Ruled Laguerre minimal surfaces

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Abstract

A *Laguerre minimal surface* is an immersed surface in $\mathbb{R}^3$ being an extremal of the functional $\int(H^2/K-1)dA$. In the present paper, we prove that the only *ruled* Laguerre minimal surfaces are the surfaces $\mathbf{R}(\varphi,\lambda)=(A\varphi,\,B\varphi,\,C\varphi+D\cos 2\varphi\,)+\lambda\,(\sin\varphi,\,\cos\varphi,\,0\,)$, where $A,B,C,D\in\mathbb{R}$ are fixed. To achieve invariance under Laguerre transformations, we also derive all Laguerre minimal surfaces that are enveloped by a family of cones. The methodology is based on the isotropic model of Laguerre geometry. In this model a Laguerre minimal surface enveloped by a family of cones corresponds to a graph of a biharmonic function carrying a family of isotropic circles. We classify such functions by showing that the top view of the family of circles is a pencil.

Contents

1 Introduction

This is the third in a series of papers [17, 16] where we develop and study a novel approach to the Laguerre differential geometry of immersed Legendre surfaces in $\mathbb{R}^3$. Laguerre geometry is the Euclidean geometry

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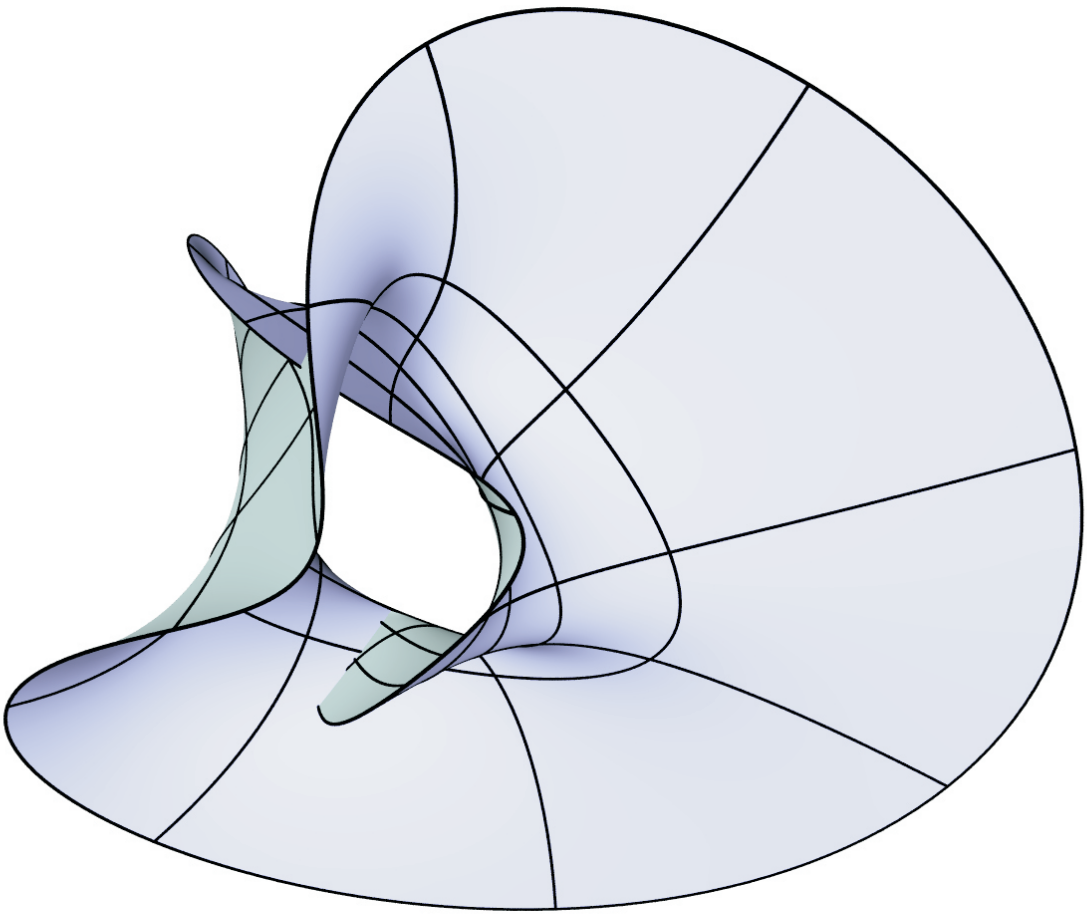

Figure 1: General L-minimal surface enveloped by an hyperbolic family of cones. For details refer to Definition 34 and Theorem 39

of oriented planes and spheres. Besides Möbius and Lie geometry, it is one member of the three classical sphere geometries in $\mathbb{R}^3$ [8].

After the seminal work [4] of Blaschke on this topic in the beginning of the 20th century, this classical topic has again found the interest of differential geometers.

For instance, the celebrated work on discrete differential geometry by Bobenko and coworkers [7, 6, 5] heavily uses this theory in developing discrete counterparts to continuous definitions.

On the practical side, recent research in architectural geometry identified certain classes of polyhedral surfaces, namely conical meshes [11, 21] and meshes with edge offsets [19], as particularly suitable for the representation and fabrication of architectural freeform structures. These types of polyhedral surfaces are actually objects of Laguerre sphere geometry [6, 11, 19, 18, 26, 16].

The aim is to study (discrete, see [16], and continuous, see [17]) minimizers of geometric energies which are invariant under Laguerre transformations. The simplest energy of this type has been introduced by Blaschke [2, 3, 4]. Using mean curvature H, Gaussian curvature K, and the surface area element dA of a surface Φ in Euclidean 3-space $\mathbb{R}^3$, it can be expressed as the surface integral

$$\Omega = \int_{\Phi} (H^2 - K)/K dA. \tag{1.1}$$

Though the quantities H, K, A used for the definition are not objects of Laguerre geometry, the functional Ω and its extremals, known as *Laguerre-minimal* (*L-minimal*) *surfaces*, are invariant under Laguerre transformations.

In 1842 Catalan proved that the only ruled Euclidean minimal surfaces are the plane and the helicoid. One of the main purposes of this paper is to describe all ruled Laguerre minimal surfaces. By a *ruled* surface we mean a surface containing an *analytic* family of lines.

The property of a surface to be ruled is not invariant under Laguerre transformations. A line in a surface may be taken to a cone or cylinder of revolution touching the image of the surface along a curve. Hence, we will also derive all Laguerre minimal surfaces which are *enveloped by a family of cones*. In the following, when speaking of a cone, we will always assume this to be a cone of revolution, including the special cases of a rotational cylinder and a line.

Our approach is based on a recent result [17] which shows that Laguerre minimal surfaces appear as graphs of biharmonic functions in the isotropic model of Laguerre geometry. This result has various corollaries on Laguerre minimal surfaces, geometric optics and linear elasticity.

A Laguerre minimal surface enveloped by a family of cones corresponds to a biharmonic function carrying a family of isotropic circles. We classify such functions. In particular we show that the top view of such a family of circles must be a pencil. In the course of the proof of this result we also develop a new symmetry principle for biharmonic functions.

1.1 Previous work

Differential geometry in the three classical sphere geometries of Möbius, Laguerre and Lie, respectively, is the subject of Blaschke's third volume on differential geometry [4]. For a more modern treatment we refer to Cecil [8]. Here we focus on contributions to L-minimal surfaces. Many L-minimal surfaces are found in the work of Blaschke [2, 3, 4] and in papers by his student König [9, 10].

Recently, this topic found again the interest of differential geometers. The stability of L-minimal surfaces has been analyzed by Palmer [14]; he also showed that these surfaces are indeed local minimizers of (1.1). Musso and Nicolodi studied L-minimal surfaces by the method of moving frames [13]. L-minimal surfaces which are envelopes of a family of cones include as special cases the L-minimal canal surfaces described by Musso and Nicolodi [12].

1.2 Contributions

Our main result is a description of all the L-minimal surfaces which are envelopes of an analytic family $\mathcal{F}$ of cones of revolution. We show that for any such surface (besides a plane, a sphere and a parabolic cyclide) the family $\mathcal{F}$ belongs to one of three simple types, see Definitions 24, 34, 41 and Corollary 19. For each type we represent the surface as a convolution of certain basic surfaces, see Examples 27–47 and Corollaries 31, 39, 48.

As an application we show the following:

Theorem 1 *A smooth ruled Laguerre minimal surface is up to motion a piece of the surface*

$$\mathbf{R}(\varphi, \lambda) = (A\varphi,\, B\varphi,\, C\varphi + D\cos 2\varphi\,) + \lambda\,(\sin\varphi,\, \cos\varphi,\, 0\,)\,, \tag{1.2}$$

for some $A, B, C, D \in \mathbb{R}$.

In other words, a ruled L-minimal surface can be constructed as a superposition of a frequency 1 rotating motion of a line in a plane, a frequency 2 “harmonic oscillation”, and a constant-speed translation.

Another result is a description of all the i-Willmore surfaces carrying an analytic family of i-circles, see Table 1 for definitions, and Corollary 18, Theorems 25, 35, 42 for the statements.

1.3 Organization of the paper

In §2 we give an introduction to isotropic and Laguerre geometries and translate the investigated problem to the language of isotropic geometry. This section does not contain new results. In §3 we state and prove the Pencil Theorem 5, which describes the possible families $\mathcal{F}$ of cones. In §4 we describe the Laguerre minimal surfaces for each type of cone family $\mathcal{F}$ and prove Theorem 1.

2 Isotropic model of Laguerre geometry

2.1 Isotropic geometry

Isotropic geometry has been systematically developed by Strubecker [23, 24, 25] in the 1940s; a good overview of the many results is provided in the monograph by Sachs [22].

The *isotropic space* is the affine space $\mathbb{R}^3$ equipped with the norm $\|(x, y, z)\|_i := \sqrt{x^2 + y^2}$. The invariants of affine transformations preserving this norm are subject of *isotropic geometry*.

The projection $(x, y, z) \mapsto (x, y, 0)$ of isotropic space onto the xy-plane is called *top view*. Basic objects of isotropic geometry and their definitions (from the point of view of Euclidean geometry in isotropic space) are given in the first two columns of Table 1. We return to the third column of the table further.

Table 1: Basic objects of isotropic geometry as images of surfaces in the isotropic model of Laguerre geometry.

Object of isotropic geometry	Definition	Corresponding surface in Laguerre geometry
point	point in isotropic space	oriented plane
non-isotropic line	line non-parallel to the z-axis	cone
non-isotropic plane	plane non-parallel to the z-axis	oriented sphere
i-circle of elliptic type	ellipse whose top view is a circle	cone
i-circle of parabolic type	parabola with z-parallel axis	cone
i-sphere of parabolic type	paraboloid of revolution with z-parallel axis	oriented sphere
i-paraboloid	graph of a quadratic function $z = F(x, y)$	parabolic cyclide *or* oriented sphere
i-Willmore surface	graph of a (multi-valued) biharmonic function $z = F(x, y)$	L-minimal surface

In isotropic space there exists a counterpart to Möbius geometry. One puts i-spheres of parabolic type and non-isotropic planes into the same class of *isotropic Möbius spheres* (*i-M-spheres*); they are given by

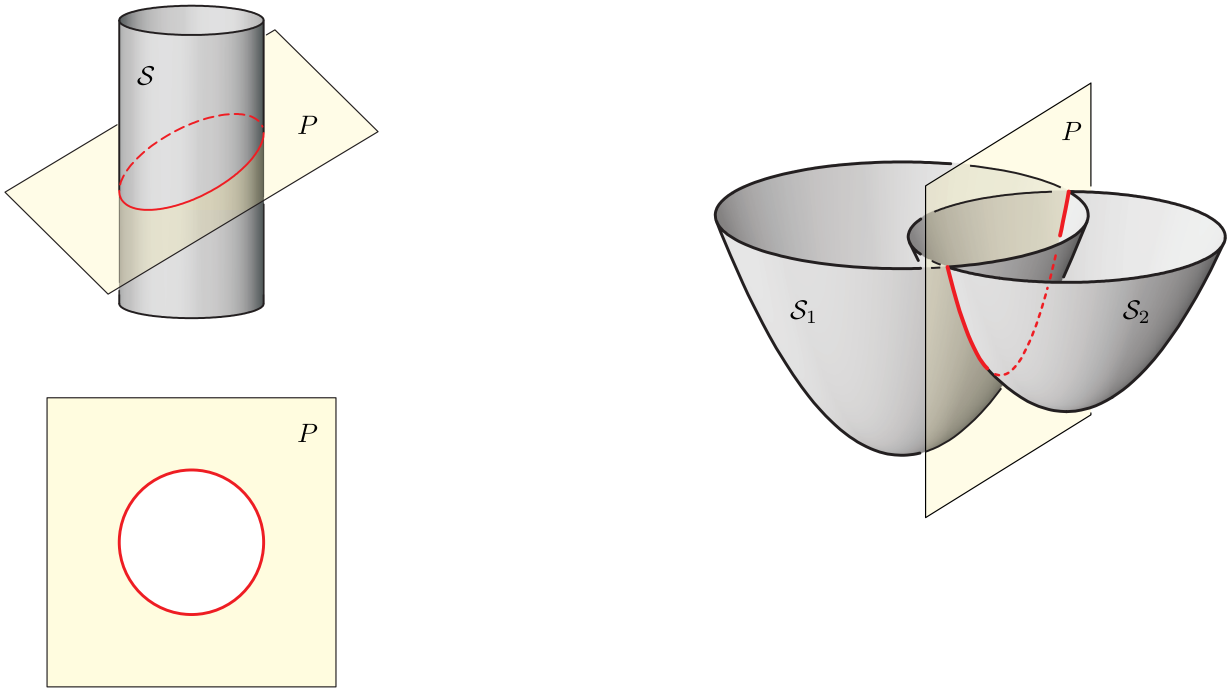

Figure 2: (Left) An i-circle of elliptic type is the intersection curve of a vertical round cylinder $\mathcal{S}$ and a nonisotropic plane P. When viewed from the top, the i-circle is a Euclidean circle. (Right) An i-circle of parabolic type is a parabola with z-parallel axis. This curve appears as the intersection curve of two i-spheres, $\mathcal{S}_1$ and $\mathcal{S}_2$, of parabolic type with the same isotropic radius. For more details, please refer to Table 1.

equation $z = a(x^2 + y^2) + bx + cy + d$ for some $a, b, c, d \in \mathbb{R}$. The coefficient a in this representation is called the *i-radius*. An intersection curve of two i-M-spheres is called an *i-M-circle*; it may be an i-circle of elliptic or parabolic type or a non-isotropic straight line.

