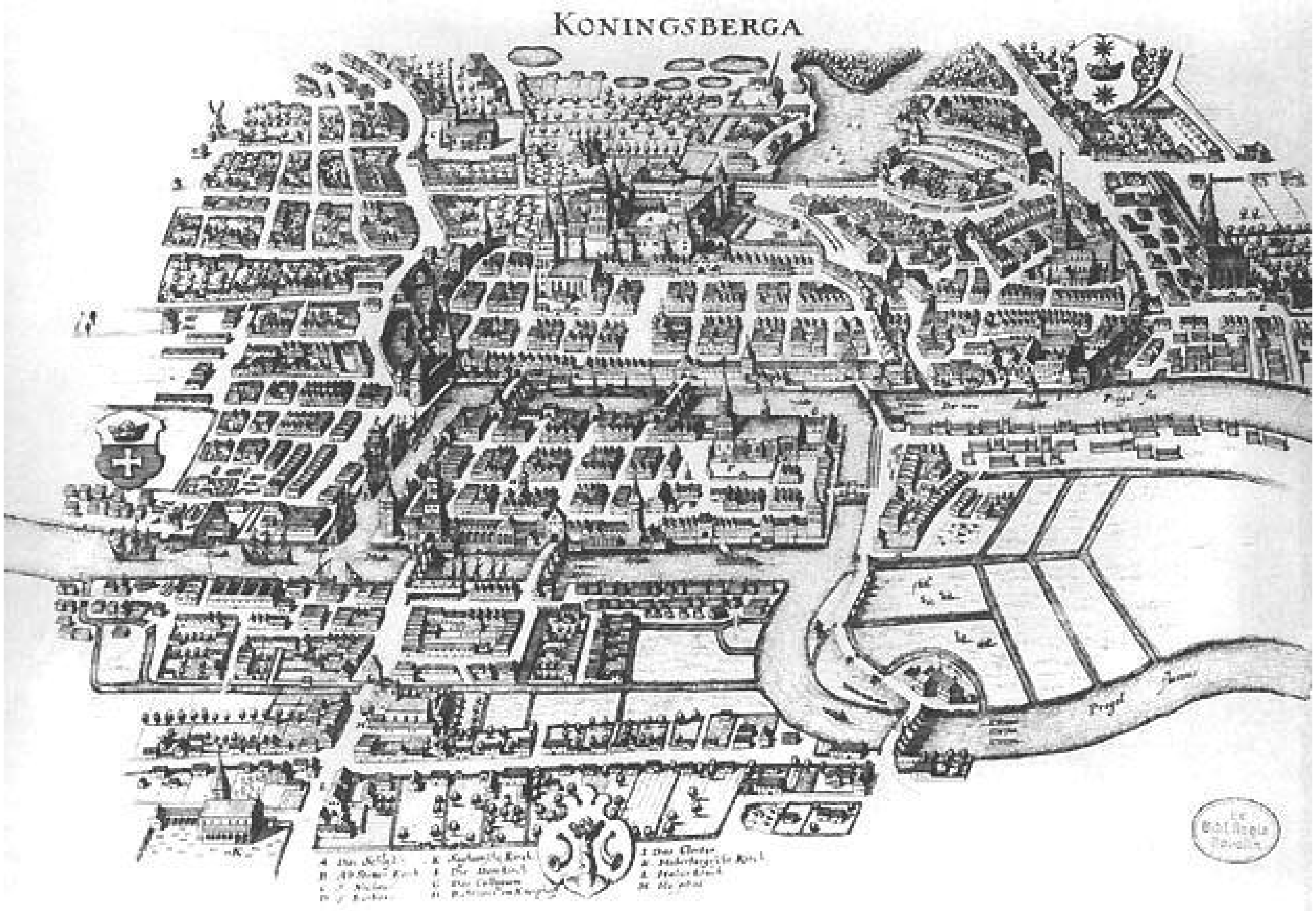
EULERIAN GRAPHS

Eulerian circuits

Seven Bridges of Königsberg problem (1736)



Eulerian circuits

An *Eulerian circuit* (or Euler circuit, Euler(ian) tour) is a closed walk that contains every edge of $G = (V, E)$ exactly once.
A graph having an Eulerian circuit is called *Eulerian graph*.

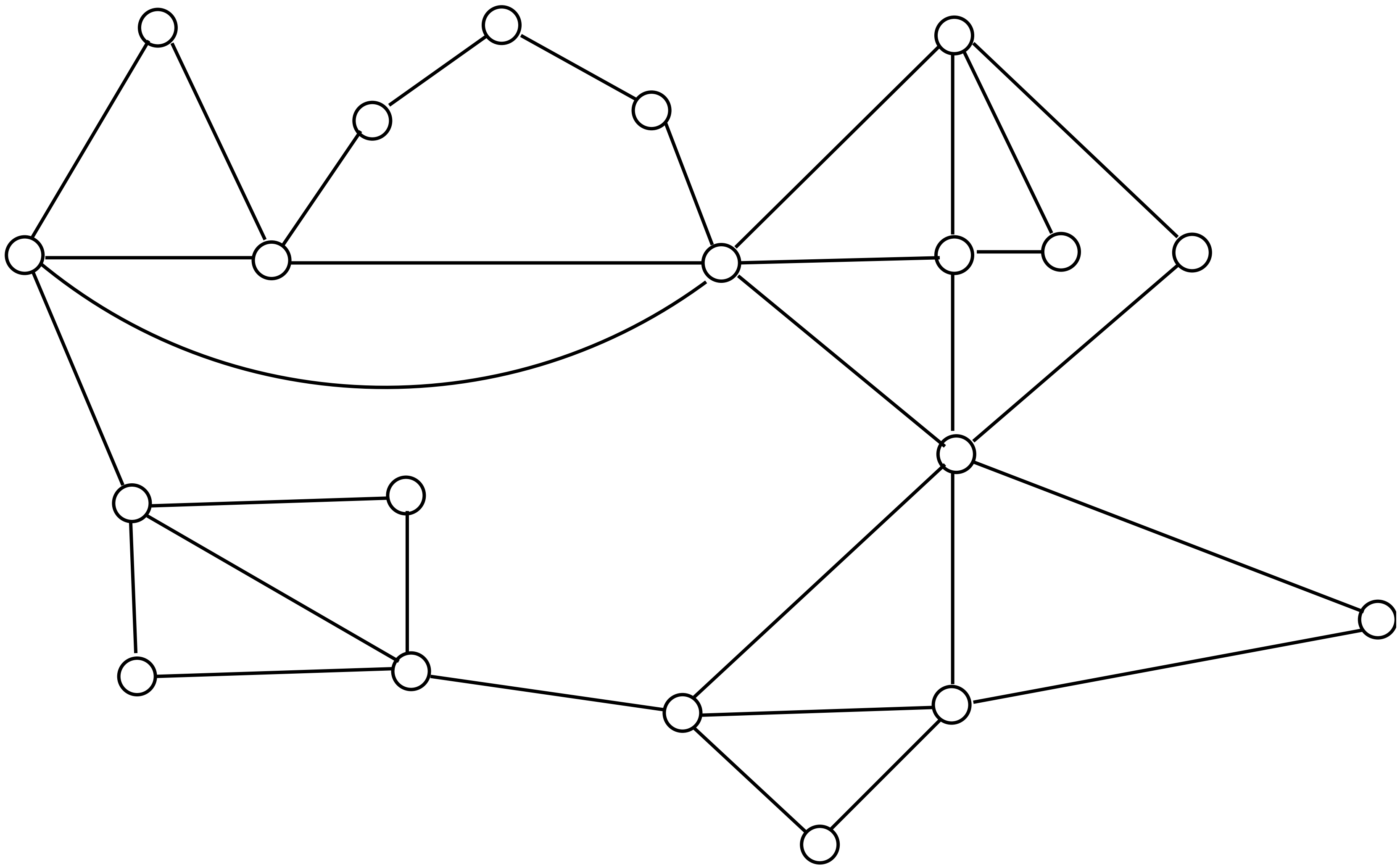
Variant: An *Eulerian trail* (or Euler trail) is an open walk that contains every edge of $G = (V, E)$ exactly once.

Theorem *An undirected connected graph is Eulerian if and only if all its vertices have even degree.*

An undirected connected graph has an open Eulerian trail if and only if all but two vertices have even degree.

Proof: induction on the number of edges.

Idea of the proof for the existence of an Eulerian circuit



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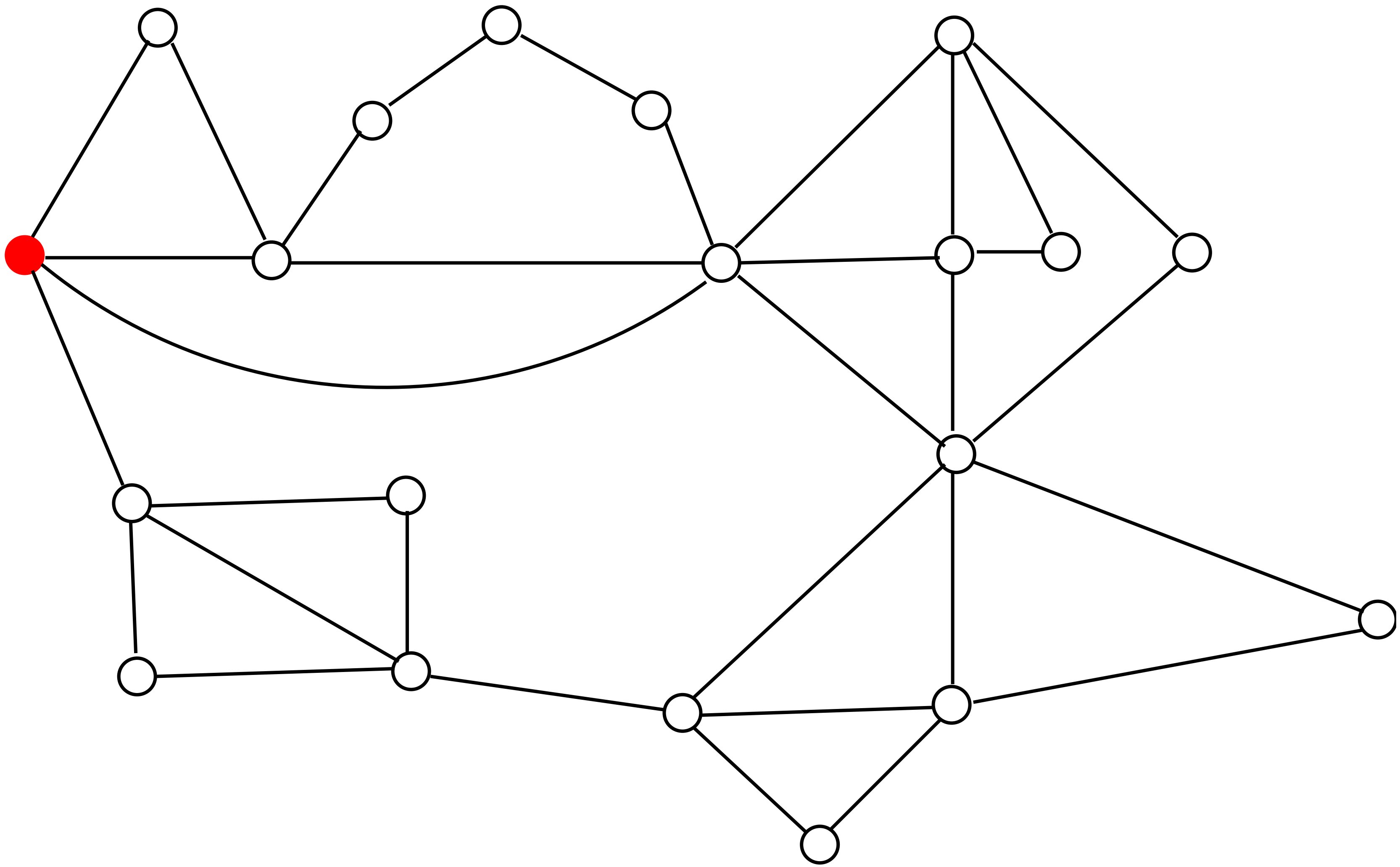

