UE Discrete Mathematics

Exercises for Oct 10, 2023

1) Use a suitable graph theoretical model to solve the following problems:

(a) Show that in every city at least two of its inhabitants have the same number of neighbours!

(b) 11 friends want to send postcards according to the following rules: (i) Each person sends and receives exactly 3 cards. (ii) Each one receives only cards from those to whom he or she sent a card and *vice versa*.

Tell how this can be done or prove that this is impossible!

(c) Determine all graphs in which all vertices have degree 1.

2) Let M be a finite set with n elements. The graph G is defined by

$$V(G) = \mathcal{P}(M) = \{U \mid U \subseteq M\},$$
$$E(G) = \{(U, V) \mid U, V \subseteq M,\ U \neq V,\ U \cap V = \emptyset\}.$$

Determine $\alpha_0(G)$ and $\alpha_1(G)$!

3) Prove that the edge set of an undirected simple graph can be partitioned into cycles if, and only if, every vertex has even degree.

Hint: To prove the existence of a cycle, consider a maximal path and use the even degree condition, i.e. the fact that all vertices have even degree.

4) Let $G = (V, E)$ and $G' = (V', E')$ be two undirected graphs. A graph isomorphism is a bijective mapping $\phi : V \to V'$ such that two vertices $x, y \in V$ are adjacent if and only if $\phi(x)$ and $\phi(y)$ are adjacent. The two graphs G and G' are called isomorphic, if there exists an isomorphism $\phi : V \to V'$.
Prove the following statements: If $G = (V, E)$ and $G' = (V', E')$ are isomorphic graphs and $\phi : V \to V'$ is an isomorphism, then $d_G(x) = d_{G'}(\phi(x))$ for all $x \in V$.
If, on the other hand, $\phi : V \to V'$ is a bijective mapping satisfying $d_G(x) = d_{G'}(\phi(x))$ for all $x \in V$, then G and G' are not necessarily isomorphic.

5) Are the following two graphs isomorphic?