Similar to the case of Euclidean Möbius geometry, where an ideal point is added to $\mathbb{R}^3$, in isotropic Möbius geometry we add a copy of $\mathbb{R}$ to $\mathbb{R}^3$, the *ideal line* ℓ_∞. The resulting space $\mathbb{R}^3 \cup \ell_\infty$ is called *extended isotropic space*. An i-M-sphere with i-radius a by definition intersects the ideal line in the point $a \in \ell_\infty$.

A map acting on $\mathbb{R}^3 \cup \ell_\infty$ is called an *isotropic Möbius (i-M) transformation*, if it takes i-M-spheres to i-M-spheres (and hence i-M-circles to i-M-circles). The top view of an i-M-transformation is a planar Euclidean Möbius transformation. Basic i-M-transformations which together with the translation $(x, y, z) \mapsto (x + 1, y, z)$ generate the whole group of i-M-transformations are given in the first column of Table 2. Here R^θ is the rotation through an angle θ around the z-axis.

Table 2: Basic isotropic Möbius transformations as images of Laguerre transformations in the isotropic model of Laguerre geometry.

i-M-transformation	Corresponding L-transformation
$(x, y, z) \mapsto R^\theta(x, y, z)$	rotation R^θ
$(x, y, z) \mapsto (x, y, z + ax + by)$	translation by vector $(a, b, 0)$
$(x, y, z) \mapsto (x, y, z + x^2 + y^2 - 1)$	translation by vector $(0, 0, 1)$
$(x, y, z) \mapsto (x, y, z + h)$	h-offset operation
$(x, y, z) \mapsto (x, y, az)$	homothety with coefficient a
$(x, y, z) \mapsto (x, y, z)/(x^2 + y^2)$	reflection with respect to the plane $z = 0$
$(x, y, z) \mapsto (x, y, z)/\sqrt{2}$	transformation Λ

2.2 Laguerre Geometry

A *contact element* is a pair (r, P), where r is a point in $\mathbb{R}^3$, and P is an oriented plane passing through the point r. Denote by $ST\mathbb{R}^3$ the space of all contact elements.

To an oriented surface Φ in $\mathbb{R}^3$ assign the set of all the contact elements (r, P) such that $r \in \Phi$ and P is the oriented tangent plane to Φ at the point r. We get a *Legendre surface*, i.e., an immersed surface $(\mathbf{r}, \mathbf{P}) : \mathbb{R}^2 \to ST\mathbb{R}^3$ such that $d\mathbf{r}(u, v) \parallel \mathbf{P}(u, v)$.

Hereafter by a *surface* we mean a Legendre surface, not necessarily obtained from a smooth oriented surface in $\mathbb{R}^3$. An example of a Legendre surface is a *point*, or a *sphere of radius* 0, which is the set of all the contact elements (r, P) such that $r = r_0$ is fixed and $P \ni r_0$ is arbitrary. A *Laguerre transformation* (*L-transformation*) is a bijective map $ST\mathbb{R}^3 \to ST\mathbb{R}^3$ taking oriented planes to oriented planes and oriented spheres (possibly of radius 0) to oriented spheres (possibly of radius 0). The invariants of Laguerre transformations are the subject of *Laguerre geometry* [4, 8].

Note that an L-transformation does not in general preserve points, since those are seen as spheres of radius 0 and may be mapped to other spheres. A simple example of an L-transformation is the *h-offset operation*, translating a contact element (r, P) by the vector $h\mathbf{n}$, where $\mathbf{n}$ is the positive unit normal vector to the oriented plane P.

A *Hesse normal form* of an oriented plane P is the equation $n_1x + n_2y + n_3z + h = 0$ of the plane such that (n_1, n_2, n_3) is the positive unit normal vector to the oriented plane. A Laguerre transformation is uniquely defined by its action on the set of oriented planes. Consider the Laguerre transformation Λ taking an oriented plane in the Hesse normal form $n_1x + n_2y + n_3z + h = 0$ to the oriented plane in the Hesse normal form $n_1x + n_2y + \frac{1}{2}(3n_3 + 1)z + h = 0$. Denote by $\tilde{\mathbf{r}}(u, v)$ the surface obtained from the surface $\mathbf{r}(u, v)$ by the transformation. Let us parametrize the surface $\tilde{\mathbf{r}}(u, v)$ so that the tangent planes to the surfaces $\tilde{\mathbf{r}}(u, v)$ and $\mathbf{r}(u, v)$ are parallel at points having the same parameters u and v. This notation turns out to be convenient in our classification results which follow.

More examples of Laguerre transformations are given in the second column of Table 2.

For a pair of parallel oriented planes P_1 and P_2 in Hesse normal forms $n_1x + n_2y + n_3z + h_1 = 0$ and $n_1x + n_2y + n_3z + h_2 = 0$, denote by $a_1P_1 \oplus a_2P_2$ the plane $n_1x + n_2y + n_3z + a_1h_1 + a_2h_2 = 0$. Define a *convolution surface* $a_1\Phi_1 \oplus a_2\Phi_2$ of two (Legendre) surfaces Φ_1 and Φ_2 to be the envelope of all the planes $a_1P_1 \oplus a_2P_2$, where (P_1, P_2) runs through all the pairs of parallel oriented tangent planes to Φ_1 and Φ_2, respectively.

2.3 Isotropic model of Laguerre geometry

To each oriented plane in the Hesse normal form $n_1x + n_2y + n_3z + h = 0$, where $n_1^2 + n_2^2 + n_3^2 = 1$ and $n_3 \neq -1$, assign the point

$$\frac{1}{n_3 + 1}(n_1, n_2, h). \tag{2.1}$$

To an oriented plane in the Hesse normal form $-z + h = 0$ assign the ideal point $h \in \ell_\infty$. This induces a map from the space $ST\mathbb{R}^3$ to the extended isotropic space $\mathbb{R}^3 \cup \ell_\infty$. The map provides the *isotropic model of Laguerre geometry.* For a more geometric definition see [20].

A non-developable surface Φ viewed as set of oriented tangent planes is mapped to a surface Φ^i in the isotropic model. Conversely, the surface Φ can be reconstructed given the surface Φ^i:

Proposition 2 cf. [17, Corollary 2] *Let Φ be a nondevelopable Legendre surface. Suppose that Φ^i is a graph of multi-valued function $z = F(x, y)$. Then the surface Φ can be parametrized as follows:*

$$\frac{1}{x^2 + y^2 + 1}\begin{pmatrix} (x^2 - y^2 - 1)F_x + 2xyF_y - 2xF \\ (y^2 - x^2 - 1)F_y + 2xyF_x - 2yF \\ 2xF_x + 2yF_y - 2F \end{pmatrix}. \tag{2.2}$$

As examples, consider the pairs of surfaces (Φ, Φ^i) given in Table 1.

An oriented sphere with center (m_1, m_2, m_3) and radius R is mapped to the isotropic Möbius sphere

$$z = \frac{R + m_3}{2}(x^2 + y^2) - m_1x - m_2y + \frac{R - m_3}{2}. \tag{2.3}$$

A cone viewed as the common tangent planes of two oriented spheres is mapped to the common points of two i-M-spheres (= i-M-circle) in the isotropic model. So if a surface Φ is the envelope of a family of cones, then the surface Φ^i contains a family of i-M-circles.

In particular, a line is mapped to an i-M-circle of the form

$$\begin{cases} z = m_3(x^2 + y^2 - 1) - m_1x - m_2y, \\ z = n_3(x^2 + y^2 - 1) - n_1x - n_2y. \end{cases} \tag{2.4}$$

Theorem 3 [17, Theorem 1] *A nondevelopable Legendre surface Φ is Laguerre minimal if and only if the surface Φ^i is a graph of a multi-valued biharmonic function $z = F(x, y)$, i.e., a function satisfying the equation $\Delta(\Delta(F)) = 0$.*

Convolution surface $a_1\Phi_1 \oplus a_2\Phi_2$ corresponds in the isotropic model to the linear combination of the two multi-valued functions whose graphs are Φ_1^i and Φ_2^i. Thus a convolution surface of two L-minimal surfaces is L-minimal [17, Corollary 3].

L-transformations correspond to i-M-transformations in the isotropic model and vice versa. Some examples are given in Table 2. Here $\mathbf{r}(u, v)$ is a parametrization of a surface Φ.

Invariance of L-minimal surfaces under L-transformations is translated in the isotropic model as follows:

Theorem 4 [17, Theorem 1] *Suppose that F is a graph of a function biharmonic in a region $U \subset \mathbb{R}^2$ and $m : \mathbb{R}^3 \cup \ell_\infty \to \mathbb{R}^3 \cup \ell_\infty$ is an isotropic Möbius transformation. Then $m(F)$ is a graph of a function biharmonic in the top view of $m(U \times \mathbb{R}) - \ell_\infty$.*

Plan of the proof of Theorem 1 To a ruled L-minimal surface there corresponds an i-Willmore surface containing a family of i-M-circles in the isotropic model.

First we show that the top view of the family of i-M-circles is a pencil. In other words, all the rulings of the L-minimal surface are parallel to one plane.

Then by appropriate choice of coordinates we transform the pencil into a pencil of lines. In the latter case we describe all possible i-Willmore surfaces by solving the biharmonic equation explicitly.

Returning to the Euclidean model we get a description of all ruled L-minimal surfaces.

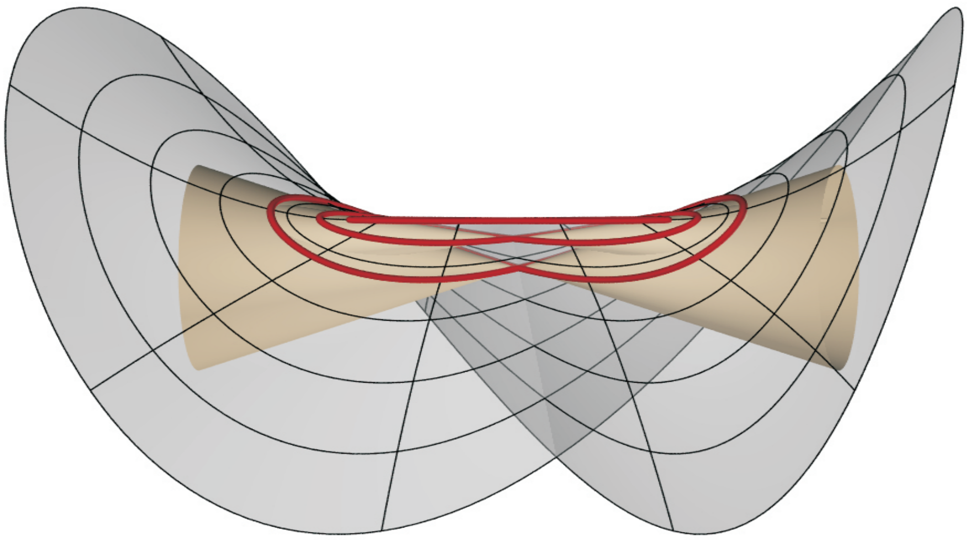

Figure 3: L-minimal surface $\mathbf{r}_5$ arising as envelope of a family of cones; see Example 37 for the details.

3 Biharmonic functions carrying a family of i-circles

3.1 Statement of the Pencil theorem

In this section we show that the top view of a family of i-M circles contained in a graph of a biharmonic function is a pencil. As a corollary we obtain the result that all the lines lying in a ruled L-minimal surface are parallel to one plane. Denote by $I = [0; 1]$.

Theorem 5 (Pencil theorem) *Let $F(x, y)$ be a biharmonic function in a region $U \subset \mathbb{R}^2$. Let S_t, $t \in I$, be an analytic family of circles in the plane. Suppose that for each $t \in I$ we have $S_t \cap U \neq \emptyset$ and the restriction $F|_{S_t \cap U}$ is a restriction of a linear function. Then either S_t, $t \in I$, is a pencil of circles or*

$$F(x,y) = A((x-a)^2 + (y-b)^2) + \frac{B(x-c)^2 + C(x-c)(y-d) + D(y-d)^2}{(x-c)^2 + (y-d)^2} \tag{3.1}$$

for some $a, b, c, d, A, B, C, D \in \mathbb{R}$.

The function given by the formula (3.1) has the following property: there is a 2-parametric family of circles S_t, $t \in I^2$, in the plane such that for each $t \in I^2$ the restriction $F|_{S_t \cap U}$ is a restriction of a linear function.

Plan of the proof of Pencil Theorem 5. We say that two circles *cross* each other if their intersection consists of 2 points. Assume that the family of circles is not a pencil. Then it contains a subfamily of one of the following types:

(1) the circles S_t, $t \in I$, pairwise cross but do not pass through one point;

(2) the circles S_t, $t \in I$, have a common point O;

(3) the circles S_t, $t \in I$, are nested.

First we prove the theorem in case when the region U is sufficiently large, i.e., $U \supset \bigcup S_t$, $U \supset \bigcup S_t - \{O\}$ and $U = \mathbb{R}^2$ for types (1), (2) and (3), respectively.

Then we reduce the theorem to the latter case by a biharmonic continuation of the function F. The continuation is in 2 steps.

In the first step we extend the function F *along* the circles S_t until we reach the envelope of the family of circles (if the envelope is nonempty). This is done easily for arbitrary real analytic function F. The main difficulty is that extending F along the circles beyond the envelope may lead to a multi-valued function.

In the second step we extend the function F *across* the circles S_t to make the region U sufficiently large while keeping the function single-valued. This is done using a new symmetry principle for biharmonic functions.

3.2 Three typical cases

First let us prove Theorem 5 in three typical particular cases treated in Lemmas 6, 7 and 8 for "suffiently large" sets U.

Lemma 6 (Crossing circles) *Let S_t, $t \in I$, be a family of pairwise crossing circles in the plane distinct from a pencil of circles. Let F be an arbitrary function defined in the set $U = \bigcup_{t \in I} S_t$. Suppose that for each $t \in I$ the restriction $F|_{S_t}$ is a restriction of a linear function. Then $F = A((x-a)^2 + (y-b)^2) + B$ for some $a, b, A, B \in \mathbb{R}$.*

Proof. Denote by l_t the linear function $F|_{S_t}$. Let $s_t = 0$ be the normalized equation of the circle S_t, i.e., $s_t = x^2 + y^2 + \dots$ and $s_1|_{S_1} = 0$. For any pair $s, t \in I$ both differences $s_t - s_s$ and $l_t - l_s$ are linear functions vanishing on $S_s \cap S_t$. Thus $l_t - l_s = k_{st}(s_t - s_s)$ for some number k_{st}.