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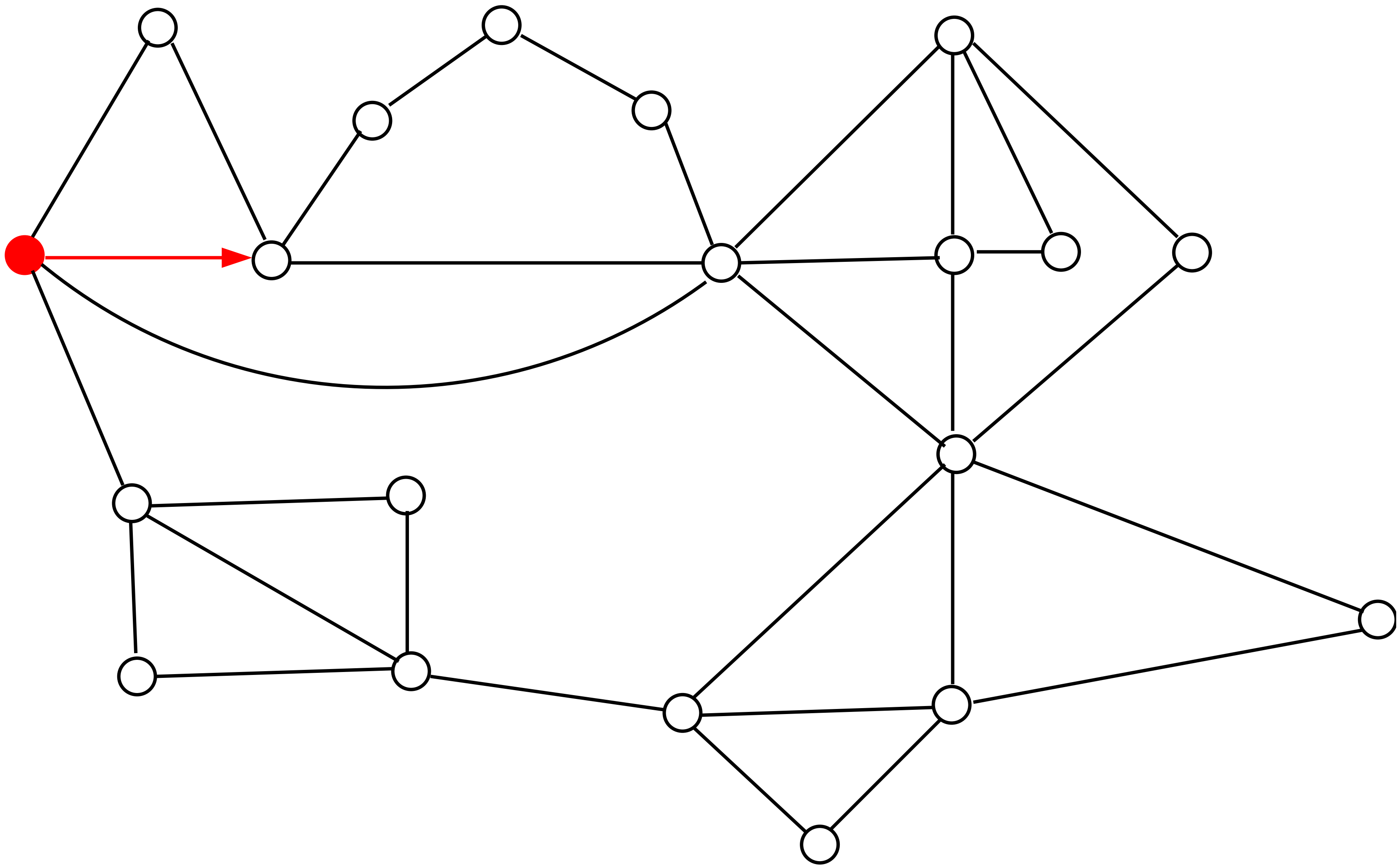

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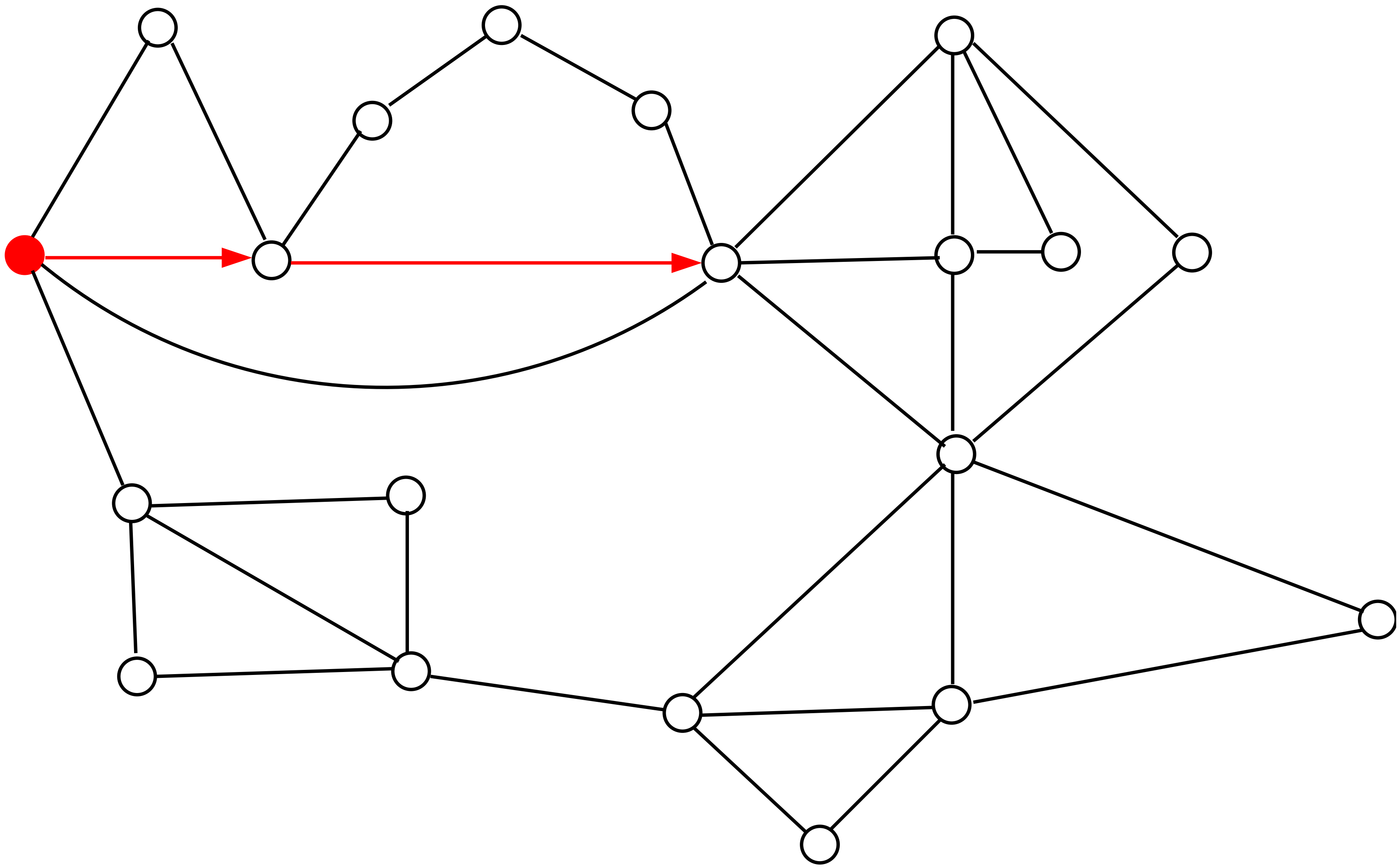

