a parallel surface s which is inscribed to the unit sphere. Here, *inscribed* means that the generating curves $s(k,\cdot)$ are contained in the surface of the unit sphere (i.e., $\|s(k,t)\|=1$ for all k and t). Thus, s itself is also circular for the following reason. The tangents $s+\mathbb{R}\partial s$ and $s_1+\mathbb{R}\partial s_1$ at the same parameter value t are both contained in the tangent plane of the strip. This tangent plane intersects the unit sphere along a circle. This circle also makes tangential contact with s and s_1 at corresponding points. In analogy to §3.3 and §3.4, in this section we study properties of semi-discrete Gauss images s of cmc surfaces f. The difference to previous sections is that f is now isothermic.

Theorem 15. *Let s be the Gauss image of a dualizable semi-discrete cmc surface f and let ν be a positive function which makes f a Koenigs net. Then s is circular if and only if*

$$\frac{\nu\nu_1}{(\nu\nu_1-1)^2}\|\Delta s\|^2 \tag{20}$$

is constant along the strip (i.e., does not depend on the continuous parameter t).

Proof. We differentiate our function with respect to t and obtain

$$\partial\Big(\frac{\nu\nu_1}{(\nu\nu_1-1)^2}\|\Delta s\|^2\Big)=\frac{2\nu\nu_1}{(\nu\nu_1-1)^2}\langle\Delta s,\Delta\partial s\rangle-\frac{(\nu\nu_1+1)\partial\log(\nu\nu_1-1)}{(\nu\nu_1-1)^2}\|\Delta s\|^2.$$

Then we look at the integrability condition for Gauss images of cmc surfaces (17) which after multiplication with Δs reads

$$\partial\log(\nu\nu_1-1)\|\Delta s\|^2=\frac{\nu_1(\nu+\nu_1)}{\nu_1^2+1}\langle\partial s_1,\Delta s\rangle-\frac{\nu(\nu+\nu_1)}{\nu^2+1}\langle\partial s,\Delta s\rangle.$$

We replace $\partial\log(\nu\nu_1-1)\|\Delta s\|^2$ in our derivative by the term on the right hand side of the last equation to obtain

$$\frac{\nu_1-\nu}{\nu\nu_1-1}\Big(\frac{\nu}{\nu^2+1}\langle\partial s,\Delta s\rangle+\frac{\nu_1}{\nu_1^2+1}\langle\partial s_1,\Delta s\rangle\Big). \tag{21}$$

It is a fact which follows from [11, Lemma 3.3] that for isothermic semi-discrete surfaces f the two vectors ∂f and ∂f_1 lie to the same side of the line spanned by f,f_1 in the tangent plane. Since ν is positive, Theorem 10 (b) implies that ∂f and ∂s point in