Since the family S_t is not a pencil it follows that there are 3 circles S_1, S_2, S_3 in the family such that the functions s_1, s_2, s_3 are linearly independent. Let us show that $F = k_{12}s_1 + l_1$.

Indeed, in the circle S_1 we have $F = l_1 = k_{12}s_1 + l_1$ because $s_1|_{S_1} = 0$. In the circle S_2 we have $F = l_2 = k_{12}s_2 + l_2 = k_{12}s_1 + l_1$ by definition of the number k_{12}.

Consider the circle S_3. We have $k_{23} = k_{31} = k_{12}$ because otherwise $k_{12}(s_1 - s_2) + k_{23}(s_2 - s_3) + k_{31}(s_3 - s_1) = (l_1 - l_2) + (l_2 - l_3) + (l_3 - l_1) = 0$ is a nontrivial linear combination of s_1, s_2, s_3. Thus in the circle S_3 we have $F = l_3 = k_{13}s_3 + l_3 = k_{13}s_1 + l_1 = k_{12}s_1 + l_1$.

Finally, take any circle S_t. We can replace one of the functions s_1, s_2, s_3 by s_t to get still a linearly independent triple. Repeating the argument from the previous paragraph we get $F = k_{12}s_1 + l_1$ in S_t. Thus $F = k_{12}s_1 + l_1$ in the whole set U. $\square$

Lemma 7 (Circles with a common point) *Let S_t, $t \in I$, be a family of pairwise crossing circles in the plane passing through the origin O. Assume that no three circles of the family belong to one pencil. Let F be an arbitrary function defined in the set $U = \bigcup_{t \in I} S_t - \{O\}$. Suppose that for each $t \in I$ the restriction $F|_{S_t - \{O\}}$ is a restriction of a linear function. Then*

$$F(x, y) = A((x-a)^2 + (y-b)^2) + \frac{Bx^2 + Cxy + Dy^2}{x^2 + y^2}$$

for some $a, b, A, B, C, D \in \mathbb{R}$.

Proof. Perform the transformation $(x, y, z) \mapsto (x, y, z)/(x^2 + y^2)$. Then the family of circles S_t transforms to a family of lines L_t. By the assumtions of the lemma any two of the lines L_t intersect each other but no three of the lines L_t pass through one point. The graph of the function F transforms to a graph of a function G defined in $V = \bigcup_{t \in I} L_t$. For each $t \in I$ the restriction $G|_{L_t}$ is a quadratic function.

Take three lines L_1, L_2, L_3 from the family. Let l_1, l_2, l_3 be nonzero linear functions vanishing in the lines L_1, L_2, L_3, respectively. Let l be a linear function such that $l = F$ in the points $L_1 \cap L_2, L_2 \cap L_3, L_3 \cap L_1$. Since $G\,|_{L_1}$ is quadratic and $G - l = 0$ in the points $L_1 \cap L_2$ and $L_1 \cap L_3$ it follows that $G\,|_{L_1} = k_{23} l_2 l_3 + l$ for some number k_{23}. Analogously, $G\,|_{L_2} = k_{31} l_3 l_1 + l$ and $G\,|_{L_3} = k_{12} l_1 l_2 + l$ for some numbers k_{12} and k_{31}.

Let us prove that $G = k_{12} l_1 l_2 + k_{23} l_2 l_3 + k_{31} l_3 l_1 + l$ in the whole set V. Indeed, consider the difference $H = k_{12} l_1 l_2 + k_{23} l_2 l_3 + k_{31} l_3 l_1 + l - G$. Then $H\,|_{L_1} = 0$, $H\,|_{L_2} = 0$, $H\,|_{L_3} = 0$ by the above. Take a line L_t distinct from L_1, L_2, L_3. Then $H\,|_{L_t}$ is a quadratic function. On the other hand, $H(L_t \cap L_1) = H(L_t \cap L_2) = H(L_t \cap L_3) = 0$. Since the points $L_t \cap L_1, L_t \cap L_2, L_t \cap L_3$ are pairwise distinct it follows that $H\,|_{L_t} = 0$. So the function H vanishes in each line L_t. Thus $H = 0$ in the set V.

We have proved that G is a polynomial of degree not greater than 2. Performing the inverse transformation $(x, y, z) \mapsto (x, y, z)/(x^2 + y^2)$ we obtain the required formula for the function F. $\square$

Lemma 8 (Nested circles) *Let S_1 and S_2 be the pair of circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 2$. Let F be a function biharmonic in the whole plane $\mathbb{R}^2$. Suppose that for each $t = 1, 2$ the restriction $F\,|_{S_t}$ is a restriction of a linear function. Then*

$$F(x, y) = (x^2 + y^2)(Ax + By + C) + ax + by + c$$

for some $a, b, c, A, B, C \in \mathbb{R}$.

The function $F(x, y) = (x^2 + y^2)\log(x^2 + y^2)$ extended by $F(0, 0) = 0$ might seem to be a counterexample to this lemma but in fact it is not: $\partial^2 F/\partial x^2$ is discontinuous at the origin.

Proof. Let l_t be the linear function such that $F\,|_{S_t} = l_t\,|_{S_t}$, where $t = 1, 2$. The function $F - l_1$ is biharmonic in $\mathbb{R}^2$. By Proposition 12 below it follows that there are functions u_1, u_2, harmonic in $\mathbb{R}^2$, such that $F - l_1 = (x^2 + y^2 - 2)u_1 + u_2$. Then $F = (x^2 + y^2 - 2)(v_1 - l_1) + (x^2 + y^2 - 1)(v_2 + l_2)$, where the functions $v_1 = u_1 - u_2$ and $v_2 = u_2 + l_1 - l_2$ are also harmonic in $\mathbb{R}^2$.

Let us prove that v_1 is a constant. We have $l_1\,|_{S_1} = F\,|_{S_1} = l_1 - v_1\,|_{S_1}$. Thus $v_1\,|_{S_1} = 0$. Then by the symmetry principle for harmonic functions it follows that $v_1(x, y) = -v_1(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$ for any $(x, y) \neq (0, 0)$. So there exists $\lim_{x^2+y^2 \to \infty} v_1(x, y) = -v_1(0, 0)$. By the Liouville theorem it follows immediately that v_1 is a constant.

Analogously v_2 is a constant and the lemma follows. $\square$

Proposition 9 *Let $F(x, y) = (x^2 + y^2)(Ax + By + C) + ax + by + c$, where $A^2 + B^2 \neq 0$. Suppose that the restriction of the function F to a circle $S \subset \mathbb{R}^2$ is linear. Then the center of the circle S is the origin.*

Proof. It suffices to consider the case when $a = b = c = C = 0$. Let $x^2 + y^2 + px + qy + r = 0$ be the equation of S. If the restriction $F\,|_S$ is linear then S is a projection of (a part of) the intersection of the surface $z = F(x, y)$ and a plane $z = \alpha x + \beta y + \gamma$. Thus there exist numbers k, l, m such that

$$(x^2 + y^2)(Ax + By) - (\alpha x + \beta y + \gamma) = (kx + ly + m)(x^2 + y^2 + px + qy + r).$$

We get

$$k = A, \quad l = B, \quad pk + m = 0, \quad ql + m = 0, \quad pl + qk = 0.$$

Thus $p = An$, $q = Bn$, $m = A^2 n = -B^2 n$ for some $n \in \mathbb{R}$. Since $A^2 + B^2 \neq 0$ it follows that $n = 0$. So S is the circle $x^2 + y^2 + r = 0$. $\square$

3.3 Biharmonic continuation

We are going to reduce Theorem 5 to Lemmas 6, 7 and 8 by "biharmonic continuation" of the function F. We say that a function F biharmonic in a region U *extends* to a function G biharmonic in a region V if there is an open set $D \subset U \cap V$ such that $F = G$ in D. Notice that F can be distinct from G in $U \cap V$ if the latter set is disconnected.

Proposition 10 (Uniqueness of a continuation) *If two functions biharmonic in a region $V \subset \mathbb{R}^2$ coincide in a region $U \subset V$ then these functions coincide in the region V.*

Proof. Let F, G be functions such that $\Delta^2 F = \Delta^2 G = 0$ in V and $F = G$ in U. Then $\Delta(F - G)$ is harmonic in V and vanishes in U. Thus $\Delta(F - G) = 0$ in V. Hence $F - G$ is harmonic in V and vanishes in U. Thus $F - G = 0$ in V. $\square$

A family of circles in the plane is *nonconstant* if it contains at least two distinct circles.

Lemma 11 (Continuation along circles) *Let $F : U \to \mathbb{R}$ be a biharmonic function defined in a region $U \subset \mathbb{R}^2$. Let S_t, $t \in I$, be a nonconstant analytic family of circles in the plane. Suppose that for each $t \in I$ we have $S_t \cap U \neq \emptyset$ and the restriction $F\,|_{S_t \cap U}$ is a restriction of a linear function. Then for some segment $J = [q, r] \subset I$ the function F extends to a function biharmonic in a region bounded by certain arcs of the circles S_q, S_r and possibly two pieces of the envelope of the family S_t, $t \in J$.*

Proof. The idea of the proof is to extend the function linearly along the circles until we reach the envelope. The obtained function will be a real analytic continuation of the initial function and hence it will be biharmonic.

Let E be the envelope of the family S_t, $t \in I$. For fixed $t \in I$ and a small $\epsilon > 0$ a *local piece* of the envelope E is the envelope of the family S_τ, $\tau \in [t - \epsilon, t + \epsilon]$. The envelope or its local piece can be empty.

Each circle S_t touches (each local piece of) the envelope E at most at 2 points, depending on the arrangement of the circles sufficiently close to S_t. Let $R_t \subset S_t$ be one of the open arcs joining the touching points of the circle S_t and (each local piece of) the envelope E. Let R_t be one of the sets $S_t - O_t$ or $\emptyset$ (respectively, $R_t = S_t$ or $\emptyset$), if there is a unique such touching point O_t (respectively, no such touching points). Choose the arcs R_t so that they form a continuous family.

One can assume that for each t in a segment $J_1 \subset I$ we have $R_t \cap U \neq \emptyset$. Indeed, if $R_t \cap U \neq \emptyset$ for at least one $t \in I$ then the same condition holds in a neighborhood J_1 of t. Otherwise replace each R_t by $S_t - R_t$ and repeat the argument.

There is a segment $[q, r] = J \subset J_1$ such that the arcs R_t, $t \in J$, are pairwise disjoint. Indeed, if $E = \emptyset$ then one can take $J = J_1$. If a local piece of the envelope E is a pair of smooth curves then one can approximate the family S_t, $t \in I$, by a family of circles touching a pair of lines and get the required segment J. If a piece of the envelope degenerates to a point then one can find a segment J analogously.

So $V = \bigcup_{t \in J} R_t$ is a region bounded by arcs R_q, R_r and possibly two pieces of the envelope E.

By the assumption of the lemma the restriction $F\,|_{R_t \cap U}$ is the restriction of a linear function. Extend the function F linearly to each arc R_t. We get a function defined in the whole region V. It remains to prove that the obtained function is biharmonic in V.

Let us show that F is real analytic in V. Parametrize the arc R_t by the functions $x(t, \phi) = x_0(t) + r(t)\cos\phi$, $y(t, \phi) = y_0(t) + r(t)\sin\phi$. Consider (t, ϕ) as coordinates in V. Since the family S_t is analytic it follows that these coordinates are analytic. Without loss of generality assume $[q, r] \times [\alpha, \beta] \subset U$ for some $\alpha, \beta \in [-\pi, \pi]$. Then $F(t, \phi)$ is real analytic in $[q, r] \times [\alpha, \beta]$. By the construction $F(t, \phi) = a(t)\cos\phi + b(t)\sin\phi + c(t)$ in the region V for some functions $a(t), b(t), c(t)$. Thus $a(t), b(t), c(t)$ are real analytic in $[q, r]$. Hence F is real analytic in the whole region V.

Then the function $\Delta^2 F$ is also real analytic in the region V and vanishes in the open set $U \cap V$. By the uniqueness theorem for analytic functions it follows that $\Delta^2 F = 0$ in the whole region V, i.e., F is biharmonic in V. $\square$

To extend the function F further we need the following preparations.

Proposition 12 (Representation [1]) *Let $s(x, y) = x^2 + y^2 + ax + by + c$. Then any function F biharmonic in a region $U \subset \mathbb{R}^2$ can be represented as $F = su_1 + u_2$ for some functions u_1, u_2 harmonic in U.*

Proposition 13 (Arc extension) *Let $S \subset R$ be a pair of circular arcs. Let F be a biharmonic function defined in a neighborhood of the arc R. Suppose that $F\,|_S$ is a restriction of a linear function. Then $F\,|_R$ is the restriction of the same linear function.*

Proof. Let l be the linear function $F\,|_S$. Let $s(x, y) = 0$ be the normalized equation of the circle containing the arc S. By Proposition 12 we have $F - l = su_1 + u_2$ for some functions u_1, u_2 harmonic in a neighborhood U of the arc R. Then $u_2\,|_S = (F - l)\,|_S = 0$. By the symmetry principle for harmonic functions it follows that $u_2(x,y) = -u_2(x',y')$ for any pair of points $(x,y),(x',y') \in U$ symmetric with respect to the circle $s(x,y)=0$. In particular, $u_2|_R = 0$. Thus $F|_R = l$. $\square$

Now we are going to give a version of a symmetry principle for biharmonic functions. The usual symmetry principle [15] is not applicable in our situation because we have no information on the growth of the function in the normal directions to the circles.

The following technical definition is required to keep the function single-valued during the continuation process with our symmetry principle.