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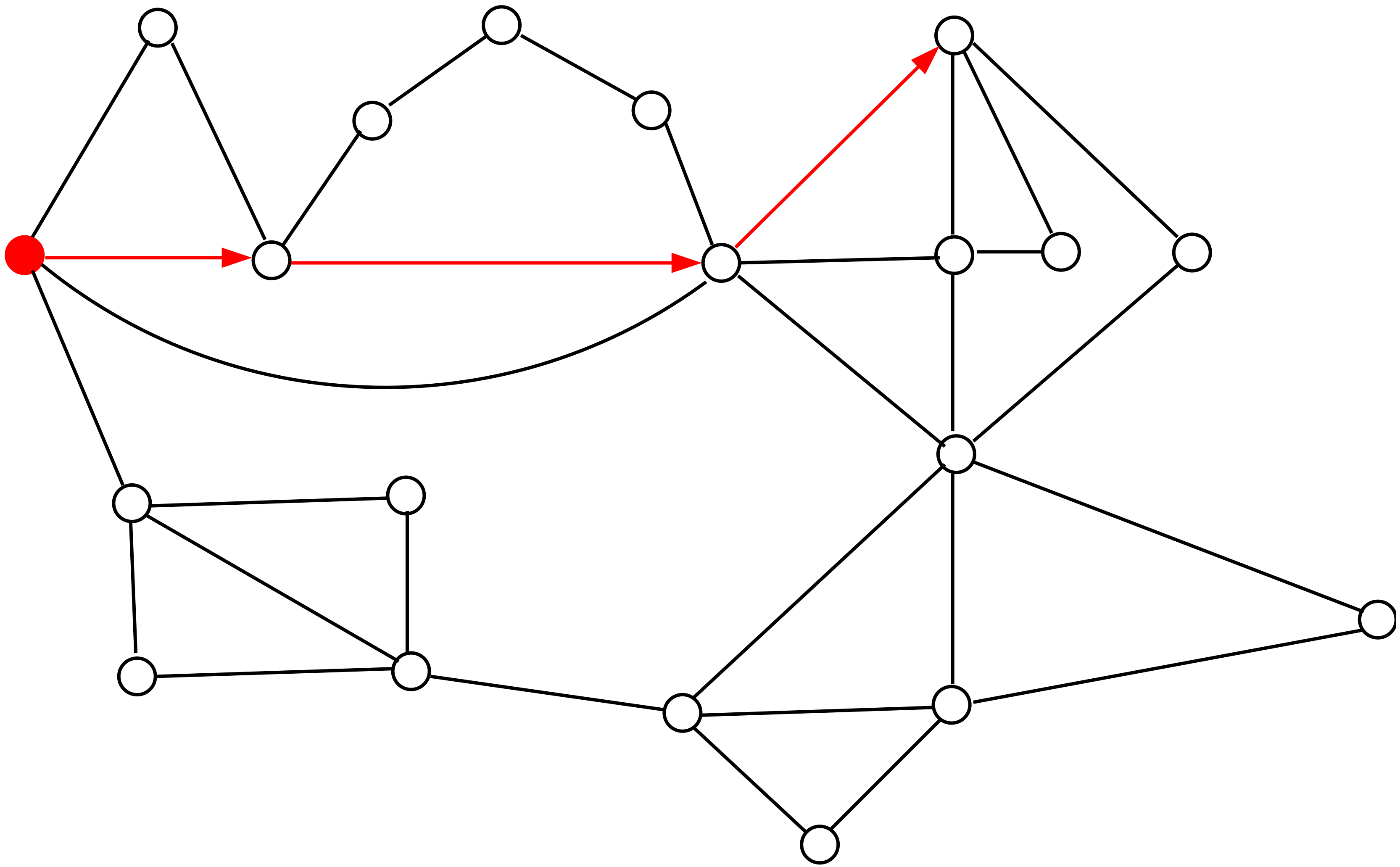

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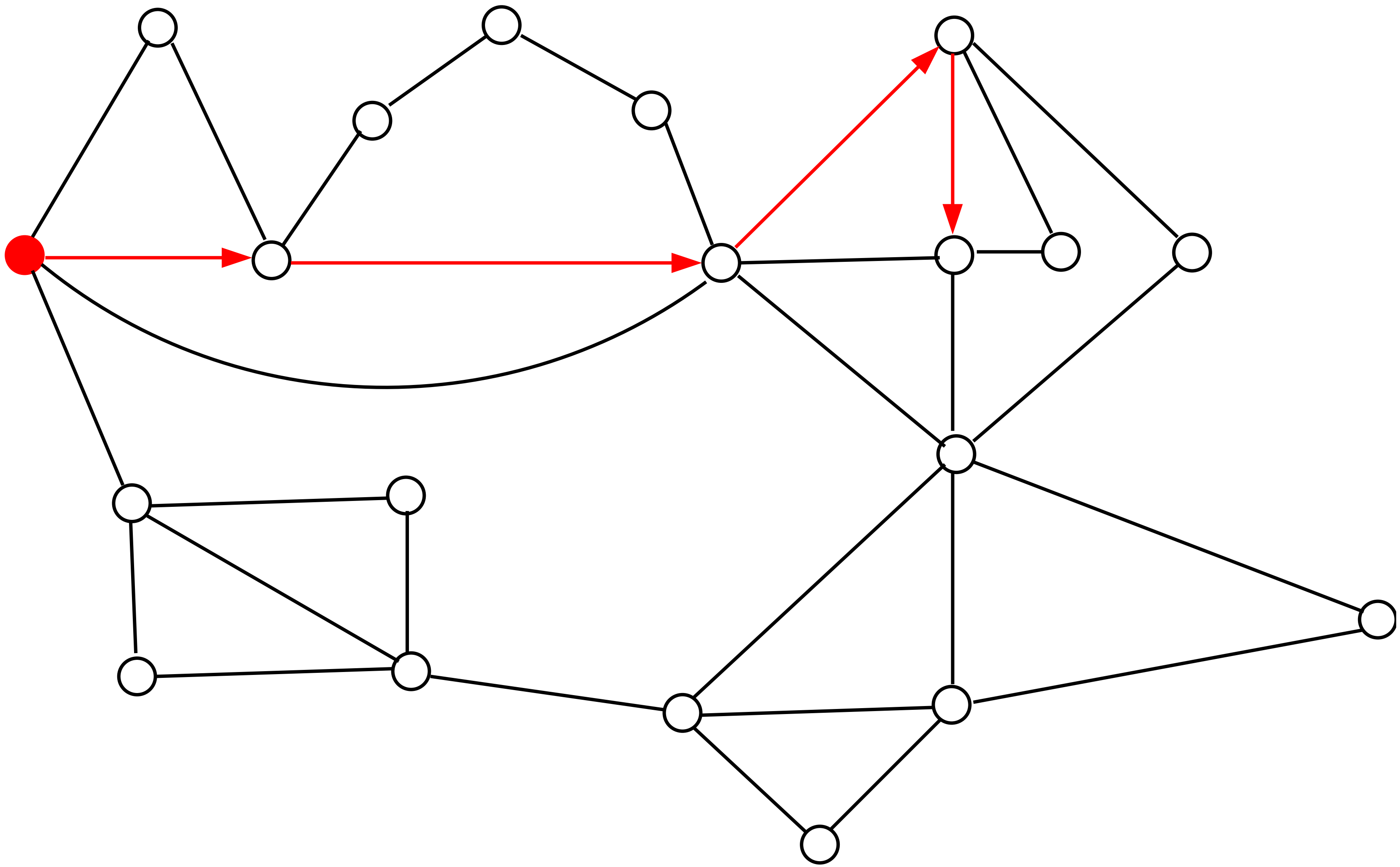

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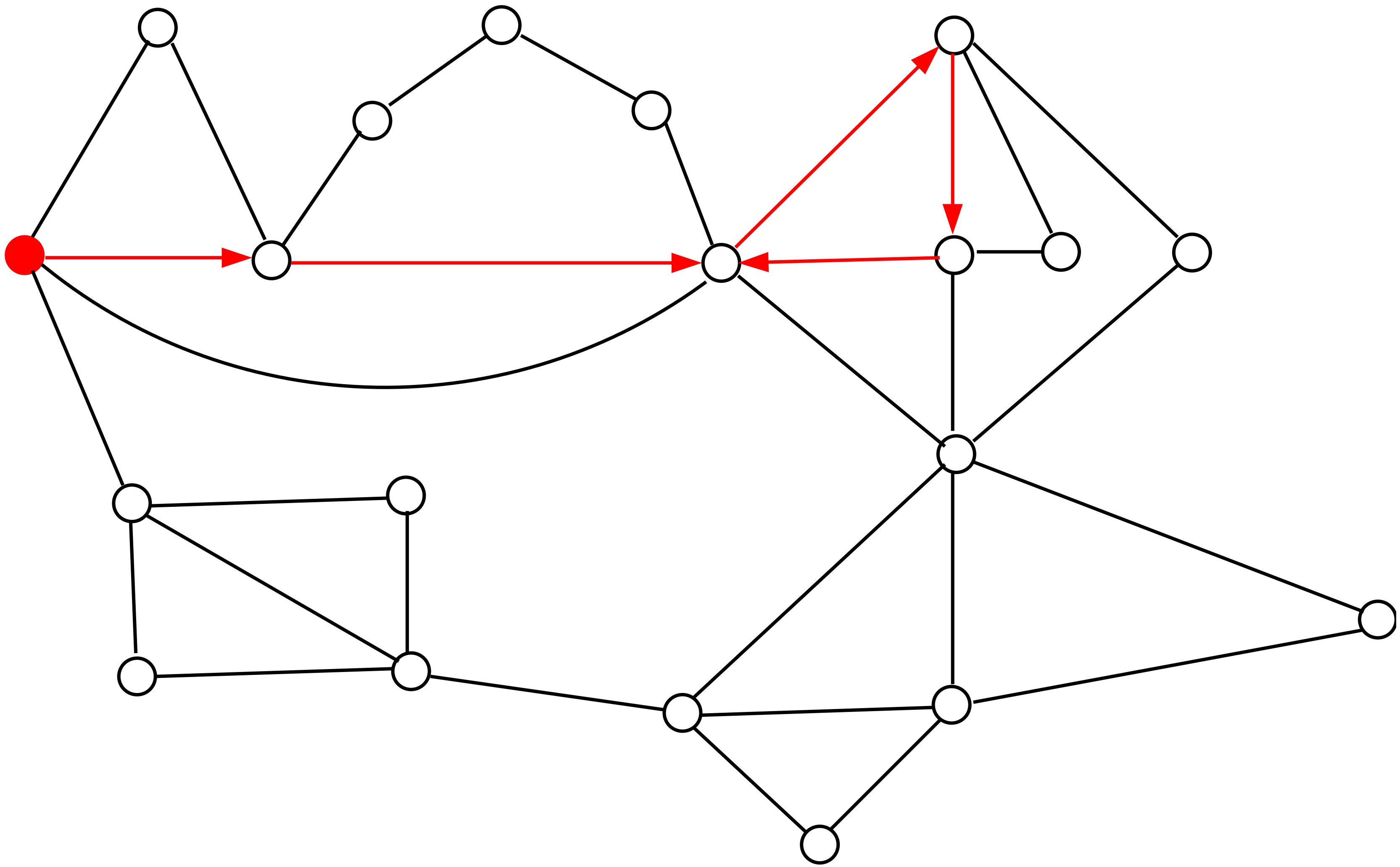