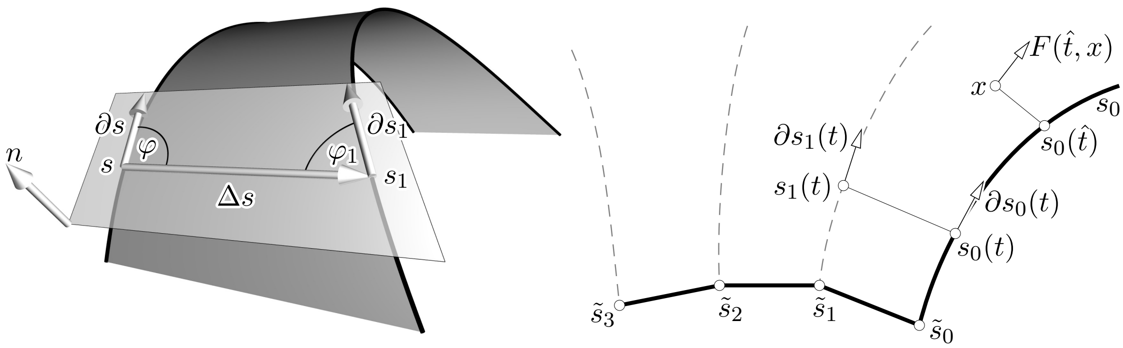


FIGURE 7. *Left:* A strip is circular if and only if $\varphi = \varphi_1$. *Right:* A semi-discrete Cauchy problem. We are given the smooth curve s_0 and a polygon $(\tilde{s}_i)_{i=0,1,\ldots}$, drawn with fat lines. Theorem 19 says that there is a unique circular semi-discrete surface which includes s_0 and $(\tilde{s}_i)_{i=0,1,\ldots}$ (as indicated with dashed lines), which is the Gauss image of an isothermic cmc surface. In the corresponding proof we construct a vector field F which maps a point $x \in \mathbb{R}^3$ at time t to a vector in $\mathbb{R}^3$ such that $\partial s_1(t) = F(t, s_1(t))$.

opposite directions. Thus, the two vectors ∂s and ∂s_1 lie to the same side of the line spanned by s, s_1 in the tangent plane. The derivative vectors ∂s and ∂s_1 enclose angles φ and φ_1 with Δs and $-\Delta s$, respectively, thus

$$\frac{\langle \partial s, \Delta s\rangle}{\|\partial s\|\|\Delta s\|} = \cos\varphi, \qquad \frac{\langle \partial s_1, \Delta s\rangle}{\|\partial s_1\|\|\Delta s\|} = -\cos\varphi_1,$$

and

$$\frac{\langle \partial s, \Delta s\times n\rangle}{\|\partial s\|\|\Delta s\|} = \sin\varphi, \qquad \frac{\langle \partial s_1, \Delta s\times n\rangle}{\|\partial s_1\|\|\Delta s\|} = \sin\varphi_1,$$

where n is the unit normal vector of the tangent plane (see Figure 7 left). Note that $\Delta s \times n$ comes out of Δs after a rotation about $\pi/2$ in the tangent plane. Further, since $\langle \partial s, \Delta s \times n\rangle = \langle \partial s \times \Delta s, n\rangle$ we rewrite Lemma 11 which now reads

$$\langle \partial s \times \Delta s, n\rangle = \langle \partial s_1 \times \Delta s, n\rangle \frac{\nu_1(\nu^2+1)}{\nu(\nu_1^2+1)}.$$

Thus, we obtain circularity if and only if $\varphi = \varphi_1$ which is further equivalent (mod π) to

$$\tan\varphi = \tan\varphi_1 \quad\Leftrightarrow\quad \frac{\sin\varphi}{\cos\varphi} = \frac{\sin\varphi_1}{\cos\varphi_1} \quad\Leftrightarrow\quad \frac{\langle \partial s, \Delta s\times n\rangle}{\langle \partial s, \Delta s\rangle} = \frac{\langle \partial s_1, \Delta s\times n\rangle}{\langle \partial s_1, -\Delta s\rangle}$$

$$\Leftrightarrow\quad \frac{\nu_1(\nu^2+1)}{\langle \partial s, \Delta s\rangle} = \frac{\nu(\nu_1^2+1)}{\langle \partial s_1, -\Delta s\rangle} \quad\Leftrightarrow\quad \frac{\nu}{\nu^2+1}\langle \partial s, \Delta s\rangle + \frac{\nu_1}{\nu_1^2+1}\langle \partial s_1, \Delta s\rangle = 0,$$

which is equivalent to vanishing of Equation (21). Thus, circularity is equivalent to

$$\partial\Big(\frac{\nu\nu_1}{(\nu\nu_1-1)^2}\|\Delta s\|^2\Big) = 0,$$

which is what we wanted to show. □

Remark 16. *In one direction, the implication of Theorem 15 is a corollary of* [11, Theorem 4.3] *which says that for circular semi-discrete surfaces isothermicity and dualizability are equivalent. Since in our case f is dualizable and circular, which implies that f is isothermic, we therefore get*

$$\frac{\nu\nu_1}{(\nu\nu_1-1)^2}\|\Delta s\|^2 = \frac{1}{\nu\nu_1}\|\Delta f\|^2 = \sigma$$

using Theorem 10 (b) and Definition 2 with $\partial\sigma = 0$. Thus, our considered function must be constant.

