UE Discrete Mathematics

Exercises for Oct 17, 2023

11) Let $G = (V, E)$ be an undirected graph with n vertices which does not have any cycle of length 3. Prove:

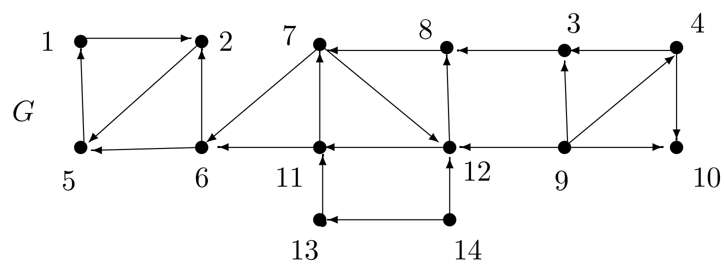
1. If $xy \in E$ then $d(x) + d(y) \leq n$.
2. The previous inequality $d(x) + d(y) \leq n$ implies that $\sum_{v \in V} d(v)^2 \leq n|E|$.
3. The graph has at most $n^2/4$ edges. Hint: Use the hand-shaking lemma, the Cauchy-Schwarz inequality $\left(\sum_{i=1}^{r} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{r} a_i^2\right)\left(\sum_{i=1}^{r} b_i^2\right)$, and what you have proved so far.

12) Let $G = (V, E)$ be a simple graph. Moreover, let G_R its reduction. Prove that G_R is acyclic!

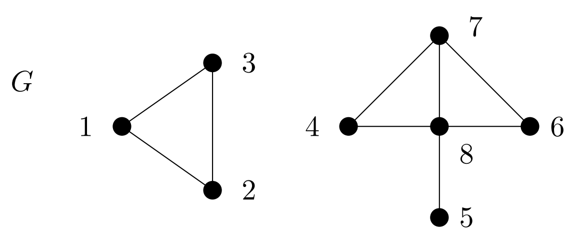
13) Let $G = (V, E)$ be a simple, directed, and acyclic graph. Prove that $B = \{v \in V \mid d^-(v) = 0\}$ is a vertex basis of G. Furthermore, prove that B is the only vertex basis of G.

14) Let $G = (V, E)$ be a simple, directed graph, G_R be its reduction and $K_1, \ldots, K_\ell$ be those strongly connected components which satisfy $d^-_{G_R}(K_i) = 0, i = 1, \ldots, \ell$. Prove that $B \subset V$ is a vertex basis of G if and only if $|B \cap K_i| = 1$ for $i = 1, \ldots, \ell$, and $|B| = \ell$.

15) Find the strongly connected components and the reduction G_R of the graph G below. Furthermore, determine all vertex bases of G.



16) Use the matrix tree theorem to compute the number of spanning forests of the graph below!



17) K_n denotes the complete graph with n vertices. Show that the number of spanning trees of K_n is n^{n-2}!

Hint: Use the matrix tree theorem and delete the first column and the first row of $D(K_n) - A(K_n)$. Then add all rows (except the first) to the first one and observe that all entries of the new first row are equal to 1. Use the new first row to transform the matrix in such a way that the submatrix built of the second to the last row and second to the last column is diagonal matrix.

18) If T is a tree having no vertex of degree 2, then T has more leaves than internal nodes. Prove this claim

(a) by induction,

(b) by considering the average degree and using the handshaking lemma.

19) List all matroids (E, S) with $E = \{1\}$, $E = \{1, 2\}$ or $E = \{1, 2, 3\}$.

20) Let $M = (E, S)$ be a matroid and $M_0 = (E_0, S_0)$ where $E_0 \subseteq E$ and $S_0 = \{X \cap E_0 \mid X \in S\}$. Prove that M_0 is a matroid.