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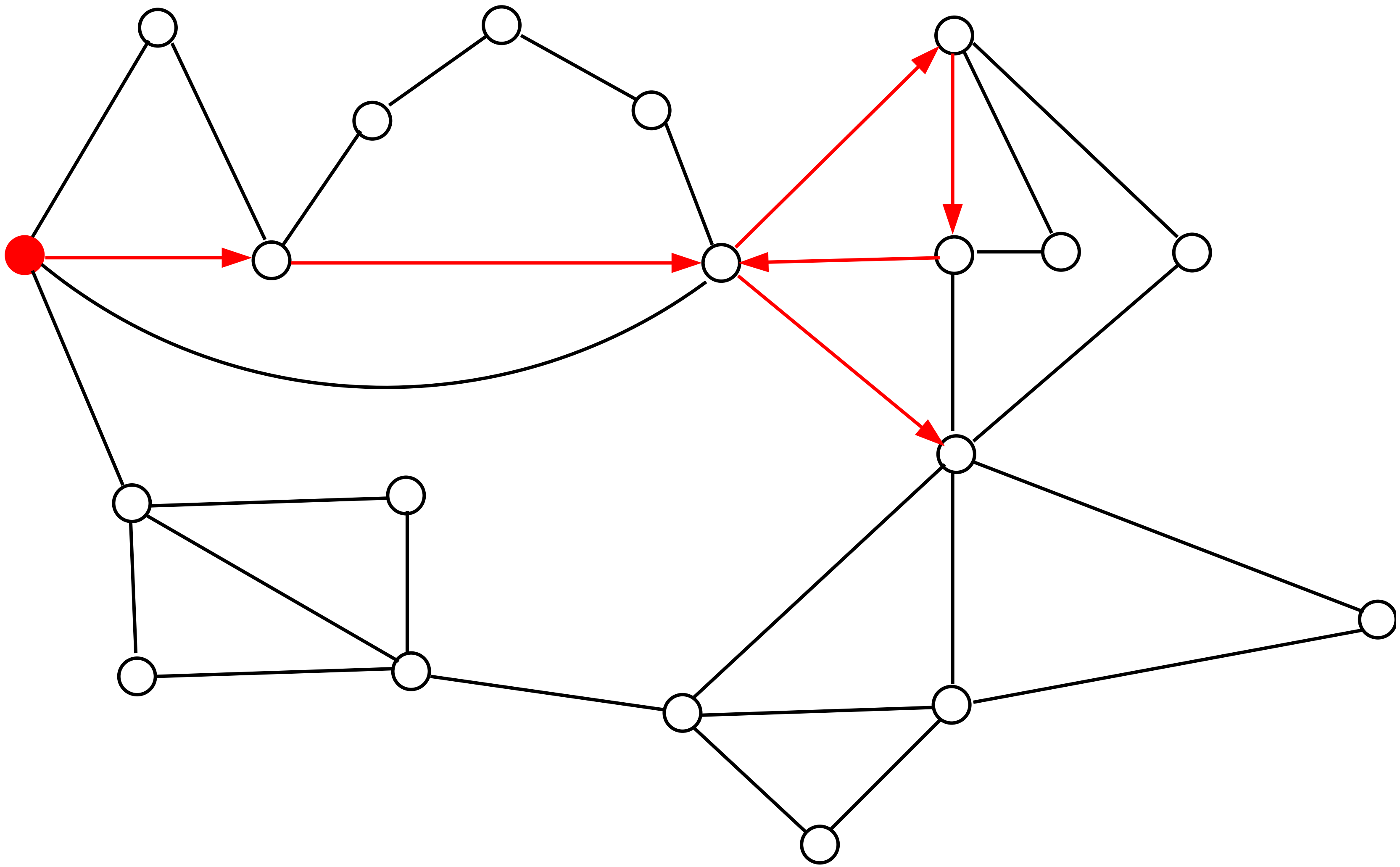

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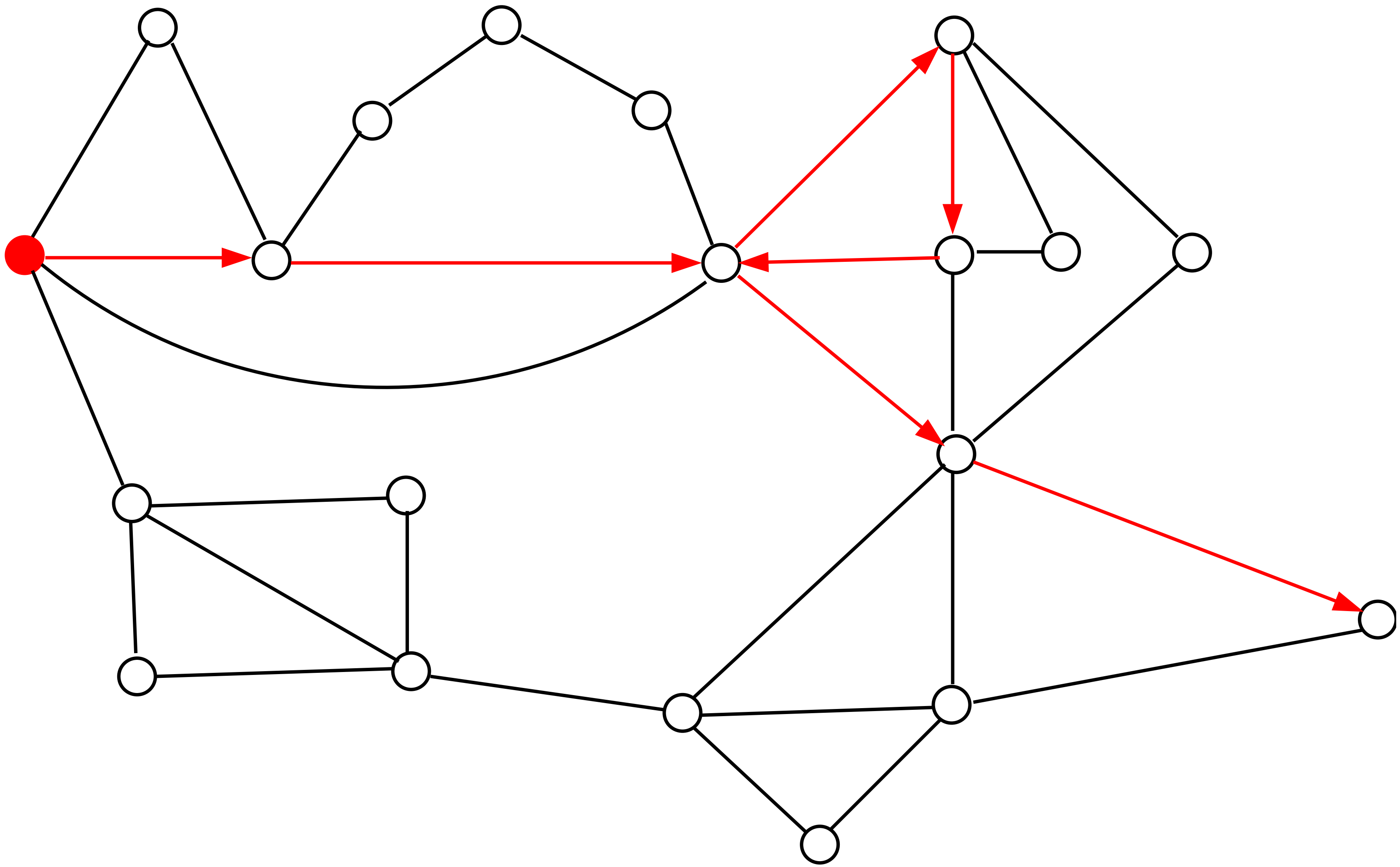

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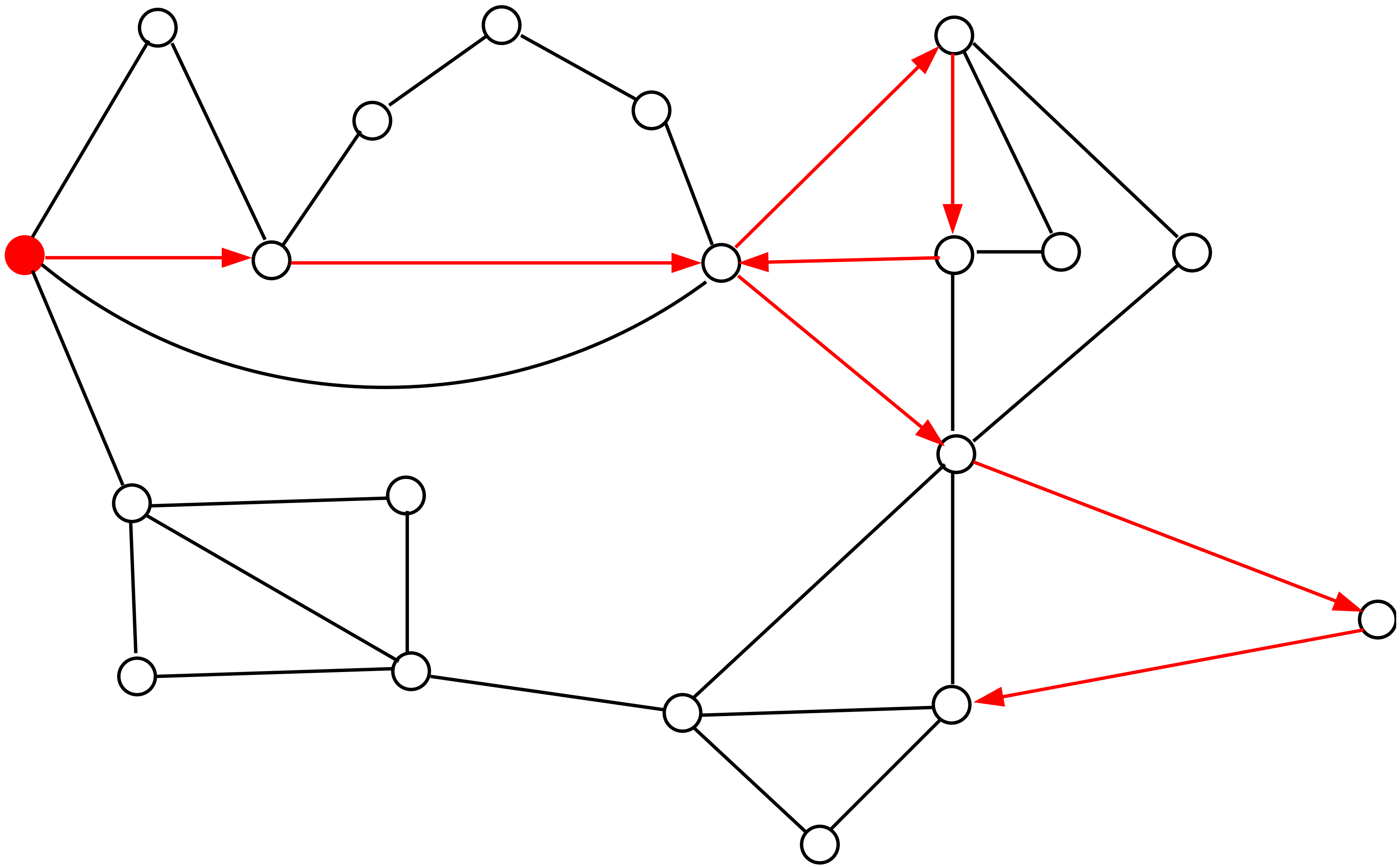