Definition 14 (Nicely arranged region) *Let S_1 and S_2 be a pair of circles with centers O_1 and O_2, respectively. Denote by $r_t : \mathbb{R}^2 - \{O_t\} \to \mathbb{R}^2 - \{O_t\}$ the reflection with respect to the circle S_t. Let $r_t(U)$ be an abbreviation for $r_t(U - \{O_t\})$. A region $U \subset \mathbb{R}^2$ is* nicely arranged *with respect to the circles S_1 and S_2, if the set $U \cap r_1(U) \cap r_2(U)$ has a connected component D such that $S_1 \cap D \neq \emptyset$ and $S_2 \cap D \neq \emptyset$.*

Notice that an arbitrary region is nicely arranged with respect to any pair of sufficiently close circles intersecting the region. For a pair of circles S_s and S_t denote by Σ_{st} the *limit set* of the pencil of circles passing through S_s and S_t, i.e., $\Sigma_{st} = \{\, x \in \mathbb{R}^2 \,:\, r_s(x) = r_t(y)\,\}$. The limit set consists of at most 2 points.

Lemma 15 (Double symmetry principle) *Let F be a function biharmonic in a simply-connected region $U \subset \mathbb{R}^2$ nicely arranged with respect to a pair of circles $S_1 \neq S_2$. Suppose that for each $t = 1,2$ the restriction $F|_{S_t \cap U}$ is a restriction of a linear function. Then F extends to a function biharmonic in the open set $r_1(U) \cap r_2(U) - \Sigma_{12}$.*

Proof. Let l_t be the linear function $F|_{S_t}$ for $t = 1,2$. Without loss of generality assume that S_1 is the unit circle $x^2+y^2=1$. By Proposition 12 it follows that $F = (x^2+y^2-1)u_2 + u_1 + l_1$ for some functions u_1 and u_2 harmonic in U.

Take functions $\nu_1(z)$ and $\nu_2(z)$ complex analytic in U such that $u_t = \nu_t(z) + \overline{\nu_t(z)}$ for $t = 1,2$. Since U is simply-connected it follows that $\nu_1(z)$ and $\nu_2(z)$ are single-valued. Let $\lambda_1(z)$ and $\lambda_2(z)$ be linear functions such that $l_t = \lambda_t(z) + \overline{\lambda_t(z)}$ for $t = 1,2$. For $t = 1,2$ represent the reflection with respect to the circle S_t as a map $z \mapsto \rho_t(\bar z)$ for a fractional linear function $\rho_t(z)$.

Let us extend the function u_1 to the open set $r_1(U)$. (What we do is the usual symmetry principle.) Let D be the open set from Definition 14. For each $z \in S_1$ we have $z = \rho_1(\bar z)$. Thus the condition $F|_{S_1} = l_1$ is equivalent to

$$\nu_1(z) = -\overline{\nu_1(\rho_1(\bar z))} \tag{3.2}$$

for each $z \in S_1 \cap D$. Both sides of formula (3.2) are complex analytic functions. By the uniqueness theorem it follows that these functions coincide in D. Thus formula (3.2) defines an extension of the function $\nu_1(z)$ to the open set $r_1(U)$. So $u_1 = \nu_1(z) + \overline{\nu_1(z)}$ is the required extension of the function u_1.

Let us extend the function u_2 to the open set $r_1(U) \cap r_2(U) - \Sigma_{12}$. For each $z \in S_2$ we have $z = \rho_2(\bar z)$. For each $z \in D$ formula (3.2) holds by the previous paragraph. Thus for each $z \in S_2 \cap D$ the condition $F|_{S_2} = l_2$ is equivalent to the condition

$$\nu_2(z) = -\overline{\nu_2(\rho_2(\bar z))} + \frac{\overline{\nu_1(\rho_1(\bar z))} - \overline{\nu_1(\rho_2(\bar z))} - \lambda_1(z) - \overline{\lambda_1(\rho_2(\bar z))} + \lambda_2(z) + \overline{\lambda_2(\rho_2(\bar z))}}{z\overline{\rho_2(\bar z)} - 1}. \tag{3.3}$$

Since both sides of formula (3.3) are complex analytic functions it follows that these functions coincide in D. If $z \in r_1(U) \cap r_2(U)$ then $\rho_1(\bar z), \rho_2(\bar z) \in U$. Thus the right-hand side of formula (3.3) defines a function complex analytic in $r_1(U) \cap r_2(U) - \Sigma_{12}$ because the denominator may vanish only in Σ_{12}. Extend the function $\nu_2(z)$ to the open set $r_1(U) \cap r_2(U) - \Sigma_{12}$ by formula (3.3). Then $u_2(z) = \nu_2(z) + \overline{\nu_2(z)}$ is the required extension of the function u_2.

Since both functions u_1 and u_2 extend to $r_1(U) \cap r_2(U) - \Sigma_{12}$ it follows that $F = (x^2+y^2-1)u_2 + u_1 + l_1$ also extends to $r_1(U) \cap r_2(U) - \Sigma_{12}$. $\square$

Lemma 16 (Continuation across nested circles) *Let S_t, $t \in I$, be a family of nested circles in the plane distinct from a pencil of circles. Let $F : U \to \mathbb{R}$ be a function biharmonic in the ring U between S_0 and S_1. Suppose that for each $t \in I$ the restriction $F|_{S_t \cap U}$ is a restriction of a linear function. Then the function F extends to a function biharmonic in the whole plane $\mathbb{R}^2$.*

Proof. The idea of the proof is to extend the function, using the double symmetry principle, to the ring between $r_0(S_1)$ and S_1, then to the ring between $r_0(S_1)$ and $r_1r_0(S_1)$, and so on.

Take any pair of circles S_t and S_s, where $s, t \in I$ are sufficiently close to 0. Draw disjoint slits T and T' such that the regions $U - T$ and $U - T'$ are simply-connected. By Lemma 15 the function F extends to both $r_t(U - T) \cap r_s(U - T)$ and $r_t(U - T') \cap r_s(U - T')$. Thus it extends to a (possibly multi-valued) function biharmonic in the ring $r_t(U) \cap r_s(U) \cup U$. The latter function is single-valued because a continuation along the closed path S_0 leads to the initial value. Approaching $t, s \to 0$ one can extend the function F to the ring between the circles $r_0(S_1)$ and S_1. Now approaching $t, s \to 1$ one can extend the function F to the larger ring between the circles $r_0(S_1)$ and $r_1r_0(S_1)$. Continuing this process one extends F to a function biharmonic in $\mathbb{R}^2$ except the limit set Σ_{01} of the pencil of circles passing through S_0 and S_1.

Since S_t, $t \in I$, is not a pencil of circles it follows that $\Sigma_{0p} \cap \Sigma_{01} = \emptyset$ for some $p \in I$. Repeating the above reflection process for the pair of circles S_0 and S_p one extends the function F to a function biharmonic in the whole plane $\mathbb{R}^2$. $\square$

Lemma 17 (Continuation across crossing circles) *Let S_t, $t \in I$, be a nonconstant analytic family of pairwise crossing circles in the plane distinct from a pencil. Let $F : U \to \mathbb{R}$ be a function biharmonic in a region U. Suppose that for each $t \in I$ we have $S_t \cap U \neq \emptyset$ and the restriction $F|_{S_t}$ is a restriction of a linear function. Then for some segment $J \subset I$ the function F extends to a function biharmonic in a neighborhood of $\bigcup_{t \in J} S_t$ possibly except a common point of all the circles S_t.*

Proof. The idea of the proof is to extend the function first along the circles until we reach the envelope E of the family, then by the double symmetry principle — to a neighborhood of the envelope, and finally — along the circles beyond the envelope.

By Lemma 11 it follows that F extends to a region bounded by certain arcs of the circles S_q and S_r and two pieces of the envelope E for some $q, r \in I$. Since the family S_t, $t \in [q, r]$, is not a pencil it follows that at least one component E_0 of the envelope E does not degenerate to a point.

Let us extend the function F to a neighborhood of the curve E_0. Use the notation from the proof of Lemma 11. Take $p \in [q, r]$ such that the curve E_0 is smooth at the point $R_p \cap E_0$. (Then in a neighborhood of the the point $R_p \cap E_0$ the family R_t, $t \in [p - \epsilon, p + \epsilon]$, is isotopic to a family of arcs tangent to one line.) Let V_p be the intersection of the open disc bounded by the circle S_p and an open disc of centered at $R_p \cap E_0$. Clearly, if the latter disc is sufficiently small then $V_p \subset U$. Without loss of generality assume that $R_t \cap V_p = \emptyset$ for each $t \in [p - \epsilon, p]$ and $R_t \cap V_p \neq \emptyset$ for each $t \in [p, p + \epsilon]$, where $\epsilon > 0$ is small enough.

Take a pair of circles S_t and S_s, where $s, t \in [p, p + \epsilon]$. By Lemma 15 the function F extends to the region $r_t(V_p) \cap r_s(V_p)$. Approaching $t, s \to p$ one extends the function F to the region $r_p(V_p)$, and hence to a neighborhood of the point $R_p \cap E_0$. So F extends to a neighborhood V of the curve E_0.

A consequence of this extension is that $F|_{S_t \cap V}$ is linear for each $t \in I$, because each intersection $S_t \cap V$ is connected and Proposition 13 can be applied.

Let us extend the function F along the arcs $S_t - R_t$. Choose $U' \subset U$ and $I' \subset I$ so that $R_t \cap U' = \emptyset$ and $S_t - R_t \cap U' \neq \emptyset$ for each $t \in I'$. Applying Lemma 11 to the region U' and the family S_t, $t \in I'$, we extend the function F to a region bounded by $S_{q'} - R_{q'}$, $S_{r'} - R_{r'}$ and certain pieces of the envelope E for some $p', q' \in I'$. Take a segment J strictly inside $[q', r']$. Then the function F extends to a (possibly multi-valued) function biharmonic in a neighborhood of $\bigcup_{t \in J} S_t$ possibly except a common point of all the circles S_t, $t \in J$. The latter function is single-valued because a continuation along any closed path S_t, $t \in J$, leads to the initial value. $\square$

3.4 Proof and corollaries of the Pencil Theorem

Proof. [of Theorem 5] Assume that S_t, $t \in I$, is not a pencil of circles. Clearly, there is a segment $J \subset I$ such that one of the following conditions hold:

(1) the circles S_t, $t \in J$, pairwise cross but do not pass through one point;

(2) the circles S_t, $t \in J$, have a common point O but no three circles S_t, $t \in J$, belong to one pencil;

(3) the circles S_t, $t \in J$, are nested.

Consider each case separately.

Case (1). By Lemma 17 the function F extends to a function biharmonic in a neighborhood of the set $\bigcup_{t\in J_1} S_t$ for some segment $J_1 \subset J$. By Proposition 13 for each $t \in J_1$ the restriction $F|_{S_t}$ is linear. Then by Lemma 6 case (1) follows.

Case (2). Analogously to the previous paragraph case (2) follows from Lemmas 17, 7 and Proposition 13.

Case (3). By Theorem 4 we may assume that the family contains a pair of circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 2$. By Lemmas 11 and 16 the function F extends to a function biharmonic in the whole plane $\mathbb{R}^2$. Then case (3) follows from Lemma 8 and Proposition 9. $\square$

The following corollaries of Theorem 5 are straightforward.

Corollary 18 *Let Φ^i be an i-Willmore surface carrying an analytic family $\mathcal{F}^i$ of i-M-circles. Then either the surface Φ^i is i-M-equivalent to an i-paraboloid or the top view of the family $\mathcal{F}^i$ is a pencil of circles or lines.*

Corollary 19 *Let Φ be an L-minimal surface enveloped by an analytic family $\mathcal{F}$ of cones. Then either the surface Φ is a parabolic cyclide, a sphere or a plane, or the Gaussian spherical image of the family $\mathcal{F}$ is a pencil of circles in the unit sphere.*

Corollary 20 *A ruled L-minimal surface is a Catalan surface, i.e., contains a family of lines parallel to one plane.*

Remark 21 *Theorem 5 does not remain true for biharmonic functions $\mathbb{C}^2 \to \mathbb{C}$. For instance, for the function $F(x, y) = (x^2 + y^2)(x + iy)$ there is a 2-parametric family of circles S_t, $t \in I^2$, such that for each $t \in I^2$ the restriction $F|_{S_t}$ is a restriction of a linear function.*

Remark 22 *Theorem 5 does not remain true for real analytic functions $\mathbb{R}^2 \to \mathbb{R}$. For instance, the restriction of the function $F(x, y) = \sqrt{(x^2 + y^2)^2 - x^2 + 1}$ to each circle of the family $x^2+y^2-tx-\sqrt{t^2-1} = 0$ is a restriction of a linear function.*

Remark 23 *The proof of Theorem 5 is simpler in the generic case when the biharmonic function F extends to a (possibly multi-valued) function in the whole plane except a discrete subset Σ. For instance, to prove Lemma 17 in this case it suffices to take a segment $J \subset I$ such that $S_t \cap \Sigma = \emptyset$ for each $t \in J$.*

4 Classification of L-minimal surfaces enveloped by a family of cones

4.1 Elliptic families of cones

The results of the previous section give enough information to describe all the L-minimal surfaces enveloped by a family of cones, in particular, ruled L-minimal surfaces. We have got to know that the top view of a family of i-M-circles in an i-Willmore surface is almost always a pencil. Let us consider separately each possible type of the pencil.

Definition 24 *An* elliptic *pencil of circles in the plane (or in a sphere) is the set of all the circles passing through two given distinct points. A 1-parametric family of cones (possibly degenerating to cylinders or lines) in space is* elliptic *if the Gaussian spherical images of the cones form an elliptic pencil of circles in the unit sphere.*

Denote by $\operatorname{Arctan} x = \arctan x + \pi k$, where $k \in \mathbb{Z}$, the multi-valued inverse of the tangent function.

Theorem 25 *Let Φ^i be an i-Willmore surface carrying a smooth family of i-M-circles. Suppose that the top view of the family is an elliptic pencil of circles. Then the surface Φ^i is i-M-equivalent to a piece of the surface*

$$z = \big(a_1(x^2+y^2)+a_2x+a_3\big)\,\mathrm{Arctan}\,\frac{y}{x}+\frac{b_1y^2+b_2xy}{x^2+y^2}+c_1y^2+c_2xy \tag{4.1}$$

for some $a_1, a_2, a_3, b_1, b_2, c_1, c_2 \in \mathbb{R}$.

Proof. [of Theorem 25] Perform an i-M-transformation taking the elliptic pencil of circles in the top view into the pencil of lines $y = tx$, where t runs through a segment $J \subset \mathbb{R}$. Denote by $z = F(x, y)$ the surface obtained from the surface Φ^i by the transformation, where F is a biharmonic function defined in a region $U \subset \mathbb{R}^2$. Assume without loss of generality that $(0, 0) \notin U$. Since an i-M-transformation takes i-M-circles to i-M-circles it follows that the restriction of the function F to (an appropriate segment of) each line $y = tx$, where $t \in J$, is a quadratic function.

