UE Discrete Mathematics

Exercises for Oct 24, 2023

21) Let E be a set, $1 \leq k \leq |E|$ an integer, and let S denote the set of all subsets $X \subseteq E$ with cardinality at most k. Examine whether (E, S) is a matroid.

22) Prove that an independence system (E, S) is a matroid if and only if for every $A \subseteq E$, all maximal independent subsets of A have the same cardinality.

23) Let E_1 and E_2 be two disjoint sets. Moreover, assume that (E_1, S_1) and (E_2, S_2) are matroids. Define $S := \{X \cup Y \mid X \in S_1 \text{ and } Y \in S_2\}$. Prove that $(E_1 \cup E_2, S)$ is a matroid.

24) Let (E, S_1) and (E, S_2) be matroids. Show that $(E, S_1 \cap S_2)$ is in general not a matroid.

25–26) Given a matroid (E, S) and $A \subseteq E$. We define $r(A) := \max\{|C| \mid C \subseteq A \text{ and } C \in S\}$. Now let $B \subseteq E$ and define $\overline{B} := \{a \in E \mid r(B) = r(B \cup \{a\})\}$.

25) Express the above definitions in your own words. Prove the following statements:

(a) For all $M \subseteq E$ we have $M \subseteq \overline{M}$.

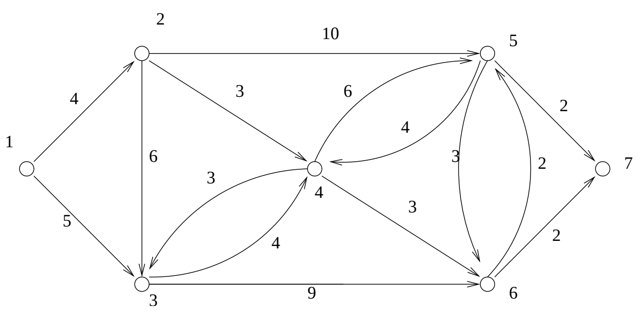
(b) For all $M \subseteq E$ and all $e, f \in E$ the equations $r(M) = r(M \cup \{e\})$ and $r(M) = r(M \cup \{f\})$ together imply $r(M) = r(M \cup \{e, f\})$

26) Prove the following statements:

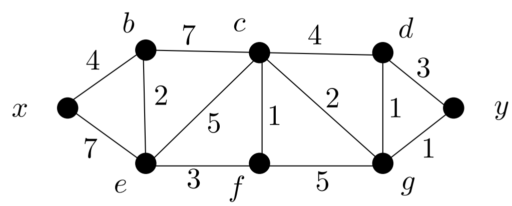
(a) Any two subsets C and D of E satisfy $r(C) + r(D) \geq r(C \cup D) + r(C \cap D)$.

(b) Let $M \subseteq N \subseteq E$ and $a \in \overline{M}$. Then $r(M) + r(N) \geq r(M) + r(N \cup \{a\})$. Use this inequality to show $\overline{M} \subseteq \overline{N}$.

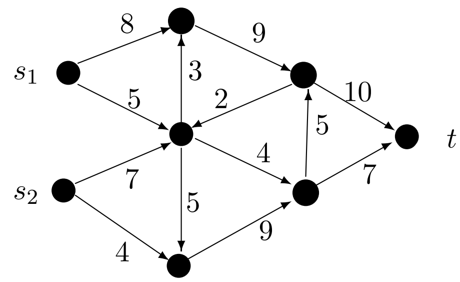
27) Use Dijkstra's algorithm to determine $d(1, 7)$ in the following graph.



28) Use Moore's algorithm to compute all distances to vertex x in the graph following graph.



29) Use the algorithm of Ford and Fulkerson to compute a maximal flow in the following network which hastwo sources s_1 and s_2!



30) Assume that we have given the graph G_1 above and that its vertices have weights (=bounded capacity), too. Is it still possible to apply the algorithm of Ford and Fulkerson to determine a maximal flow?