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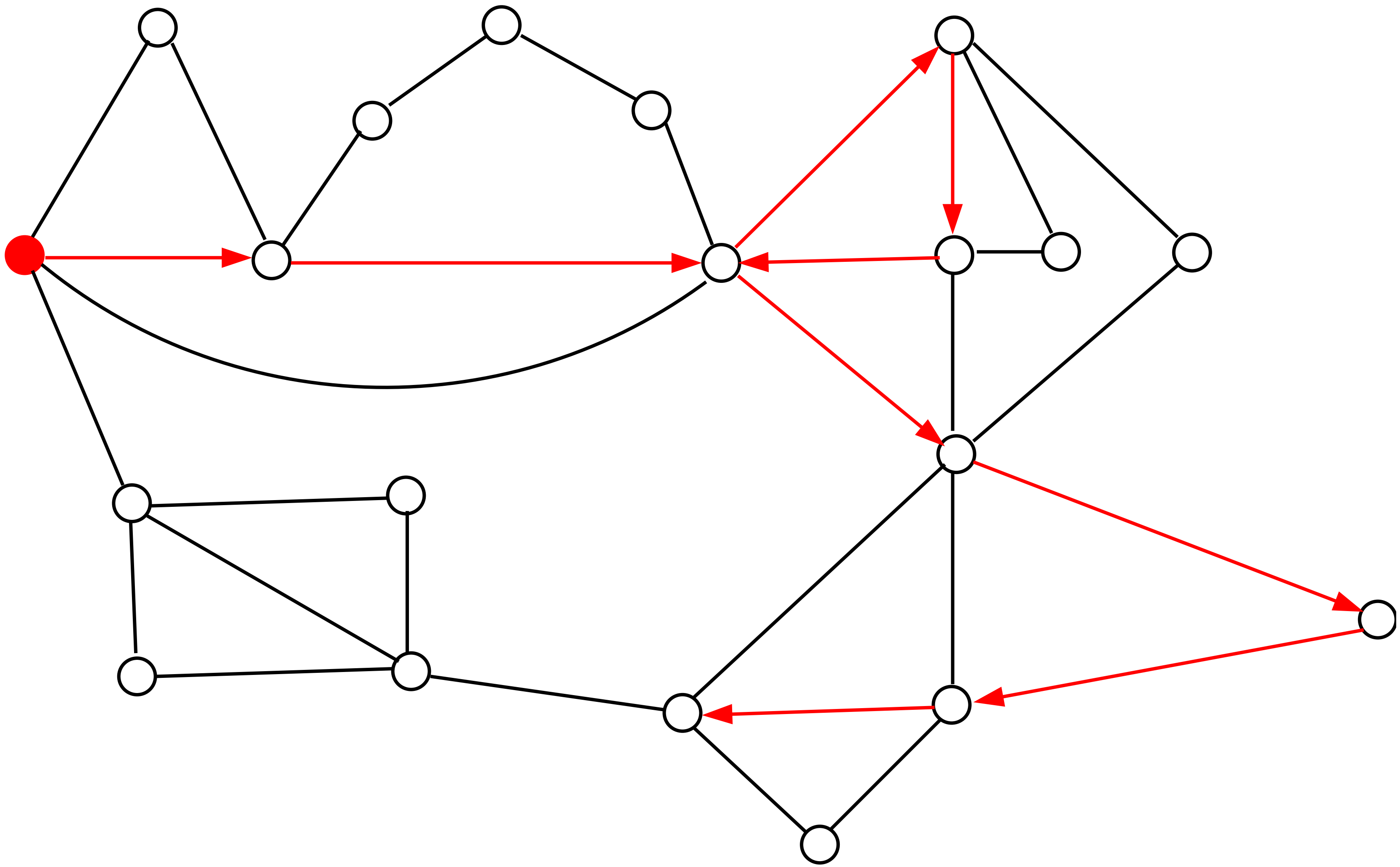

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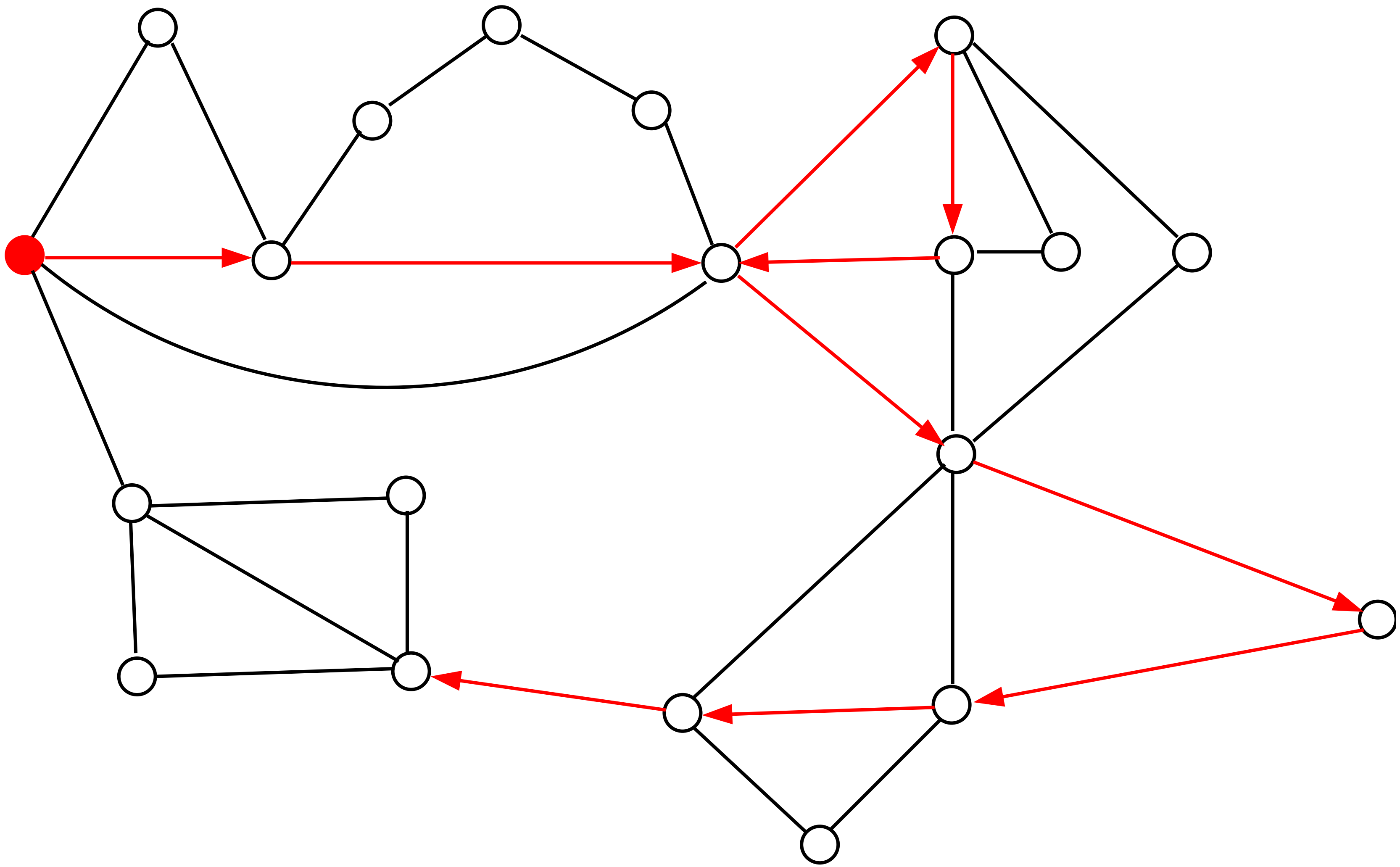

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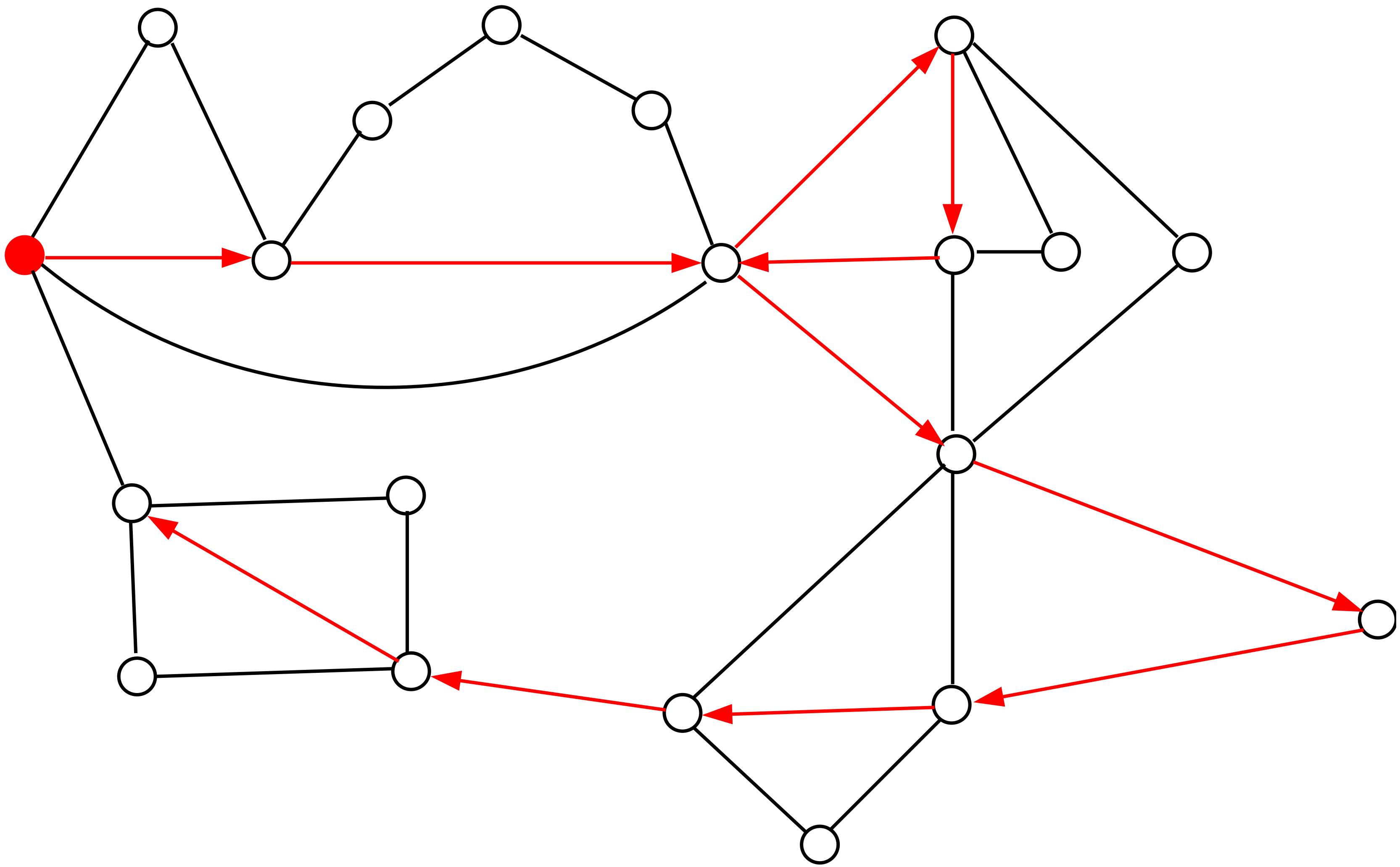

