UE Discrete Mathematics

Exercises for Jan 16, 2024

121) Let $I = \{(x^2+1) \cdot p(x) \mid p(x) \in \mathbb{R}[x]\}$. Prove that I is an ideal of $\mathbb{R}[x]$ and compute $((2x+3)+I) \cdot ((x-2)+I)$ in $\mathbb{R}[x]/I$.

122) From Exercise 121 we know that $I = \{(x^2+1) \cdot p(x) \mid p(x) \in \mathbb{R}[x]\}$ is an ideal of $\mathbb{R}[x]$. Prove that $x^2 + I = -1 + I$ holds and that $\mathbb{R}[x]/I \cong \mathbb{C}$.

123) Show that $(\mathbb{Z}[x], +, \cdot)$ is a ring and that $1 \notin (\{x, x+2\})$.

Remark: It can be shown that a principal ideal which is generated by $a_1, a_2, \ldots, a_k$ can be alternatively generated by $\gcd(a_1, a_2, \ldots, a_k)$. Therefore this example shows that $\mathbb{Z}[x]$ is a ring where not every ideal is a principal ideal. As a consequence, $\mathbb{Z}[x]$ cannot be a Euklidean ring.

124) Show that the set $R = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$ with the usual addition and multiplication is an integral domain but not a field. Furthermore, prove that there are infinitely many units in R and give three concrete examples.

125) Let R be a commutative ring with $R \neq \{0\}$ that has only the trivial ideals $\{0\}$ and R. Prove the following assertions:

(a) The set $I = \{x \in R \mid (x) = \{0\}\}$ is an ideal of R.

(b) If $I = R$ then $xy = 0$ for all $x, y \in R$.

126) Consider the ring R from Exercise 125 and its ideal $I = \{x \in R \mid (x) = \{0\}\}$. Prove the following assertions under the assumption that $I \neq R$:

(a) If $a \neq 0$ then $(a) = R$. This implies that there is a $b \in R$ with $ab = a$.

(b) R is a field.

Remarks and hints: Note that what you show in (a) does not yet imply that $b = 1$! To show that b is indeed the multiplicative identity, consider an arbitrary $c \in R$ and the expression $a(bc - c)$. Having done this, in (b) you only have to show that a^{-1} exists.

127) Let $(L, +, \cdot)$ be a field and K be a subfield of L. Show that $(L, +)$ together with the scalar multiplication $\odot : K \times L \to K$, $(x, z) \mapsto x \odot z := x \cdot z$ is a vector space over the scalar field K.

128) Show that the set $\{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ with the usual addition and multiplication is a field. Compute $(3 - 5\sqrt{2})^{-1}$.

129) Determine all elements of the ring $R = \mathbb{Z}_3[x]/_{(x^2+x+1)}$ as well as its table for the multiplication. Is this ring a field?

130) Examine whether $x^2 + 1$ is a unit of $\mathbb{Q}[x]/_{(x^3+x^2-3x+1)}$. If so, determine its multiplicative inverse!