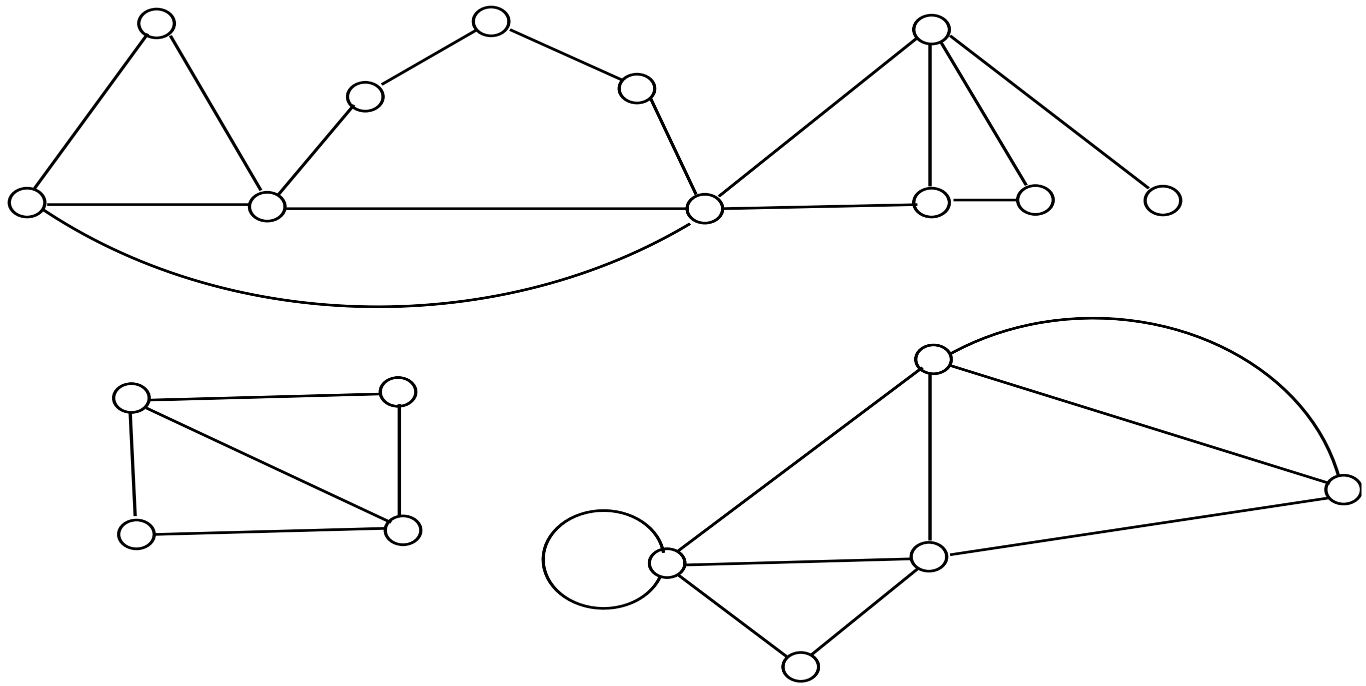
BASIC CONCEPTS OF GRAPH THEORY

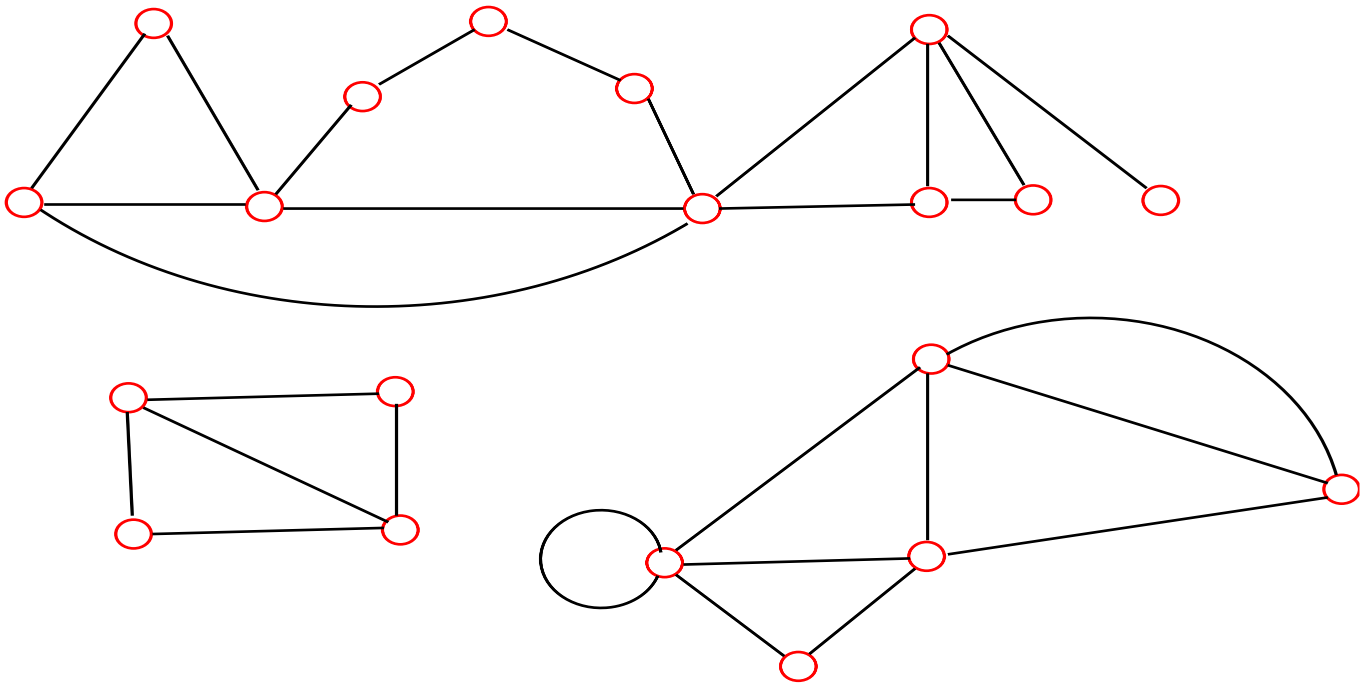
Basic Concepts of Graph Theory

Undirected graph



Basic Concepts of Graph Theory

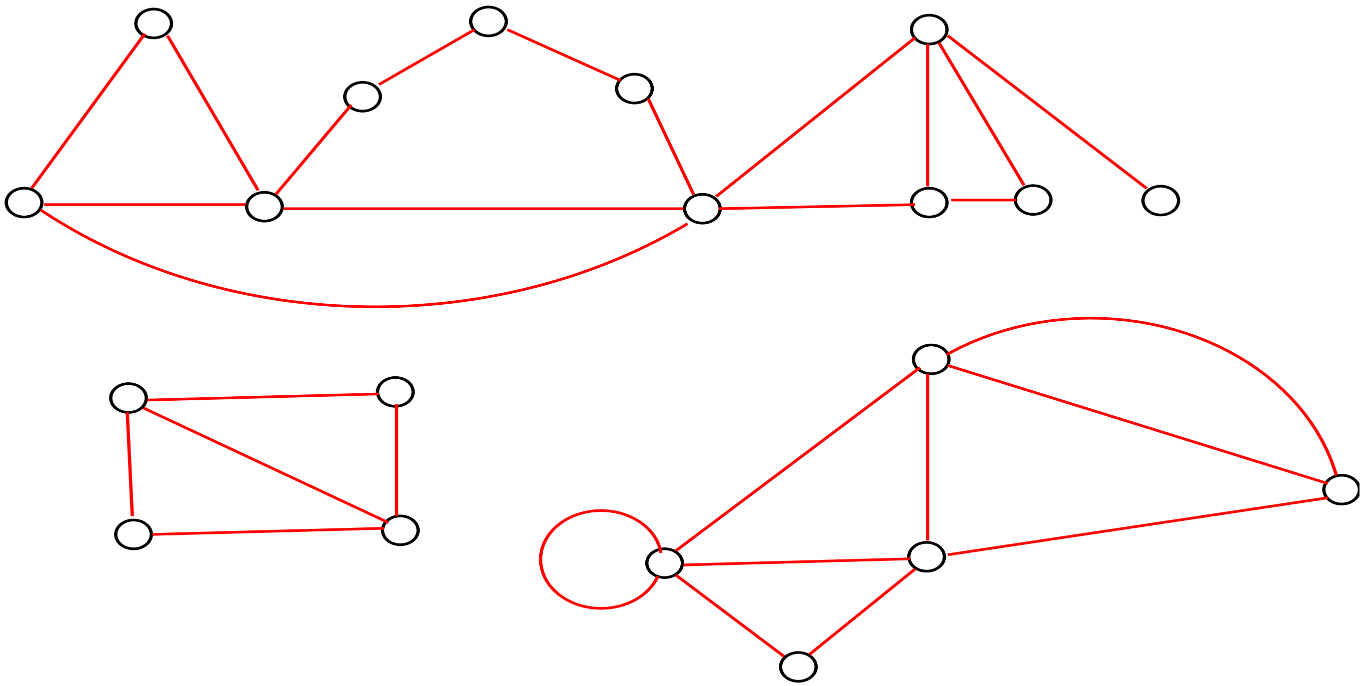
The vertices of a graph