Proposition 26 *Let $F(x, y)$ be a biharmonic function in a region $U \subset \mathbb{R} - \{(0, 0)\}$. Suppose that the restriction of the function F to the intersection of each line $y = tx$, where $t \in J$, with the region U is a quadratic function. Then*

$$\begin{aligned}F(x,y) = \big(a_1(x^2+y^2)+a_2x+a_3+a_4y\big)\,\mathrm{Arctan}\,\frac{y}{x}+&\\ +\frac{b_1y^2+b_2xy+b_3x^2}{x^2+y^2}+c_1y^2+c_2xy+c_3x^2+d_1x+d_2y&\end{aligned} \tag{4.2}$$

for some $a_1, a_2, a_3, a_4, b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2 \in \mathbb{R}$.

Proof. Consider the polar coordinates in U. Restrict the function F to a subregion of the form $(r_1, r_2) \times (\phi_1, \phi_2) \subset U$. Then $F(r, \phi) = a(\phi)r^2 + b(\phi)r + c(\phi)$ in the region $(r_1, r_2) \times (\phi_1, \phi_2)$. Thus $r^4\Delta^2F = \big(4a'' + a^{(4)}\big)\, r^2 + \big(b + 2b'' + b^{(4)}\big)\, r + \big(4c'' + c^{(4)}\big)$. Since $\Delta^2F = 0$ it follows that the coefficients of this polynomial in r vanish. Solving the obtained ordinary differential equations we get:

$$\begin{aligned}a(\phi) &= \alpha_1+\alpha_2\phi+\alpha_3\cos 2\phi+\alpha_4\sin 2\phi;\\ b(\phi) &= \beta_1\cos\phi+\beta_2\sin\phi+\beta_3\phi\cos\phi+\beta_4\phi\sin\phi;\\ c(\phi) &= \gamma_1+\gamma_2\phi+\gamma_3\cos 2\phi+\gamma_4\sin 2\phi,\end{aligned}$$

for some $\alpha_1, \dots, \alpha_4, \beta_1, \dots, \beta_4, \gamma_1, \dots, \gamma_4 \in \mathbb{R}$. Returning to the cartesian coordinate system we get the required formula. □

To complete the proof of Theorem 25 perform an appropriate rotation around the z-axis to achieve $a_4 = 0$ in formula (4.2) and then the i-M-transformation $z \mapsto z - c_3(x^2 + y^2) - d_1x - d_2y - b_3$ to achieve $b_3 = c_3 = d_1 = d_2 = 0$. □

Table 3: Biharmonic functions whose restrictions to each line $y = tx$, $t \in I$, are quadratic functions and corresponding Laguerre minimal surfaces

Biharmonic function	Laguerre minimal surface
$(x^2+y^2-1)\mathrm{Arctan}(y/x)$	$\mathbf{r}_1(u,v)$
$(x^2+y^2-2)\mathrm{Arctan}(y/x)/2\sqrt{2}$	$\tilde{\mathbf{r}}_1(u,v)$
$-x\mathrm{Arctan}(y/x)$	$\mathbf{r}_2(u,v)$
$(x\cos\theta_1+y\sin\theta_1)^2(1-1/(x^2+y^2))/2$	$R^{\theta_1}\mathbf{r}_3(u,v)$
$(x\cos\theta_2+y\sin\theta_2)^2(1-2/(x^2+y^2))/4\sqrt{2}$	$R^{\theta_2}\tilde{\mathbf{r}}_3(u,v)$
$a(x^2+y^2)+bx+cy+d$	oriented sphere

A Laguerre minimal surface enveloped by an elliptic family of cones is obtained from the surface (4.1) by transformation (2.2). Let us give some typical examples obtained from graphs of the functions in the left column of Table 3, see also Figure 4. These examples are "building blocks" whose convolutions form all the surfaces in question. We represent all the surfaces in special parametric form $\mathbf{r}(u,v)$, where the map $\mathbf{r}(u,v)$ is the inverse of the composition of the Gaussian spherical map and the stereographic projection. This is convenient to get easy expressions for the convolution surfaces.

Example 27 *The first building block is the well-known helicoid which is given implicitly by $x = -y\tan(z/2)$. It can be parametrized via*

$$\mathbf{r}_1(u,v) = \left(u - \frac{u}{u^2+v^2},\ \frac{v}{u^2+v^2} - v,\ 2\mathrm{Arctan}\,\frac{u}{v} \right) \qquad \textit{or as a ruled surface via} \tag{4.3}$$

$$\mathbf{R}_1(\varphi,\lambda) = (\,0,\ 0,\ -2\varphi\,) + \lambda\,(\,\sin\varphi,\ \cos\varphi,\ 0\,)\,. \tag{4.4}$$

Example 28 *The next example is the cycloid $\mathbf{r}(t) = (t - \sin t, 1 - \cos t, 0)/2$. One should think of a* cycloid *as a Legendre surface formed by all the contact elements (r, P) such that the plane P passes through the line tangent to the curve $\mathbf{r}(t)$ at the point r (see the definitions in §2). We use the parametrization:*

$$\mathbf{r}_2(u,v) = \left(\mathrm{Arctan}\,\frac{u}{v} - \frac{uv}{u^2+v^2},\ \frac{u^2}{u^2+v^2},\ 0 \right). \tag{4.5}$$

The family of tangent lines to the cycloid can be parametrized via

$$\mathbf{R}_2(\varphi,\lambda) = (\,\varphi,\ 1,\ 0\,) + \lambda\,(\,\sin\varphi,\ \cos\varphi,\ 0\,)\,. \tag{4.6}$$

Example 29 *The third building block is the Plücker conoid $z = y^2/(x^2+y^2)$. In parametric form it can be written as:*

$$\mathbf{r}_3(u,v) = \frac{uv}{u^2+v^2}\left(\frac{v}{u^2+v^2} - v,\ u - \frac{u}{u^2+v^2},\ \frac{u}{v} \right) \qquad \textit{or as} \tag{4.7}$$

$$\mathbf{R}_3(\varphi,\lambda) = (\,0,\ 0,\ \cos 2\varphi + 1\,)/2\ +\ \lambda\,(\,\sin\varphi,\ \cos\varphi,\ 0\,)\,. \tag{4.8}$$

The Plücker conoid has the special property that it arises as an L-minimal ruled surface and at the same time it is an i-Willmore surface carrying a 2-parametric family of i-M-circles.

We shall now see that an arbitrary ruled L-minimal surface is up to motion a convolution of these building blocks.

Theorem 30 (Classification of ruled L-minimal surfaces) *A ruled Laguerre minimal surface distinct from a plane is up to motion a piece of the surface*

$$\mathbf{r}(u,v) = a_1\mathbf{r}_1(u,v) + a_2\mathbf{r}_2(u,v) + a_3R^\theta\mathbf{r}_3(u,v) \tag{4.9}$$

for some $a_1, a_2, a_3, \theta \in \mathbb{R}$. (Here R^θ is the rotation through the angle θ around the z-axis.)

Proof. [of Theorem 30] Let Φ be a ruled L-minimal surface. By Corollary 20 it follows that all the lines lying in Φ are parallel to one plane. Choose a coordinate system so that the plane is Oxy.

Consider the corresponding surface Φ^i in the isotropic model. By Theorem 3 it follows that (a piece of) the surface Φ^i is a graph of a function F biharmonic in a region $U \subset \mathbb{R}^2$. Since the surface Φ carries a family of lines parallel to the plane Oxy it follows that the surface Φ^i carries a family of i-M-circles of the form (2.4) with $m_3 = n_3$. Thus the restriction of the function F to the intersection of the region U with each line $y = tx$, where t runs through a segment $J \subset \mathbb{R}$, is a quadratic function $m_3(t)(x^2+y^2-1)-m_1(t)x-m_2(t)y$.

By Proposition 26 it follows that formula (4.2) holds. In this formula $a_1 + a_3 = b_1 + c_1 = b_2 + c_2 = b_3 + c_3 = 0$ because the restriction of the function F to the lines $y = tx$ has special form $m_3(t)(x^2 + y^2 - 1) - m_1(t)x - m_2(t)y$.

Let us simplify expression (4.2) by appropriate motions of $\mathbb{R}^3$ (corresponding to i-M-transformations of the isotropic model, see Table 2). First perform an appropriate rotation of $\mathbb{R}^3$ around the z-axis to achieve $a_4 = 0$ in formula (4.2) and appropriate translations along the x- and y-axes to achieve $d_1 = d_2 = 0$. Bringing to the diagonal form one gets $c_1x^2 + c_2xy + c_3y^2 = a\,(x\sin\theta + y\cos\theta)^2 + c\left(x^2 + y^2\right)$ for some numbers $a, \theta, c \in \mathbb{R}$. Perform the translation by vector $(0, 0, -c)$ along the z-axis.

After all the above motions the function (26) becomes a linear combination in the first, third and fourth functions in the left column of Table 3. By Proposition 2 transformation (2.2) takes the functions in the left column of Table 3 to the surfaces in the right column. Since the transformation (2.2) is linear in F the theorem follows. □

Proof of Theorem 1 By Theorem 30 a ruled L-minimal surface distinct from a plane can be parametrized via (4.9). Let us show that the surface can be also parametrized via

$$\mathbf{R}(\varphi, \lambda) = a_1\mathbf{R}_1(\varphi, \lambda) + a_2\mathbf{R}_2(\varphi, \lambda) + a_3R^\theta\mathbf{R}_3(\varphi, \lambda) \tag{4.10}$$

with the same $a_1, a_2, a_3, \theta \in \mathbb{R}$ unless $a_1 = a_3 = 0$.

Indeed, consider three parallel lines L_1, L_2, L_3 such that L_1 and L_3 are contained in the surfaces $\mathbf{r}_1(u, v)$ and $R^\theta\mathbf{r}_3(u, v)$, respectively, and L_2 is tangent to the cycloid $\mathbf{r}_2(u, v)$. Take their convolution $L = a_1L_1 \oplus a_2L_2 \oplus a_3L_3$ as Legendre surfaces. The surface $\mathbf{r}(u, v)$ is the convolution of the surfaces $\mathbf{r}_1(u, v)$, $\mathbf{r}_2(u, v)$, and $R^\theta\mathbf{r}_3(u, v)$. Thus L is a line lying in the surface $\mathbf{r}(u, v)$ unless $a_1 = a_3 = 0$. This shows that the latter surface can be parametrized via (4.10).

It remains to notice that up to motion formulas (4.10) and (1.2) define the same class of surfaces, unless $C = D = 0$. For $C = D = 0$ we get a plane. □

A Laguerre minimal surface enveloped by an elliptic family of cones can be obtained from Examples 27–29 by performing Laguerre transformations and taking convolution surfaces. Notice that the converse is not true: the convolution operation preserves the class of Laguerre minimal surfaces but *not* the class of surfaces enveloped by a family of cones. Recall that $\tilde{\mathbf{r}}(u, v)$ is the surface obtained from a surface $\mathbf{r}(u, v)$ by the Laguerre transformation Λ, see §2.2 for the details.

Corollary 31 (Classification Elliptic Type) *A Laguerre minimal surface enveloped by an elliptic family of cones is Laguerre equivalent to a piece of the surface*

$$\mathbf{r}(u, v) = a_1\mathbf{r}_1(u, v) + a_2\mathbf{r}_2(u, v) + a_3R^{\theta_1}\mathbf{r}_3(u, v) + a_4\tilde{\mathbf{r}}_1(u, v) + a_5R^{\theta_2}\tilde{\mathbf{r}}_3(u, v) \tag{4.11}$$

for some $a_1, a_2, a_3, a_4, a_5, \theta_1, \theta_2 \in \mathbb{R}$.

Proof. [of Corollary 31] Let Φ be a Laguerre minimal surface enveloped by an elliptic family of cones. Then the surface Φ^i carries a family of i-M-circles such that the top view of the family is an elliptic pencil. By Theorem 25 it follows that the surface Φ^i is i-M-equivalent to surface (4.1).

The right-hand side of formula (4.1) is a linear combination in the expressions in the left column of Table 3. Performing an i-M-transformation $z \mapsto z + a(x^2 + y^2) + bx + cy + d$ one can eliminate the last expression from the linear combination. By Proposition 2 and Table 2 transformation (2.2) takes the functions in the left column of Table 3 to the surfaces in the right column. Since the transformation (2.2) is linear in F the result follows. □

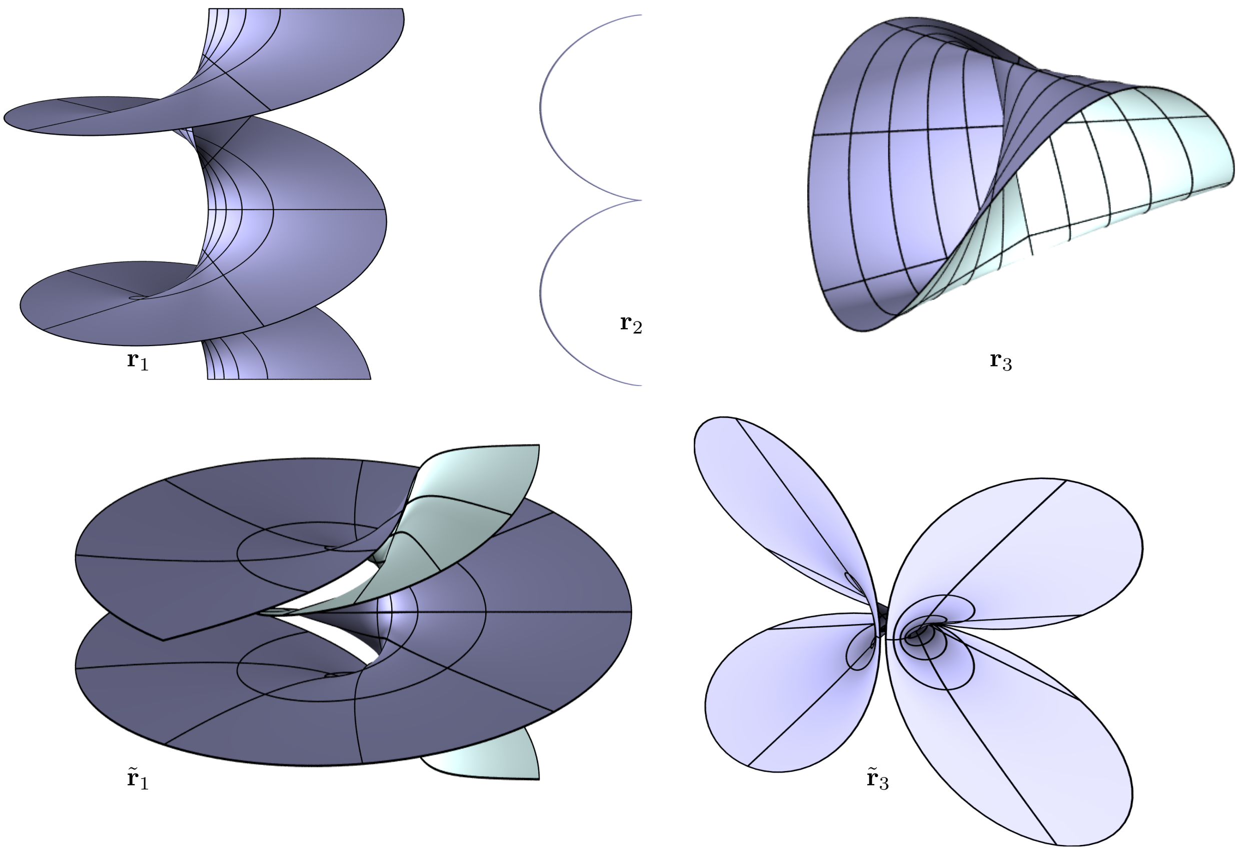

Figure 4: Building blocks for L-minimal surfaces enveloped by an elliptic family of cones. Starting from the top left we show the surfaces $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, $\tilde{\mathbf{r}}_3$ and $\tilde{\mathbf{r}}_1$ in clockwise direction. For details refer to Examples 27, 28, 29 and Section 2.2. Note that the cycloid $\mathbf{r}_2$ lies in a plane orthogonal to the z-axis.