Remark 17. *We rewrite the constant term* (15) *of Theorem 15. We obtain the following constant expressions as we use Theorem 10 (b) at* $(*)$ *and β from Lemma 12 at* $(**)$ *and Equations* (18) *at* (§)

$$\begin{aligned}\text{const.} &= \frac{\nu\nu_1}{(\nu\nu_1-1)^2}\|\Delta s\|^2 = \frac{\nu\nu_1}{\nu\nu_1-1}\|\Delta s\|\cdot\Big(\frac{\nu\nu_1}{\nu\nu_1-1}-1\Big)\|\Delta s\| \\ &\overset{(*)}{=} \|\Delta f\|\cdot\big(\|\Delta f\| - \operatorname{sgn}(\nu\nu_1-1)\|\Delta s\|\big) \overset{(**)}{=} \beta\|\Delta s\|\cdot(\beta-1)\|\Delta s\| \\ &\overset{(\S)}{=} \langle q-s, s-p\rangle.\end{aligned}$$

Theorem 18. *Let s be the Gauss image of an isothermic cmc surface f and let ν be a positive function which makes f a Koenigs net. Further, let m be the curve traced by the center of the family of circles that makes contact with s, s_1 and let p, q be curves as defined by Equations* (18)*. Then the strip generated by p and q is circular and the centers of the circles are identical to m for each parameter value t.*

Proof. Let c be the midpoint of s and s_1, i.e., $c = (s+s_1)/2$. Then, there are two right angled triangles m, c, p and m, c, s and we therefore get

$$\begin{aligned}&\|m-p\|^2 - \|m-s\|^2 = \|m-c\|^2 + \|c-p\|^2 - \|m-c\|^2 - \|c-s\|^2 \\ &= \|p-q\|^2/4 - \|\Delta s\|^2/4 \overset{(*)}{=} ((1-2\beta)^2-1)\|\Delta s\|^2/4 = (\beta^2-\beta)\|\Delta s\|^2.\end{aligned}$$

Equality at $(*)$ follows from the affine combinations representing p and q in Equation (18). Consequently, Remark 17 implies that $\|m-p\|^2 - \|m-s\|^2$ is constant which yields

$$0 = \partial\big(\|m-p\|^2 - \|m-s\|^2\big) = 2\langle m-p, \partial m - \partial p\rangle - 2\langle m-s, \partial m - \partial s\rangle. \tag{22}$$

Since m is the center of the circle that makes tangential contact with s and s_1 we have $\langle m-s, \partial s\rangle = 0$ and $\langle m-s_1, \partial s_1\rangle = 0$. From $\|m-s\|^2 = \|m-s_1\|^2$ we conclude

$$\langle \partial m - \partial s, m-s\rangle = \langle \partial m - \partial s_1, m-s_1\rangle.$$

After expanding the last equation we get $\langle \partial m, s_1 - s\rangle = 0$. Since $s-p$ is parallel to $s_1 - s$ we also get $\langle \partial m, s-p\rangle = 0$. Finally, inserting all the derived equations into (22), we get

$$\langle m-p, \partial p\rangle = 0,$$

which means that the circle with center m and radius $\|m-p\|$ makes tangential contact with the curve p. Analogously, this circle also makes tangential contact with the curve q. ∎

5. A Cauchy problem for the Gauss image of a cmc surface

In this section we investigate a Cauchy problem in the setting of circular semi-discrete surfaces. The following theorem says that for some given initial values there is a unique circular semi-discrete surface which is the Gauss image of a cmc surface.

