

UE Discrete Mathematics

Exercises for Nov 7, 2023

41–42) Use a suitable graph model to reformulate the exercises as graph theoretical problems and solve them.

41) Given a subset $A \subseteq \mathbb{R}^2$ which has area a and two decompositions of A into pairwise disjoint subsets A_1, A_2, \dots, A_m and B_1, B_2, \dots, B_m such that all the A_i 's and all the B_i 's have the same area a/m . Prove that there exists a permutation π of $\{1, 2, \dots, m\}$ such that for all $i = 1, \dots, m$ we have $A_i \cap B_{\pi(i)} \neq \emptyset$.

42) Given a set A with n elements and $B = \{A_1, A_2, \dots, A_n\} \subseteq 2^A$. Prove that there exists an injective mapping $f : B \rightarrow A$ such that $f(A_i) \in A_i$ for all $i \in \{1, 2, \dots, n\}$ if and only if for all $I \subseteq \{1, 2, \dots, n\}$ the cardinality of $\bigcup_{i \in I} A_i$ is at least equal to the cardinality of I .

43) Let $n \in \mathbb{N}$ and $G = (V, E) = (V_1 \cup V_2, E)$ be a bipartite graph with $\min_{x \in V} d(x) \geq n/2$ and $|V_1| = |V_2| = n$. Use Hall's theorem to prove that G has a perfect matching.
 $\alpha_1(G) \geq \binom{k}{2}$.

44) Show that every edge colouring of the K_6 with two colors contains at least two monochromatic triangles.

Hint: Estimate the number of bichromatic triangles by estimating the number of bichromatic paths of length two.

45) Let $2 \leq p \leq q$ and $2 \leq r \leq s$. Prove that the Ramsey numbers satisfy $R(p, r) \leq R(q, s)$ and that equality holds if and only if $p = q$ and $r = s$.

46) Prove that $R(4, 3) = 9$.

Hint: If some vertex v is incident to at least 6 red edges or to at least 4 blue edges, then show that there is either a red K_4 or a blue K_3 ; otherwise use the handshaking lemma.

47) Show the following inequality for Ramsey numbers: If $r \geq 3$ then

$$R(n_1, \dots, n_{r-2}, n_{r-1}, n_r) \leq R(n_1, \dots, n_{r-2}, R(n_{r-1}, n_r))$$

Hint: Let $n = R(n_1, \dots, n_{r-2}, R(n_{r-1}, n_r))$ and consider an edge colouring of K_n with r colours, say c_1, \dots, c_r . Identify the colours c_{r-1} and c_r and apply the Ramsey property for $r - 1$ colours.

48) Let A be a non-empty set. Show that A has as many subset with an odd numbers of elements as subsets with an even number of elements.

49) In how many ways can the letters a, a, b, b, c, d, e be listed such that the letters c and d are not in consecutive positions?

50) Find the number of ways to place n rooks on an $n \times n$ checkerboard such that no two of them attack each other.