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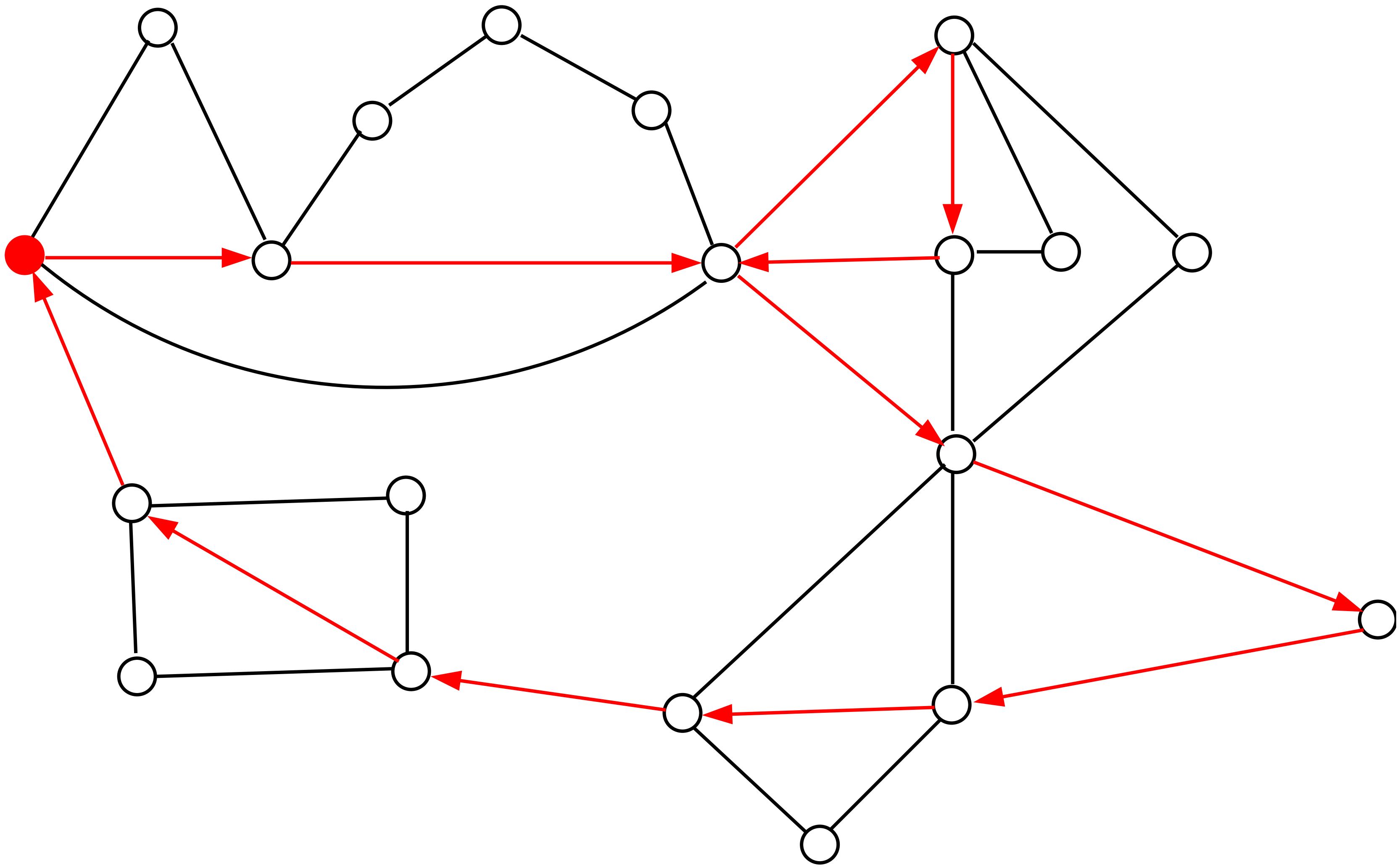

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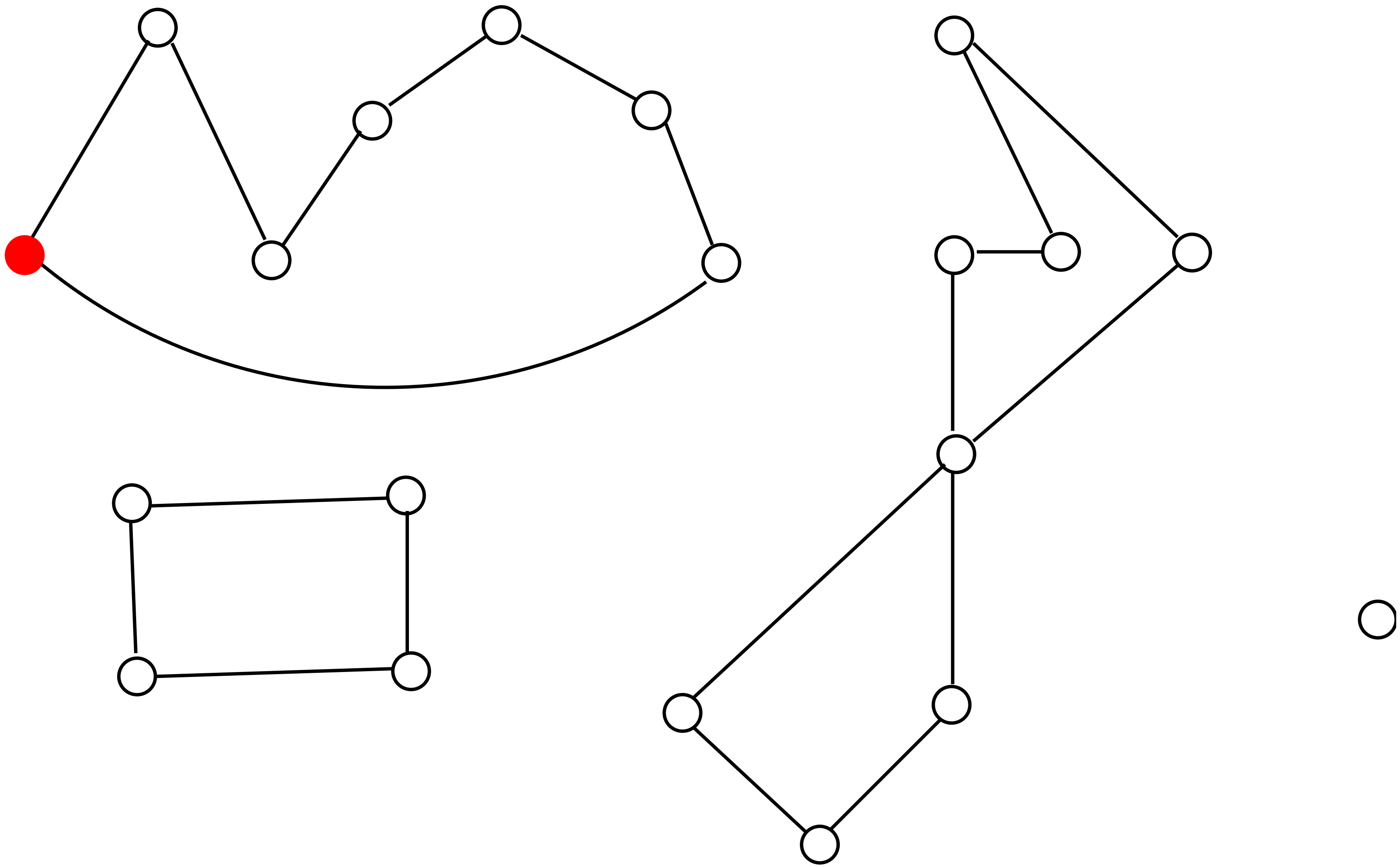

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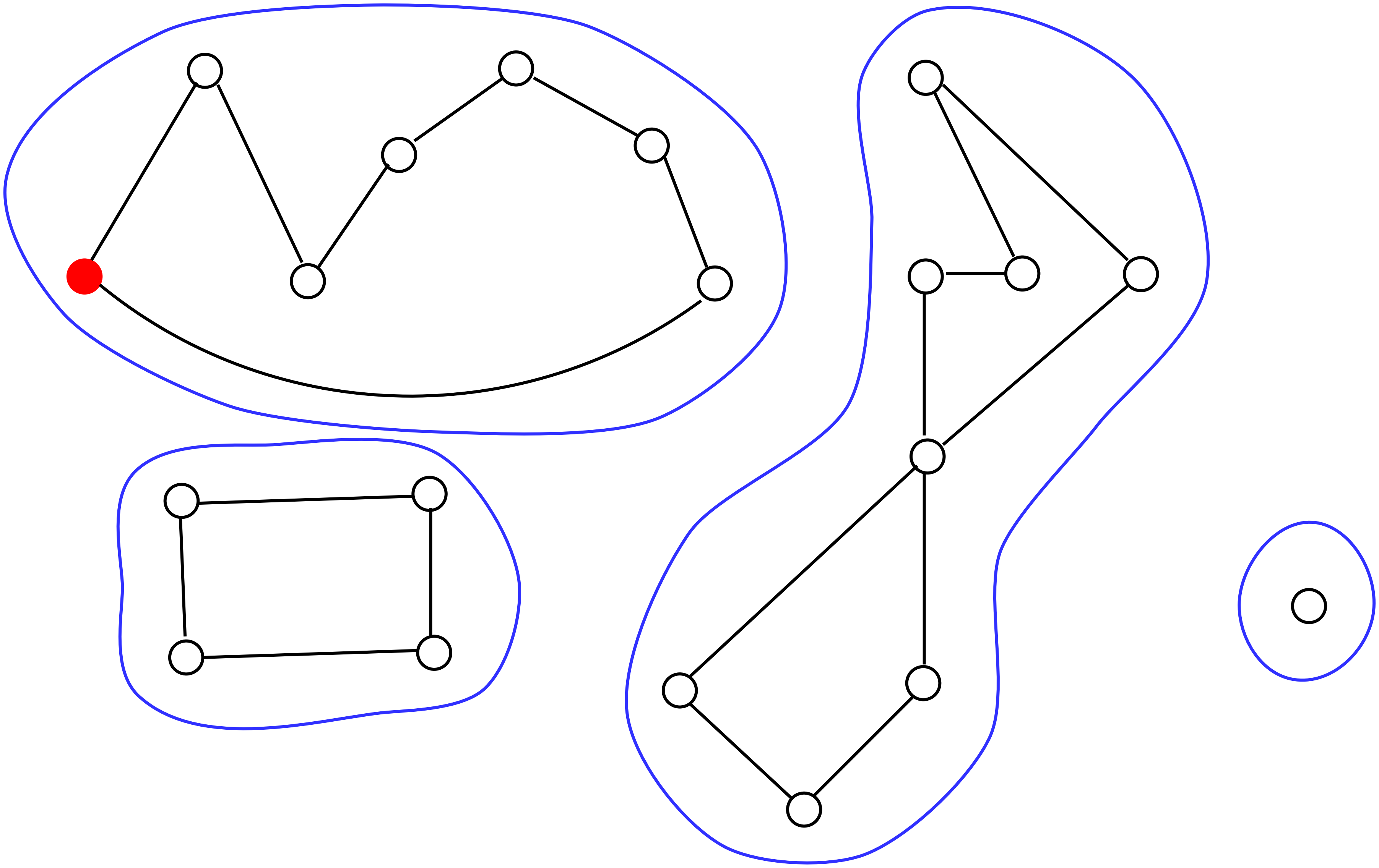