Theorem 19. *Let $I \subset \mathbb{R}$ be an interval with $t_0 \in I$. Suppose we are given a curve $s_0 : I \to \mathbb{R}^3$, a positive function $\nu : I \to \mathbb{R}^+$, and a polygon $(\tilde{s}_k)_{k=0,1,\dots}$ with $\tilde{s}_0 = s_0(t_0)$ (see Figure 7 right). Further, let $(\tilde{\nu}_k)_{k=1,\dots}$ be a sequence with positive entries.*

Then there are a unique circular semi-discrete surface

$$s : \{0, 1, \dots\} \times I \to \mathbb{R}^3$$

and positive functions $\nu_0, \nu_1, \dots : I \to \mathbb{R}^+$ such that

- *the first curve of s, namely $s(0,t)$, equals the given curve s_0,*
- *the k-th curve passes through $\tilde{s}_k$ for all k, i.e., $s_k(t_0) = \tilde{s}_k$,*
- *$\nu_k(t_0) = \tilde{\nu}_k$ for all k,*
- *the functions ν_k are of the form required by Theorem 10, which means that s is the Gauss image of a cmc surface.*

Proof. As a first step we construct s_1 and ν_1 from the given data and show that the two curves s, s_1 together with ν, ν_1 fulfill the integrability condition (17). Then the rest follows by induction.

We aim at a construction of a Lipschitz continuous function $F : I \times \mathbb{R}^3 \to \mathbb{R}^3$ such that the unique solution $c(t)$ of the initial value problem

$$\frac{\partial c}{\partial t} = F(t, c), \qquad c(t_0) = \tilde{s}_1 \tag{23}$$

is the curve $c = s_1$ we are looking for. That is, F is a time-dependent vector field such that for all $t \in I$ and $x \in \mathbb{R}^3$ the vector $F(t, x)$ equals $\partial s_1(t)$. Here, x plays the role of the point $s_1(t)$ (see Figure 7 right). Theorem 15 implies that the value (20) has to be constant along the strip which yields

$$\frac{\nu\nu_1}{(\nu\nu_1 - 1)^2}\|x - s\|^2 = \frac{\nu(t_0)\tilde{\nu}_1}{(\nu(t_0)\tilde{\nu}_1 - 1)^2}\|\tilde{s}(t_0) - \tilde{s}_1\|^2 =: \sigma = \text{const}.$$

Consequently, $\nu\nu_1$ is either smaller than 1 or greater than 1 along the strip because otherwise σ would have a pole and would no longer be constant. Further,

$$\nu\nu_1 = \frac{2a + 1 \pm \sqrt{(2a+1)^2 - 4a^2}}{2a} =: h^{\pm}(a) > 0, \quad \text{where } a = \frac{\sigma}{\|x - s\|^2}.$$

It is easy to see that $0 < h^-(a) < 1 < h^+(a)$ for all $a > 0$. Thus, we define

$$\nu_1(t, x) = \begin{cases} \frac{h^+(a)}{\nu(t)} & \text{if } \nu(t_0)\tilde{\nu}_1 > 1 \\ \frac{h^-(a)}{\nu(t)} & \text{if } \nu(t_0)\tilde{\nu}_1 < 1, \end{cases}$$

which yields a differentiable function ν_1. We take this function and define a smooth vector field

$$F(t, x) = \frac{\nu_1^2 + 1}{\nu_1}\Big(\frac{\partial \log(\nu\nu_1 - 1)}{\nu + \nu_1}(x - s) + \frac{\nu}{\nu^2 + 1}\partial s\Big),$$

by setting $F(t,x)$ equal to ∂s_1 from the equation of the integrability condition (17). Thus, the initial value problem (23) has a unique solution s_1 automatically fulfilling the integrability condition (17). Since we chose ν_1 in such a way that value (20) is constant along the strip s, s_1, Theorem 15 implies circularity of the strip. Therefore, we proved the existence of the first strip in the desired way. The rest follows by induction. □

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