UE Discrete Mathematics

Exercises for Oct 31, 2023

31) Find an Eulerian graph with an even/odd number of vertices and an even/odd number of edges or prove that there is no such graph (for each of the four cases).

32) Let G_n denote the n-dimensional hypercube. Show that G_n is Hamiltonian if $n \geq 2$.

33) Prove that a graph G is bipartite if and only if each cycle in G has even length.

34) For which m and n is the complete bipartite graph $K_{m,n}$ an Eulerian graph?

35) Let G be an Eulerian graph and H be a subdivision of G. Is H Eulerian? Suppose that G is Hamiltonian. Does this imply that H is Hamiltonian as well?

36) For which m and n does the complete bipartite graph $K_{m,n}$ have a Hamiltonian cycle?

37)

a) Prove that every simple, connected and planar graph with at least 3 vertices satisfies $\alpha_1(G) \leq 3\alpha_0(G) - 6$. Show that this implies that K_5 is not planar.

b) Prove the following statement or find a counter-example: For all n, m, f with $n - m + f = 2$ and for which there exists a simple graph with $\alpha_0 = n$, $\alpha_1 = m$ there exists also a simple planar graph with $\alpha_0 = n$, $\alpha_1 = m$ and $\alpha_2 = f$.

38) Let G and H be conected graphs with no vertices in commen and $x \in V(G)$, $y \in V(H)$. Define the graph $K = (V, E)$ by $V = V(G) \cup V(H)$ and $E = E(G) \cup E(H) \cup \{xy\}$.

a) Prove the following statement or find a counter-example! If G and H are Eulerian, then K has an Eulerian trail, but is not Eulerian itself.

b) Prove the following statement or find a counter-example! If K has an Eulerian trail, then G and H are Eulerian.

c) Choose a concrete example with $\alpha_0(G) = \alpha_0(H) = 7$ and $\alpha_1(G) = \alpha_1(H) = 21$ and find an Eulerian trail for K, using your proof of a) or b).

39) Prove that a connected planar graph always has a vertex v with $d(v) \leq 5$.

40) Let G be a connected planar graph. Suppose that every vertex has the same degree d and that boundary of every face is a cycle of length ℓ.

a) Prove that $2\alpha_1(G) = \ell \cdot \alpha_2(G)$.

b) Find an equation relating $\alpha_0(G)$, $\alpha_1(G)$ and d and prove it.

c) Use Euler's polyhedron formula and what you proved so far to show that $\ell \geq 6$ implies $d = \alpha_2(G) = 2$.