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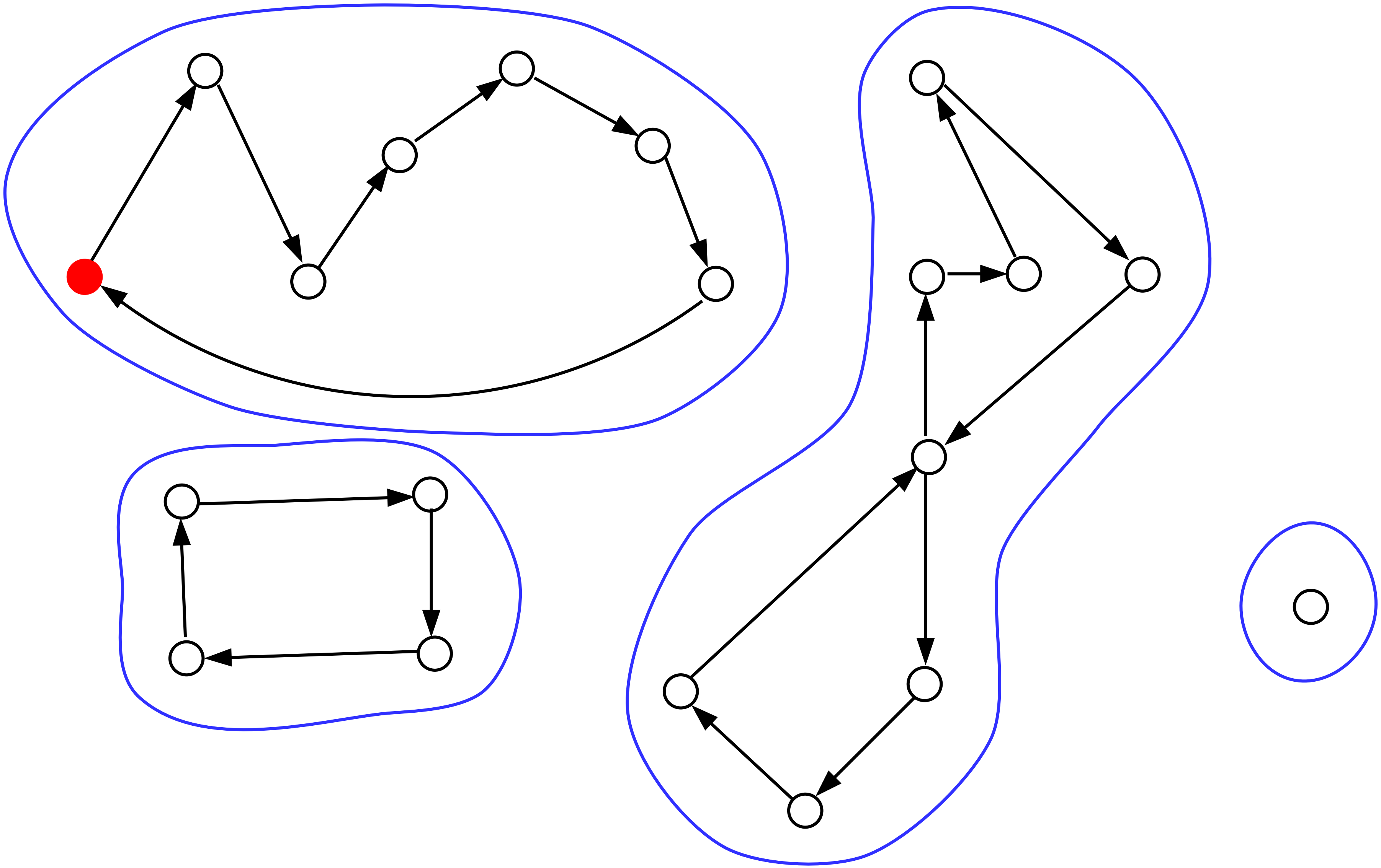

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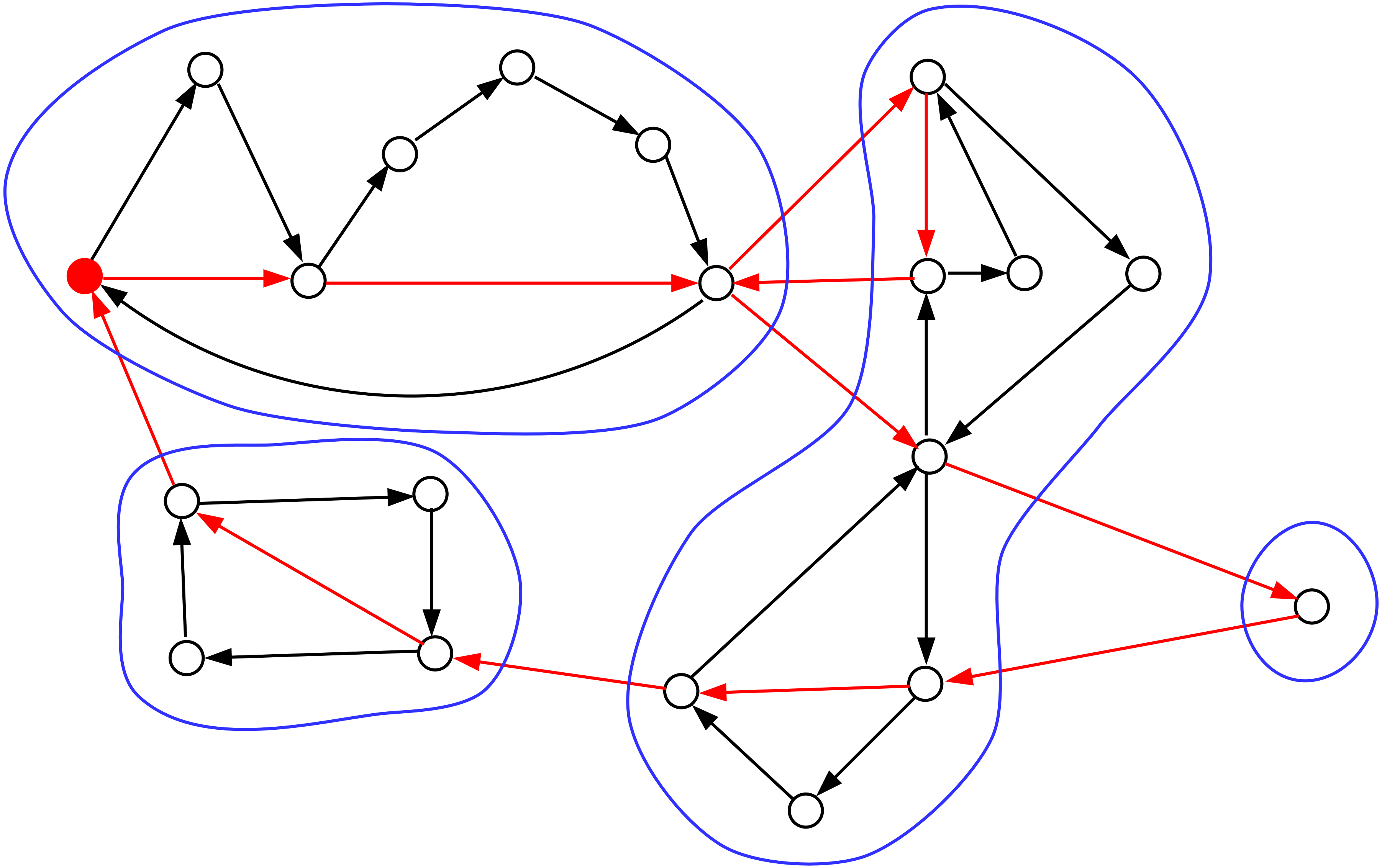


Eulerian circuits in directed graphs

Theorem *A directed, weakly connected graph* $G = (V, E)$ *is Eulerian if and only if for all vertices in-degree and out-degree coincide, i.e., e*

$$\forall\, x \in V:\; d^{+}(x) = d^{-}(x).$$

A directed, weakly connected graph $G = (V, E)$ *has an Eulerian trail if and only if there are vertices* $x, y \in V$ *such that*

$$\begin{aligned} &d^{+}(x) = d^{-}(x) + 1, \\ &d^{+}(y) = d^{-}(y) - 1, \\ &\forall\, z \in V \setminus \{x, y\}:\; d^{+}(z) = d^{-}(z). \end{aligned}$$

HAMILTONIAN GRAPHS

Hamiltonian graphs

A path in a graph that visits every vertex exactly once is called *Hamiltonian path*.

A cycle in a graph that visits every vertex exactly once is called *Hamiltonian cycle*.

A graph having a Hamiltonian cycle is called *Hamiltonian graph*.

Let $G = (V, E)$ be a graph. We construct another graph $[G] = (V, \tilde{E})$, called the *closure* of G: Let

$$A(E) := \{vw \mid vw \in E \text{ or } d(v) + d(w) \geq |V|\}.$$

Then $\tilde{E} = A^{\infty}(E)$.
(In fact, there is a k s.t. $A^{\infty}(E) = A^{\ell}$ for all $\ell \geq k$.)

Hamiltonian graphs

Theorem *G is Hamiltonian if and only if $[G]$ is Hamiltonian.*

Proof: "$\Longrightarrow$": Obvious.

"$\Longleftarrow$": Let $v, w \in V$ with $vw \notin E$, $d(v)+d(w) \geq |V|$; $H := (V, E \cup \{vw\})$. Assume: H Hamiltonian, G not. Then there is a Hamiltonian cycle in H containing vw, say $v = x_1, x_2, \ldots, x_n = w, x_1$, where $n = |V|$.
Let

$X = \{x_i \mid x_{i-1} \in \Gamma(w), 3 \leq i \leq n-1\}$, $\qquad Y = \{x_i \mid x_i \in \Gamma(v), 3 \leq i \leq n-1\}$.

