UE Discrete Mathematics

Exercises for Nov 28, 2023

71) Use generating functions to answer the following question: What is the number of solutions of the equation $a + b + c + d = 25$ if $a, b, c, d \in \{0, 1, 2, \ldots, 9\}$?

72) Prove the following identity:

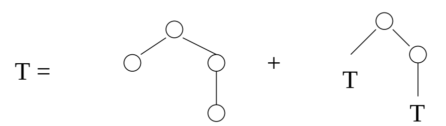
$$\sum_{n \geq 0} \binom{2n}{n} z^n = \frac{1}{\sqrt{1 - 4z}}.$$

73) Compute

$$[z^n] \frac{2 + 3z^2}{\sqrt{1 - 5z}}.$$

74) A ternary tree is a plane rooted tree such that every node has either 3 or 0 succesors. A node with 3 succesors is called internal nodes. How many leaves has a ternary tree with n internal nodes? Moreover, let a_n be the number of ternary trees with n internal nodes and $A(z)$ the generating function of this sequence. Find a functional equation for $A(z)$!

75) Compute the numbers t_n of plane rooted trees with n nodes specified by the equation



76) Compute the number of plane rooted trees with n nodes.

77) Consider the following context-free grammar: $S \to aSbS|\varepsilon$. This defines a formal language $\mathcal{L}$ which consists of all words w over the alphabet $\Sigma = \{a, b\}$ such that either (a) w starts with a followed by a word from $\mathcal{L}$, then a b follows, which is itself followed by another word of $\mathcal{L}$, or (b) w is the empty word. Compute the number of words in $\mathcal{L}$ that consist of n letters. Do this by finding a combinatorial structure that specifies $\mathcal{L}$ and analyzing the generating function of that structure.

78) Let $\mathcal{W}$ denote the set of words over the alphabet $\{a, b\}$ that contain exactly k occurrences of b. Obviously, the number of words in $\mathcal{W}$ which have exactly n letters is $\binom{n}{k}$. Prove this by finding a specification of $\mathcal{W}$ as combinatorial construction and translating this specification into generating functions.

79) Use exponential generating functions to determine the number a_n of ordered choices of n balls such that there are 2 or 4 red balls, an even number of green balls and an arbitrary number of blue balls.

80) Determine all solutions of the recurrence relation:

$$a_n - 2na_{n-1} + n(n-1)a_{n-2} = 2n \cdot n!, \; n \geq 2, \; a_0 = a_1 = 1.$$

Hint: Use exponential generating functions.