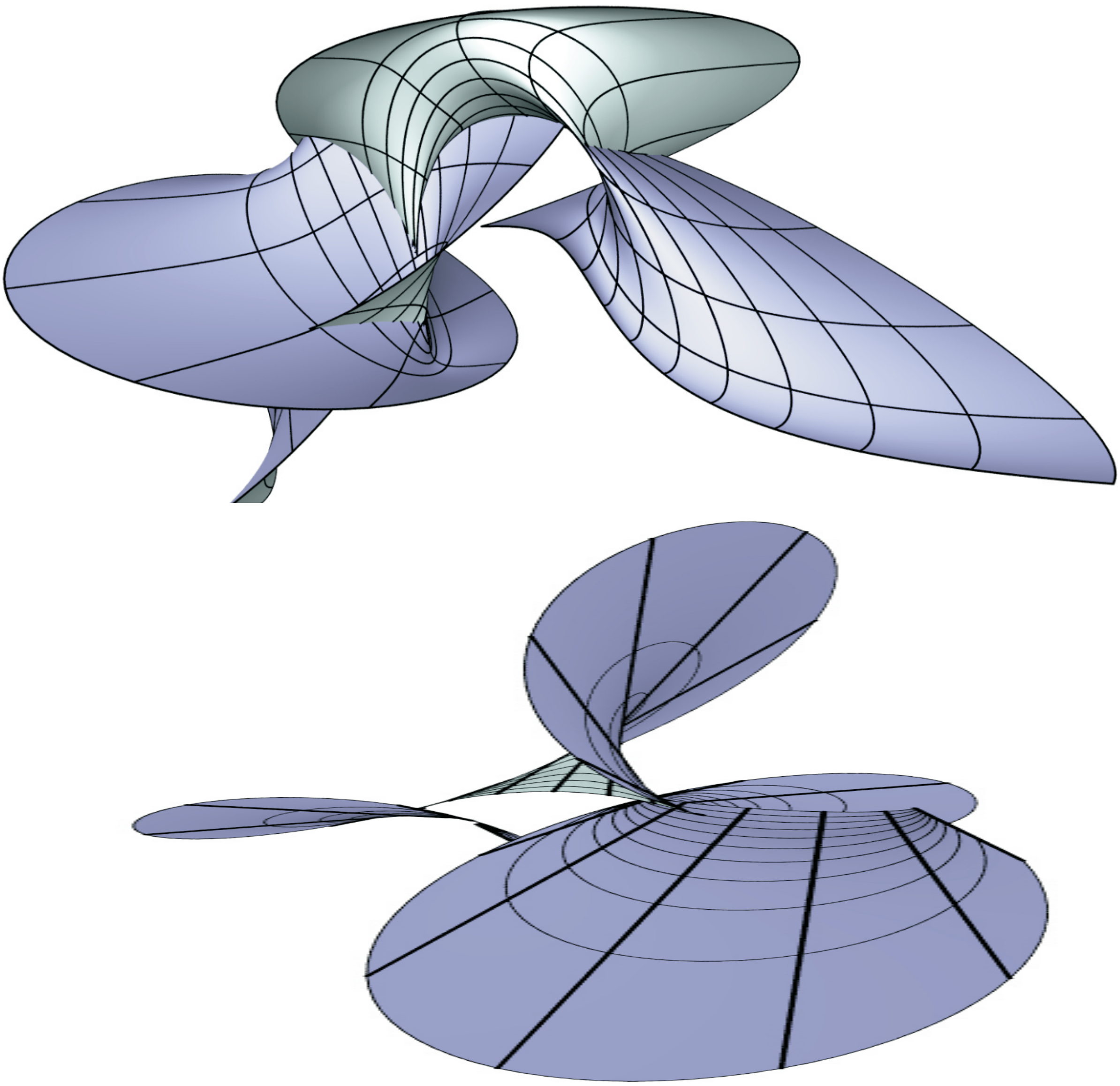

Figure 5: (Top) A general L-minimal surface enveloped by a hyperbolic family of cones. For more information refer to Definition 34 and Theorem 39. (Bottom) A general ruled L-minimal surface is a convolution surface of the surfaces $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$. The rulings are depicted in black. For more information refer to Theorem 30.

Description of the families of cones. Let describe the families of cones which make up the L-minimal surfaces in question. We view a cone as a linear family of oriented spheres. If we map a sphere with midpoint (m_1, m_2, m_3) and (signed) radius R to the point $(m_1, m_2, m_3, R) \in \mathbb{R}^4$, we get a correspondence between cones in $\mathbb{R}^3$ and lines in $\mathbb{R}^4$. Surfaces enveloped by a family of cones can be regarded as ruled 2-surfaces in $\mathbb{R}^4$. Laguerre transformations of $\mathbb{R}^3$ correspond to Lorentz transformations under this mapping. This is known as the *cyclographic model* of Laguerre geometry, see [20] for more information. We will refer to the ruled 2-surface in $\mathbb{R}^4$ corresponding to a surface enveloped by a family of cones as the *cyclographic preimage.*

Example 32 *The cyclographic preimage of the surface $\tilde{\mathbf{r}}_1(u,v)$ can be parametrized as*

$$\tilde{\mathbf{R}}_1(\varphi,\lambda) = (0,\, 0,\, -3\varphi,\, \varphi)/2\sqrt{2} \,+\, \lambda\,(\sin\varphi,\, \cos\varphi,\, 0,\, 0)\,. \tag{4.12}$$

The cyclographic preimage of the surface $\tilde{\mathbf{r}}_3(u,v)$ can be parametrized as

$$\tilde{\mathbf{R}}_3(\varphi,\lambda) = \big(0,\, 0,\, -\cos^2\varphi, 3\cos^2\varphi\big)/4\sqrt{2} \,+\, \lambda\,(\sin\varphi,\, \cos\varphi,\, 0,\, 0)\,. \tag{4.13}$$

Theorem 33 *The cyclographic preimage of a Laguerre minimal surface enveloped by an elliptic family of cones is up to Lorentz transformations a piece of the surface*

$$\mathbf{R}(\varphi,\lambda) = (\,A\varphi,\, B\varphi,\, C\varphi + D\cos 2\varphi,\, E\varphi + F\cos 2\varphi + G\sin 2\varphi\,) + \lambda\,(\,\sin\varphi,\, \cos\varphi,\, 0,\, 0\,) \tag{4.14}$$

for some $A, B, C, D, E, F, G \in \mathbb{R}$.

In other words, an L-minimal surface enveloped by an elliptic family of cones can be interpreted as a frequency 1 rotation of a line in a plane, plus a constant-speed translation and a frequency 2 "harmonic oscillation"; this time in $\mathbb{R}^4$.

Proof. Notice that the cyclographic preimage of a ruled surface in $\mathbb{R}^3$ is the surface itself, if $\mathbb{R}^3$ is identified with subspace $R = 0$ of $\mathbb{R}^4$. Analogously to the proof of Theorem 1 one can show that the cyclographic preimage of surface (4.11) can be parametrized via

$$\mathbf{R}(\varphi,\lambda) = a_1\mathbf{R}_1(\varphi,\lambda) + a_2\mathbf{R}_2(\varphi,\lambda) + a_3R^{\theta_1}\mathbf{R}_3(\varphi,\lambda) + a_4\tilde{\mathbf{R}}_1(\varphi,\lambda) + a_5R^{\theta_2}\tilde{\mathbf{R}}_3(\varphi,\lambda), \tag{4.15}$$

with the same $a_1, a_2, a_3, a_4, a_5, \theta_1, \theta_2 \in \mathbb{R}$. Such a parametrization gives the same class of surfaces as the required parametrization (4.14). □

4.2 Hyperbolic families of cones

Here we consider the second kind of L-minimal surfaces enveloped by a family of cones.

Definition 34 *A* hyperbolic *pencil of circles in the plane (or in a sphere) is the set of all the circles orthogonal to two given crossing circles. A 1-parametric family of cones (possibly degenerating to cylinders or lines) in space is* hyperbolic *if the Gaussian spherical images of the cones form a hyperbolic pencil of circles in the unit sphere.*

Theorem 35 *Let Φ^i be an i-Willmore surface carrying a smooth family of i-M-circles. Suppose that the top view of the family is a hyperbolic pencil of circles. Then the surface Φ^i is i-M-equivalent to a piece of the surface*

$$z = \big(a_1(x^2+y^2) + a_2x + a_3\big)\ln(x^2+y^2) + \frac{b_1y + b_2x}{x^2+y^2} + (c_1y + c_2x)(x^2+y^2) \tag{4.16}$$

for some $a_1, a_2, a_3, b_1, b_2, c_1, c_2 \in \mathbb{R}$.

Table 4: Biharmonic functions whose restrictions to each circle $x^2 + y^2 = t$, $t \in I$, are linear functions and corresponding Laguerre minimal surfaces

Biharmonic function	Laguerre minimal surface
$(x^2+y^2-1)(\ln(x^2+y^2)-2)/2-2$	$\mathbf{r}_4(u,v)$
$(x^2+y^2-2)(\ln(x^2+y^2)-2-\ln 2)/4\sqrt{2}-\sqrt{2}$	$\tilde{\mathbf{r}}_4(u,v)$
$x\ln(x^2+y^2)+x(x^2+y^2-1)$	$\mathbf{r}_5(u,v)$
$(x\cos\theta_1+y\sin\theta_1)(x^2+y^2-2+1/(x^2+y^2))$	$R^{\theta_1}\mathbf{r}_6(u,v)$
$(x\cos\theta_2+y\sin\theta_2)(x^2+y^2-4+4/(x^2+y^2))/4$	$R^{\theta_2}\tilde{\mathbf{r}}_6(u,v)$
$a(x^2+y^2)+bx+cy+d$	oriented sphere

Proof. [of Theorem 35] Perform an i-M-transformation taking the hyperbolic pencil of circles in the top view into the pencil of concentric circles $x^2 + y^2 = t$, where t runs through a segment $J \subset \mathbb{R}$. Denote by $z = F(x, y)$ the surface obtained from the surface Φ^i by the transformation, where F is a biharmonic function defined in a region $U \subset \mathbb{R}^2$. Since an i-M-transformation takes i-M-circles to i-M-circles it follows that the restriction of the function F to (an appropriate arc of) each circle $x^2 + y^2 = t$, where $t \in J$, is a linear function.

Without loss of generality assume that $(0, 0) \notin U$. Consider the polar coordinates in U. Then $F(r, \phi) = a(r)\cos\phi + b(r)\sin\phi + c(r)$. Thus

$$\begin{aligned} r^4\Delta^2 F = r^4F_{rrrr} + 2r^3F_{rrr} - r^2F_{rr} + rF_r + 2r^2F_{rr\phi\phi} - 2rF_{r\phi\phi} + 4F_{\phi\phi} + F_{\phi\phi\phi\phi} = \\ = \left(r^4a^{(4)} + 2r^3a^{(3)} - 3r^2a'' + 3ra' - 3a\right)\cos\phi + \\ + \left(r^4b^{(4)} + 2r^3b^{(3)} - 3r^2b'' + 3rb' - 3b\right)\sin\phi + \\ + \left(r^4c^{(4)} + 2r^3c^{(3)} - r^2c'' + rc'\right). \end{aligned}$$

Since $\Delta^2 F = 0$ it follows that the coefficients of this trigonometric polynomial vanish. Solving the obtained ordinary differential equations we get:

$$\begin{aligned} a(r) &= \alpha_1 r + \alpha_2 r\ln r + \alpha_3/r + \alpha_4 r^3; \\ b(r) &= \beta_1 r + \beta_2 r\ln r + \beta_3/r + \beta_4 r^3; \\ c(r) &= \gamma_1 + \gamma_2 r^2 + \gamma_3\ln r + \gamma_4 r^2\ln r. \end{aligned}$$

One can achieve $\beta_2 = 0$ by an appropriate rotation of the coordinate system around the origin. One can also achieve $\alpha_1 = \beta_1 = \gamma_1 = \gamma_2 = 0$ by the i-M-transformation $z \mapsto z - \gamma_2(x^2 + y^2) - \alpha_1 x - \beta_1 y - \gamma_1$ Returning to the cartesian coordinate system we get the required formula. □

A Laguerre minimal surface enveloped by a hyperbolic family of cones is obtained from the surface (4.16) by transformation (2.2). Let us give some typical examples obtained from the graphs of the functions in the left column of Table 4, see also Figure 6. These examples are "building blocks" forming all the surfaces in question.