Note: $v — x_2 — \cdots — x_{n-1} — w$ is a path in G.

$v \notin \Gamma(w)$ implies $|X| = d(w)-1$ and $|Y| = d(v)-1$ and so $|X|+|Y| \geq n-2$.

Thus there exists $3 \leq i \leq n-1$ such that $x_{i-1} \in \Gamma(w)$ and $x_i \in \Gamma(v)$.
Hence

$$v, x_i, x_{i+1}, \ldots, x_{n-1}, w, x_{i-1}, x_{i-2}, \ldots, v$$

is a Hamiltonian cycle in G. ↯ $\square$

Hamiltonian graphs

Consequences:

Theorem (Ore's theorem) *A graph with $n \geq 3$ vertices, in which the sum of the degrees of any two non-adjacent vertices is at least n is Hamiltonian.*

Theorem (Dirac's theorem) *A graph with n vertices in which the degree of every vertex is at least $n/2$ is Hamiltonian.*

Generalization: Travelling salesman problem, where one has to find an optimal Hamiltonian cycle in a weighted graph.

PLANAR GRAPHS

Planar graphs

Two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$ are called *isomorphic*, notation $G \cong H$, if there is a bijection $f : V_G \to V_H$ that preserves adjacency, i.e.,

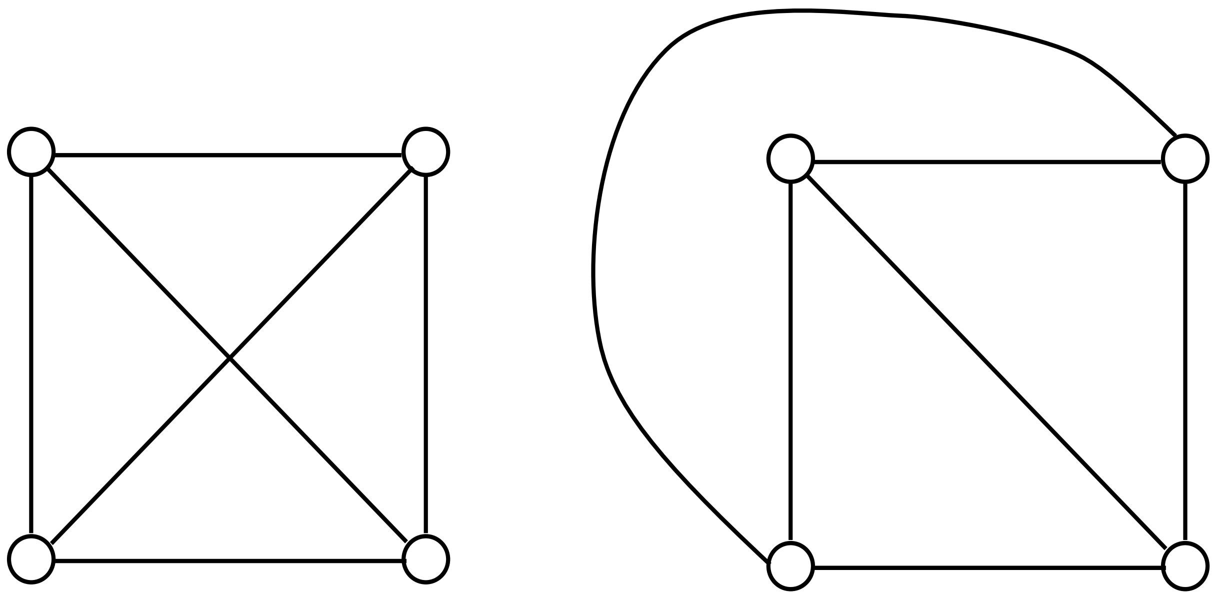
$$\forall\, x, y \in V_G :\ xy \in E_G \iff f(x)f(y) \in E_H.$$

A graph $G = (V, E)$ is called a *plane graph* if $V \subseteq \mathbb{R}^2$ and each edge is a simple curve (like a polygonal chain) that connect two vertices and no two edge cross.

A graph G is called *planar graph* if there is a plane graph H with $G \cong H$.

Planar graphs

Example:



The edges and vertices of a plane graph G enclose areas in $\mathbb{R}^2$. These are called the *faces* of G, their number is denoted by $\alpha_2(G)$.

Planar graphs

Theorem (Euler's polyhedron formula) *If G is connected and planar, then we have $\alpha_0(G) - \alpha_1(G) + \alpha_2(G) = 2$.*

Proof: Induction on α_2:

$\alpha_2 = 1$: ✓

If $\alpha_2(G) = n + 1 \geq 2$, then there must exist an edge separating two faces. Remove this edge such that the two faces collapse into one face. Call the resulting graphG'.

The induction hypothesis implies

$$\underbrace{\alpha_0(G')}_{\alpha_0(G)} - \underbrace{\alpha_1(G')}_{\alpha_1(G)-1} + \underbrace{\alpha_2(G')}_{\alpha_2(G)-1} = 2$$

Corollary *In a planar graph $\alpha_1 \leq 3\alpha_0 - 6$ holds.*
If a planar graph has no cycles of length 3 then $\alpha_1 \leq 2\alpha_0 - 4$ holds.

Planar graphs

If a planar graph has no cycles of length 3 then $\alpha_1 \leq 2\alpha_0 - 4$ holds.

Proof: $f_j :=$ # faces with boudary of j edges

Then $f_3 = 0$ and $\sum_{j \geq 4} f_j = \alpha_2(G)$.

Moreover,

$$\underbrace{4 \sum_{j \geq 4} f_j}_{4\alpha_2(G)} \leq \sum_{j \geq 4} j f_j \leq 2\alpha_1(G).$$

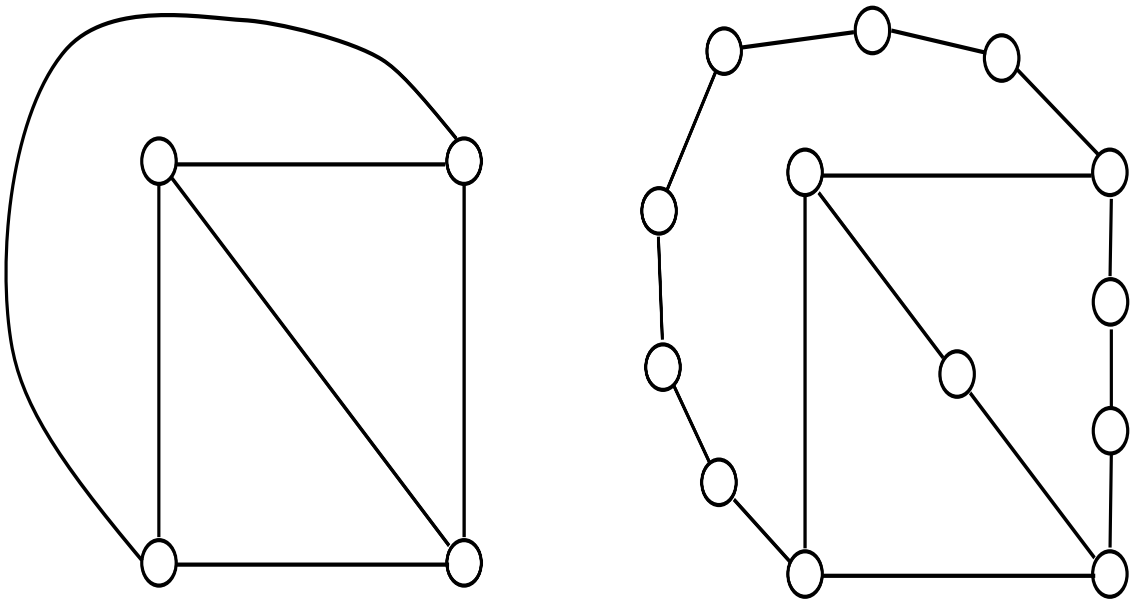
As $\alpha_0 - \alpha_1 + \alpha_2 \geq 2$ we get

$$4 \leq 2\alpha_0 - 2\alpha_1 + 2\alpha_2 \leq 2\alpha_0 - \alpha_1.$$

□

Planar graphs

A graph G' is called *subdivision* of G if each edge of G corresponds to a path in G'.

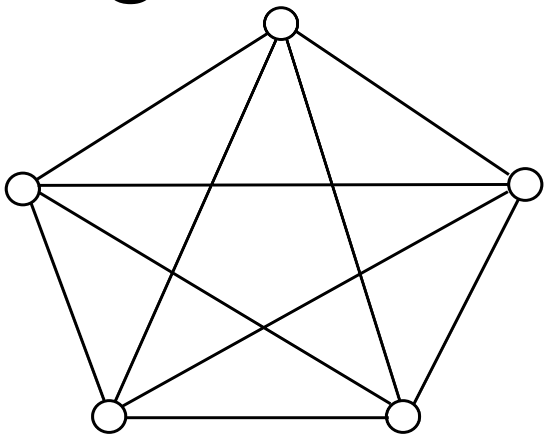


A graph H is called *topological minor* of G if there is a subdivision H' of H such that H' is a subgrah of G.

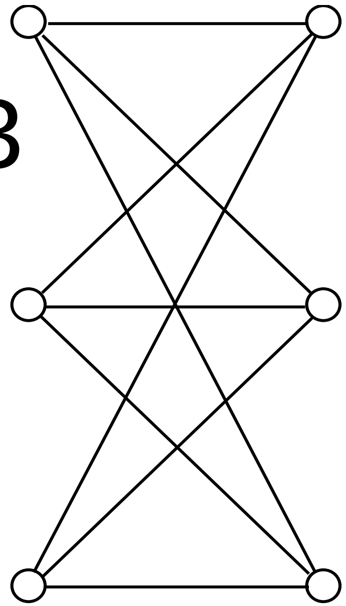
Planar graphs

Theorem (Kuratowski's theorem) *A graph G is planar if and only if neither K_5 nor $K_{3,3}$ are topological minors of G*

K5



K3,3



Planar graphs

Let $G = (V, E)$ be a planar graph and F its set of faces. The *topological dual* $G^* = (V^*, E^*)$ of G is defined as follows:

$V^* = F$ and for each edge e that separates two faces f_1 and f_2 put $f_1 f_2$ into E^*.

Note: In general, G^* is a multigraph and $|E| = |E^*|$.

Theorem *If $G = (V, E)$ is a connected and planar multigraph, then a set of edges $A \subset E$ is a cycle if and only if $A^* = \{e^* \mid e \in A\}$ is a minimal cut of G^*.*