UE Discrete Mathematics

Exercises for Nov 14, 2023

51) Let $p_n(k)$ be the number of permutations of $\{1, 2, \ldots, n\}$ having exactly k fixed points. Use the method of double counting to prove the identity $\sum_{k=0}^{n} kp_n(k) = n!$.

52) Let A be a set of 11 positive integers such that for all $x \in A$ we have $20 \nmid x$. Prove that there are two integers $a, b \in A$ such that $20|(a + b)$ or $20|(a - b)$.

53) Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, \ldots, 100\}$. All points of the plane having coordinates (x, y) which satisfy $(x, y) \in A \times B$ are coloured with one of the colours red, green or blue. Prove that there exists a monochromatic rectangle.

Remark: A rectangle is called monochromatic if all its four vertices have the same colour.

54) Place the numbers 1,2,...,12 around a circle, in any order. Then there are three consecutive numbers which sum to at least 19.

55) Let $n \in \mathbb{N}$. Prove the identities

$$\sum_{k=0}^{n} 2^k = 2^{n+1} - 1 \qquad \text{and} \qquad \sum_{k=1}^{n} (n-k)2^{k-1} = 2^n - n - 1$$

by using a combinatorial interpretation.

56) Let $D_n \subseteq S_n$ be the fixed-point-free permutations of $\{1, 2, \ldots, n\}$, i.e. all permutations $\pi \in S_n$ with $\pi(i) \neq i$ for all i. The *derangement numbers* are defined as $d_n := |D_n|$. Prove the recurrence

$$\begin{cases} d_n &= (n-1)(d_{n-1} + d_{n-2}) \quad \text{for } n \geq 2, \\ d_0 &= 1, d_1 = 0, \end{cases}$$

using a combinatorial interpretation. Furthermore, prove that this recurrence relation implies

$$d_n = nd_{n-1} + (-1)^n \qquad \text{and} \qquad d_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

57) Compute the number of elements of the set
$\{x \in \mathbb{N} \mid 1 \leq x \leq 100\,000 \text{ and } x \text{ is neither a square nor a 3rd, 4th or 5th power of some } y \in \mathbb{N}\}$.

58) Prove Pascal's recurrence

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

algebraically using the closed formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ of the binomial coefficient, and prove

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

using a combinatorial interpretation of the binomial coefficient.

59) Let $n \in \mathbb{N}$. Prove the identity

$$\sum_{m=0}^{n} \binom{m}{k} = \binom{n+1}{k+1}.$$

60) Let $t_{n,k}$ denote the number of permutations of an n-element set that have exactly k cycles. Prove that $t_{n,2} = (n-1)! H_{n-1}$ for $n \geq 2$, where $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.