Example 36 *The first example is the catenoid. It can be parametrized as*

$$\mathbf{r}_4(u,v) = \left(u + \frac{u}{u^2+v^2},\ v + \frac{v}{u^2+v^2},\ \ln(u^2+v^2)\right). \tag{4.17}$$

Its cyclographic preimage can be written as

$$\mathbf{R}_4(\varphi,\lambda) = (\,0,\ 0,\ -2\varphi,\ -2\,) + \lambda\,(\,0,\ 0,\ \cosh\varphi,\ \sinh\varphi\,). \tag{4.18}$$

The cyclographic preimage of the surface $\tilde{\mathbf{r}}_4$ *can be written as*

$$\tilde{\mathbf{R}}_4(\varphi,\lambda) = -\,(\,0,\,0,\,6\varphi+3\ln 2+14,\,2\varphi+\ln 2-6\,)/4\sqrt{2}\;+\;\lambda\,(\,0,\,0,\,\cosh\varphi,\,\sinh\varphi\,)\,. \tag{4.19}$$

We immediately see that the surface $\tilde{\mathbf{r}}_4(u,v)$ *is a canal surface, a fact that can also be seen by observing that it is a Laguerre transformed catenoid.*

Example 37 *Another building block is given by the surface* $\mathbf{r}_5$ *parametrized by:*

$$\mathbf{r}_5(u,v) = \left(\,(u^2-v^2)\left(1-\frac{1}{u^2+v^2}\right)-\ln(u^2+v^2),\;2uv\left(1-\frac{1}{u^2+v^2}\right),\;4u\,\right). \tag{4.20}$$

Its cyclographic preimage is the surface parametrized by

$$\mathbf{R}_5(\varphi,\lambda) = \big(\,1-e^{-2\varphi}+2\varphi,\,0,\,0,\,0\,\big)\;+\;\lambda\,(\,0,\,0,\,\cosh\varphi,\,\sinh\varphi\,)\,. \tag{4.21}$$

Example 38 *Finally we have the surface* $\mathbf{r}_6$ *given implicitly by* $z^2(1-x+z^2/4)=y^2$*. In parametric form it can be written as:*

$$\mathbf{r}_6(u,v) = \left(\,(u^2-v^2)\left(1-\frac{1}{u^2+v^2}\right)^2,\;2uv\left(1-\frac{1}{u^2+v^2}\right)^2,\;4u\left(1-\frac{1}{u^2+v^2}\right)\right). \tag{4.22}$$

Its cyclographic preimage can be written as

$$\mathbf{R}_6(\varphi,\lambda) = \big(\,2-2\cosh^2\varphi,\,0,\,0,\,0\,\big)+\lambda\,(\,0,\,0,\,\cosh\varphi,\,\sinh\varphi\,)\,. \tag{4.23}$$

The cyclographic preimage of the surface $\tilde{\mathbf{r}}_6$ *is given by*

$$\tilde{\mathbf{R}}_6(\varphi,\lambda) = \big(\,1-e^{2\varphi}-e^{-2\varphi}/4,\,0,\,0,\,0\,\big)+\lambda\,(\,0,\,0,\,\cosh\varphi,\,\sinh\varphi\,)\,. \tag{4.24}$$

A Laguerre minimal surface enveloped by a hyperbolic family of cones can be obtained from Examples 36–38 by performing Laguerre transformations and taking convolution surfaces:

Corollary 39 (Classification Hyperbolic Type) *A Laguerre minimal surface enveloped by a hyperbolic family of cones is Laguerre equivalent to a piece of the surface*

$$\mathbf{r}(u,v) = a_1\mathbf{r}_4(u,v)+a_2\mathbf{r}_5(u,v)+a_3R^{\theta_1}\mathbf{r}_6(u,v)+a_4\tilde{\mathbf{r}}_4(u,v)+a_5R^{\theta_2}\tilde{\mathbf{r}}_6(u,v)$$

for some $a_1,a_2,a_3,a_4,a_5,\theta_1,\theta_2\in\mathbb{R}$*.*

Proof. [of Corollary 39] Let Φ be a Laguerre minimal surface enveloped by a hyperbolic family of cones. Then the surface Φ^i carries a family of i-M-circles such that the top view of the family is a hyperbolic pencil. By Theorem 35 it follows that the surface Φ^i is i-M-equivalent to surface (4.16).

The right-hand side of formula (4.16) is a linear combination in the expressions in the left column of Table 4. Performing an i-M-transformation $z\mapsto z+a(x^2+y^2)+bx+cy+d$ one can eliminate the last expression from the linear combination. By Proposition 2 and Table 2 transformation (2.2) takes the functions in the left column of Table 4 to the surfaces in the right column. Since the transformation (2.2) is linear in F the result follows. □

There is a simple parametrization of the cyclographic preimage (the proof is analogous to the proof of Theorem 33).

Theorem 40 *The cyclographic preimage of a Laguerre minimal surface enveloped by a hyperbolic family of cones is up to Lorentz transformations a piece of the surface*

$$\mathbf{R}(\varphi,\lambda) = (\,A\varphi + B\cosh 2\varphi,\; C\varphi + D\cosh 2\varphi + E\sinh 2\varphi,\; F\varphi,\; G\varphi\,) + \lambda\,(\,0,\;0,\;\cosh\varphi,\;\sinh\varphi\,) \tag{4.25}$$

for some $A, B, C, D, E, F, G \in \mathbb{R}$.

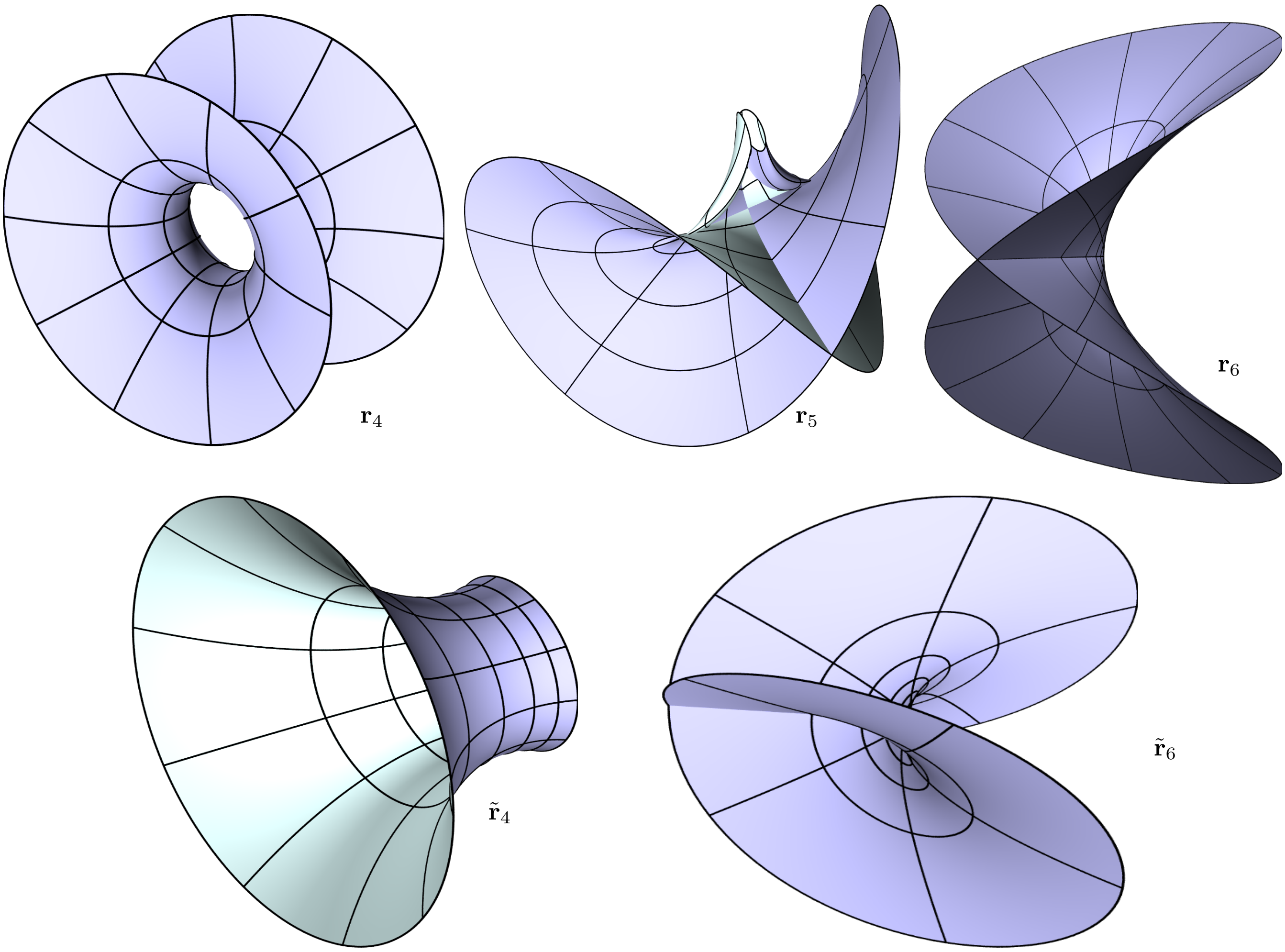

Figure 6: Building blocks for L-minimal surfaces enveloped by a hyperbolic family of cones. Starting from the top left we show the surfaces $\mathbf{r}_4$, $\mathbf{r}_5$, $\mathbf{r}_6$, $\tilde{\mathbf{r}}_6$ and $\tilde{\mathbf{r}}_4$ in clockwise direction. For details refer to Examples 36, 37, 38 and Section 2.2.

4.3 Parabolic families of cones

Definition 41 *A* parabolic *pencil of circles in the plane (or in a sphere) is the set of all the circles touching a given circle at a given point. A* 1*-parametric family of cones (possibly degenerating to cylinders or lines) in space is* parabolic *if the Gaussian spherical images of the cones form a hyperbolic pencil of circles in the unit sphere.*

Theorem 42 *Let Φ^i be an i-Willmore surface carrying a smooth family of i-M-circles. Suppose that the top view of the family is a parabolic pencil of circles. Then the surface Φ^i is i-M-equivalent to a piece of the surface*

$$z = a_1(5y^2 - x^2)x^3 + a_2(3y^2 - x^2)x^2 + (b_1y^2 + b_2xy + b_3x^2)x + c_1y^2 + c_2xy \tag{4.26}$$

for some $a_1, a_2, b_1, b_2, b_3, c_1, c_2 \in \mathbb{R}$.

Proof. [of Theorem 42] Perform an i-M-transformation taking the parabolic pencil of circles in the top view into the pencil of parallel lines $x = t$, where t runs through a segment $J \subset \mathbb{R}$. Denote by $z = F(x, y)$ the surface obtained from the surface Φ^i by the transformation, where F is a biharmonic function defined in a region $U \subset \mathbb{R}^2$. Since an i-M-transformation takes i-M-circles to i-M-circles it follows that the restriction of the function F to (an appropriate segment of) each line $x = t$, where $t \in J$, is a quadratic function.

So $F(x, y) = a(x)y^2 + b(x)y + c(x)$. Thus $\Delta^2 F = a^{(4)}y^2 + b^{(4)}y + c^{(4)} + 4a''$. Since $\Delta^2 F = 0$ it follows that the coefficients of this polynomial in y vanish. Hence

$$\begin{aligned} a(x) &= \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3; \\ b(x) &= \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3; \\ c(x) &= \gamma_0 + \gamma_1 x + \gamma_2 x^2 + \gamma_3 x^3 - \alpha_2 x^4/3 - \alpha_3 x^5/5. \end{aligned}$$

One can achieve $\beta_0 = \gamma_0 = \gamma_1 = \gamma_2 = 0$ by the i-M-transformation $z \mapsto z - \gamma_2(x^2 + y^2) - \gamma_1 x - \beta_0 y - \gamma_0$. We get the required formula. □

Table 5: Biharmonic functions whose restrictions to each line $x = t$, $t \in I$, are quadratic functions and corresponding Laguerre minimal surfaces

Biharmonic function	Laguerre minimal surface
$(x \cos\theta_1 + y \sin\theta_1)^2/2$	$R^{\theta_1}\mathbf{r}_7(u, v)$
x^3	$\mathbf{r}_8(u, v)$
x^2y	$\mathbf{r}_9(u, v)$
xy^2	$R^{\pi/2}\mathbf{r}_9(u, v)$
$x^2(x^2 - 3y^2)/2$	$\mathbf{r}_{10}(u, v)$
$x^3(x^2 - 5y^2)$	$\mathbf{r}_{11}(u, v)$
$a(x^2 + y^2) + bx + cy + d$	oriented sphere

A Laguerre minimal surface enveloped by a parabolic family of cones is obtained from the surface (4.26) by transformation (2.2). Let us give some typical examples obtained from the graphs of the functions in the left column of Table 5, see also Figure 7. These examples are "building blocks" forming all the surfaces in question.

Example 43 *The first example is the parabolic horn cyclide $\left(y^2 + z^2\right)(1 - z) = x^2 z$. In parametric form it can be written as:*

$$\mathbf{r}_7(u, v) = \frac{1}{1 + u^2 + v^2}\left(-u - uv^2,\, u^2v,\, u^2\right). \tag{4.27}$$

The cyclographic preimage of $R^\theta\mathbf{r}_7$ can be parametrized as

$$R^\theta\mathbf{R}_7(\varphi, \lambda) = \left(0,\, -\varphi\sin 2\theta,\, -\varphi^2\cos 2\theta + \sin^2\theta,\, \varphi^2\cos 2\theta + \sin^2\theta\right)/2 \,+\, \lambda\,(1,\, 0,\, -\varphi,\, \varphi) \tag{4.28}$$

The surface $\mathbf{r}_7$ has the following property: there is a 2-parametric family of cones touching the cyclide along certain curves. In particular, there are both parabolic and elliptic 1-parametric families of cones touching the surface along curves. One of the elliptic families of cones can be parametrized as

$$\left(0,\, 0,\, \cos^2\varphi,\, \cos^2\varphi\right)/2 \,+\, \lambda\,(\sin\varphi,\, \cos\varphi,\, 0,\, 0) \tag{4.29}$$

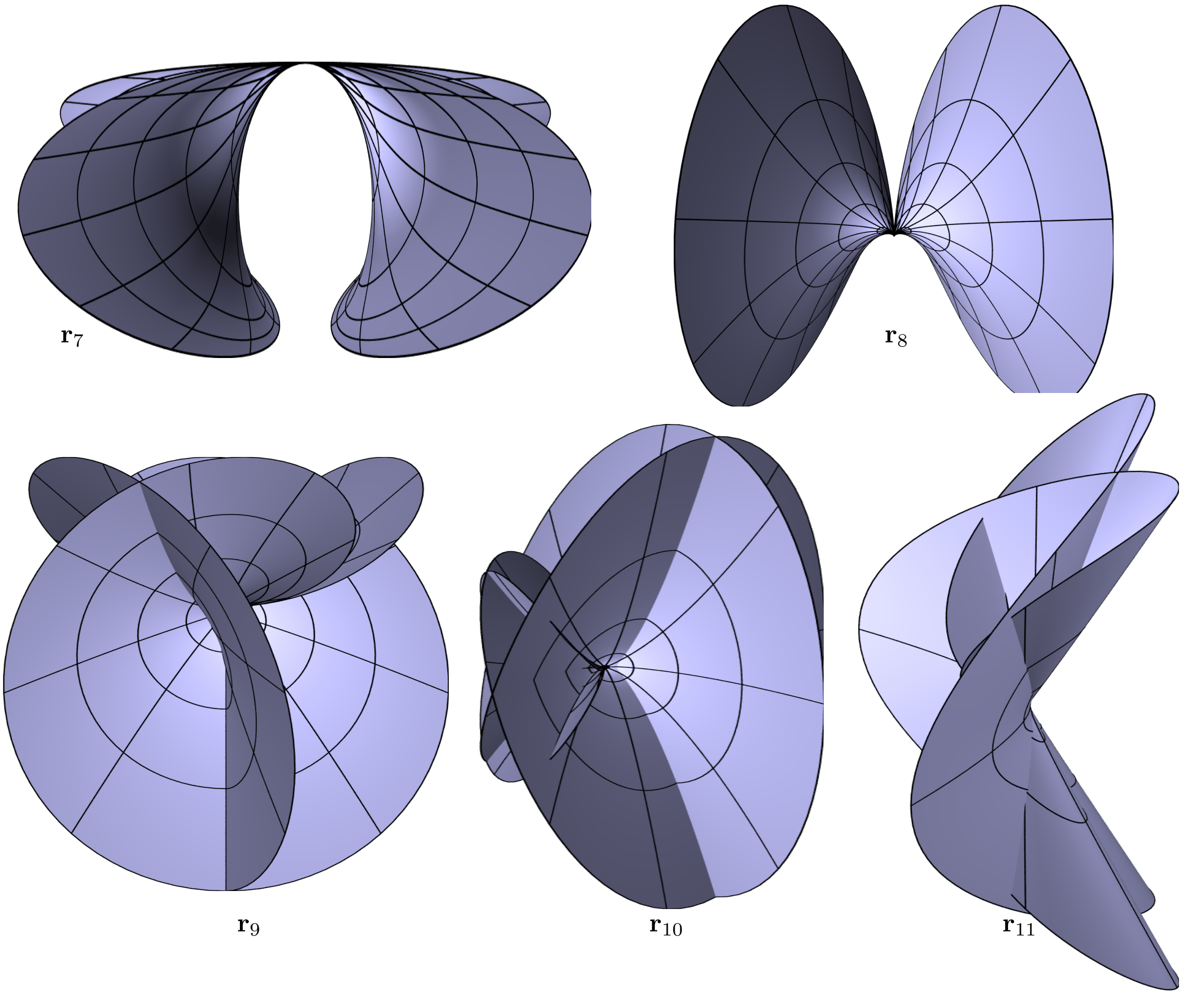

Figure 7: Building blocks for L-minimal surfaces enveloped by a parabolic family of cones. Starting from the top left we show the surfaces $\mathbf{r}_7$, $\mathbf{r}_8$, $\mathbf{r}_{11}$, $\mathbf{r}_{10}$ and $\mathbf{r}_9$ in clockwise direction. For details refer to Examples 43, 44, 45, 46, 47.

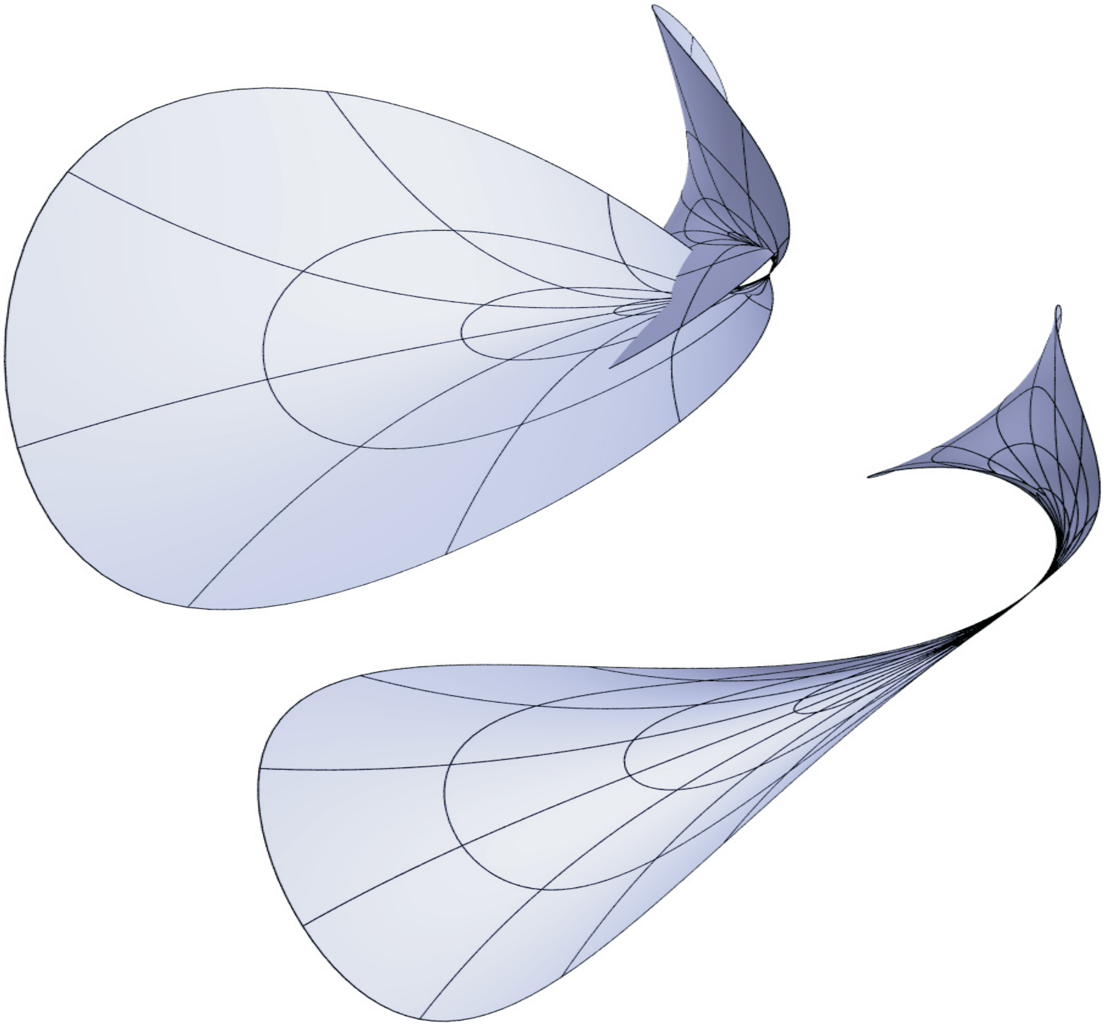

Figure 8: Two general L-minimal surfaces enveloped by parabolic families of cones. For details refer to Definition 41 and Theorem 48.

Example 44 *The next building block is given by the algebraic surface of degree 6 with implicit equation*

$$(x^2+y^2+z^2)z^4 - 2(8x^2+9y^2+9z^2)xz^2 - 27(y^2+z^2)^2 = 0.$$

In parametric form it can be written as:

$$\mathbf{r}_8(u,v) = \frac{1}{1+u^2+v^2}\left(u^4 - 3u^2v^2 - 3u^2,\, 4u^3v,\, 4u^3\right). \tag{4.30}$$

Its cyclographic preimage can be parametrized as

$$\mathbf{R}_8(\varphi,\lambda) = \left(0,\, 0,\, -\varphi^3,\, \varphi^3\right) + \lambda\,(1,\, 0,\, -\varphi,\, \varphi\,) \tag{4.31}$$

Example 45 *Another example is the algebraic surface of degree 8, given implicitly by:*

$$z^2(y^2+z^2)\left(z^2-4y-4\right)^2 + x^2\left(64y^3 - 24(y-3)yz^2 - 6(y+6)z^4 + z^6\right) - 27x^4z^2 = 0.$$

In parametric form:

$$\mathbf{r}_9(u,v) = \frac{1}{1+u^2+v^2}\left(2uv(u^2-v^2-1),\, u^2(3v^2-u^2-1),\, 4u^2v\right). \tag{4.32}$$

Its cyclographic preimage can be parametrized as

$$\mathbf{R}_9(\varphi,\lambda) = \left(0,\, -\varphi^2,\, 0,\, 0\right) + \lambda\,(1,\, 0,\, -\varphi,\, \varphi\,) \tag{4.33}$$

This surface has the following property: there are two 1-parametric families of cones touching the surface along certain curves. The other family can be parametrized as

$$\left(0,\, 0,\, \varphi-\varphi^3,\, \varphi+\varphi^3\right) + \lambda\,(0,\, 1,\, -\varphi,\, \varphi\,). \tag{4.34}$$

Thus we can write the cyclographic preimage of the surface $R^{\pi/2}\mathbf{r}_9$ as

$$R^{\pi/2}\mathbf{R}_9(\varphi,\lambda) = \left(0,\, 0,\, \varphi-\varphi^3,\, \varphi+\varphi^3\right) + \lambda\,(1,\, 0,\, -\varphi,\, \varphi\,). \tag{4.35}$$

Finally, we have the following two "monsters". We do not write their implicit equations because this would take a few pages.

Example 46 *First the algebraic surface of degree not greater than 14 described by*

$$\mathbf{r}_{10}(u,v) = \frac{1}{1+u^2+v^2}\begin{pmatrix} u^5 - 2u^3(1+4v^2) + 3u(v^2+v^4) \\ 3u^2v(1+2u^2-2v^2) \\ 3u^2(u^2-3v^2) \end{pmatrix}. \tag{4.36}$$

Its cyclographic preimage can be parametrized as

$$\mathbf{R}_{10}(\varphi,\lambda) = \left(0,\, 0,\, -3\varphi^2-4\varphi^4,\, -3\varphi^2+4\varphi^4\right)/2 \,+\, \lambda\,(1,\, 0,\, -\varphi,\, \varphi\,). \tag{4.37}$$

Example 47 *The second monster is the algebraic surface of degree not greater than* 18 *with parametrization*

$$\mathbf{r}_{11}(u,v) = \frac{1}{1+u^2+v^2}\begin{pmatrix} 3u^6 - 5u^4(1+6v^2) + 15u^2(v^2+v^4) \\ 2u^3v(5+9u^2-15v^2) \\ 8u^3(u^2-5v^2) \end{pmatrix}. \tag{4.38}$$

Its cyclographic preimage can be parametrized as

$$\mathbf{R}_{11}(\varphi,\lambda) = \left(0,\, 0,\, -5\varphi^3 - 6\varphi^5,\, -5\varphi^3 + 6\varphi^5\right) + \lambda\,(1,\, 0,\, -\varphi,\, \varphi\,). \tag{4.39}$$

A Laguerre minimal surface enveloped by a parabolic family of cones can be obtained from Examples 43–47 by performing rotations and taking convolution surfaces:

Corollary 48 (Classification Parabolic Type) *A Laguerre minimal surface enveloped by a parabolic family of cones is Laguerre equivalent to a piece of the surface*

$$\mathbf{r}(u,v) = a_1 R^{\theta}\mathbf{r}_7(u,v) + a_2\mathbf{r}_8(u,v) + (a_3 + a_4 R^{\pi/2})\mathbf{r}_9(u,v) + a_5\mathbf{r}_{10}(u,v) + a_6\mathbf{r}_{11}(u,v)$$

for some $a_1, a_2, a_3, a_4, a_5, a_6, \theta \in \mathbb{R}$.

Proof. [of Corollary 48] Let Φ be a Laguerre minimal surface enveloped by a parabolic family of cones. Then the surface Φ^i carries a family of i-M-circles such that the top view of the family is a parabolic pencil. By Theorem 42 it follows that the surface Φ^i is i-M-equivalent to surface (4.26).

Right-hand side of formula (4.26) is a linear combination in the expressions in the left column of Table 5. Performing an i-M-transformation $z \mapsto z + a(x^2+y^2) + bx + cy + d$ one can eliminate the last expression from the linear combination. By Propositions 2 and Table 2 transformation (2.2) takes the functions in the left column of Table 5 to the surfaces in the right column. Since the transformation (2.2) is linear in F the result follows. □

Finally we describe the cyclographic preimage (the proof is analogous to the proof of Theorem 33).

Theorem 49 *The cyclographic preimage of a Laguerre minimal surface enveloped by a parabolic family of cones is up to Lorentz transformations a piece of the surface*

$$\mathbf{R}(\varphi,\lambda) = \begin{pmatrix} 0 \\ A\varphi + B\varphi^2 \\ C\varphi + D\varphi^2 + E\varphi^3 + F(3\varphi^2 + 4\varphi^4) + G(5\varphi^3 + 6\varphi^5) \\ C\varphi - D\varphi^2 - E\varphi^3 + F(3\varphi^2 - 4\varphi^4) + G(5\varphi^3 - 6\varphi^5) \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ -\varphi \\ \varphi \end{pmatrix} \tag{4.40}$$

for some $A, B, C, D, E, F, G \in \mathbb{R}$.

4.4 Open problems

Conjecture 50 *A surface such that there is a 2-parametric family of cones of revolution touching the surface along certain curves distinct from directrices is either a sphere or a parabolic cyclide.*

Problem 51 *Describe all surfaces such that there are two* 1*-parametric families of cones of revolution touching the surface along curves.*

Problem 52 *Describe all Willmore surfaces such that there is a* 1*-parametric family of circles lying in the surface.*

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