vertex set V, $\alpha_0 := |V|$

Basic Concepts of Graph Theory

The edges of a graph



edge set E, $\alpha_1 := |E|$, $\longrightarrow$ Graph $G = (V, E)$

density $\varepsilon(G) = \frac{|E|}{|V|}$

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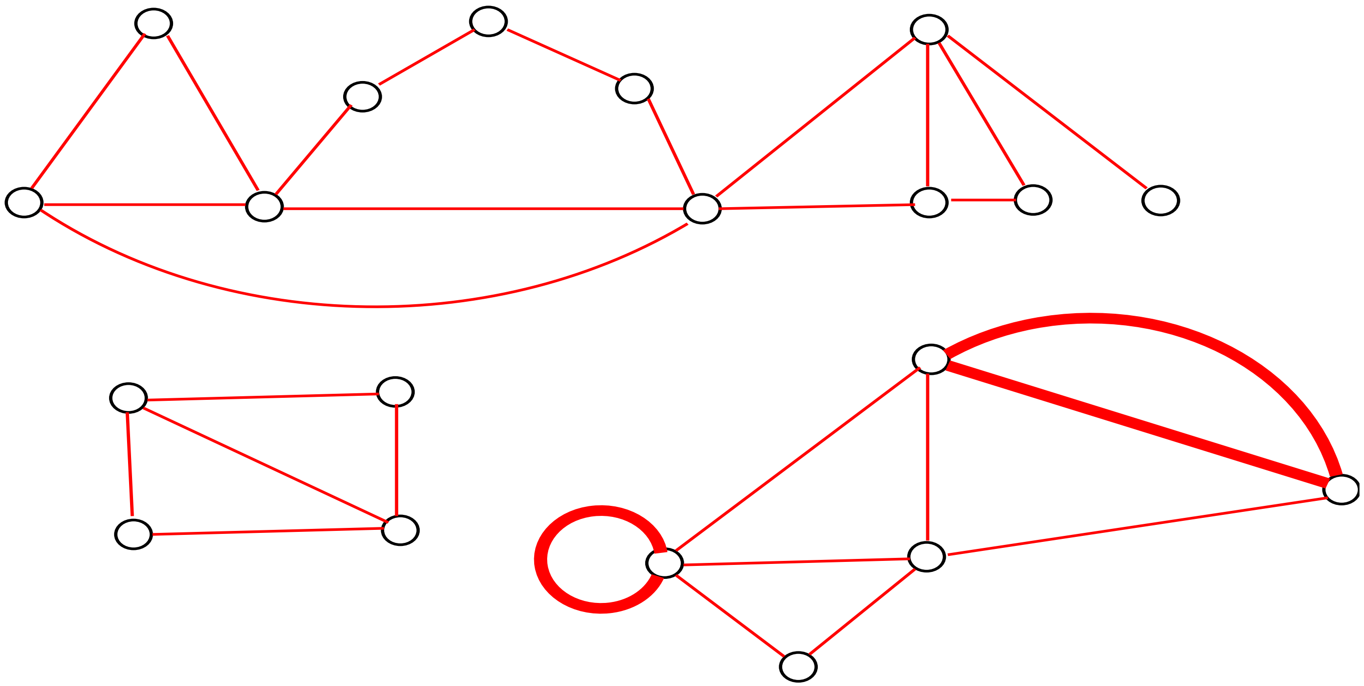
Special edges: loops and multiple edges



Graphs without loops and multiple edges: **simple** graphs

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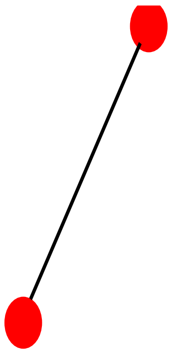
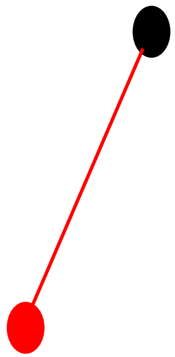
Adjacency and incidence

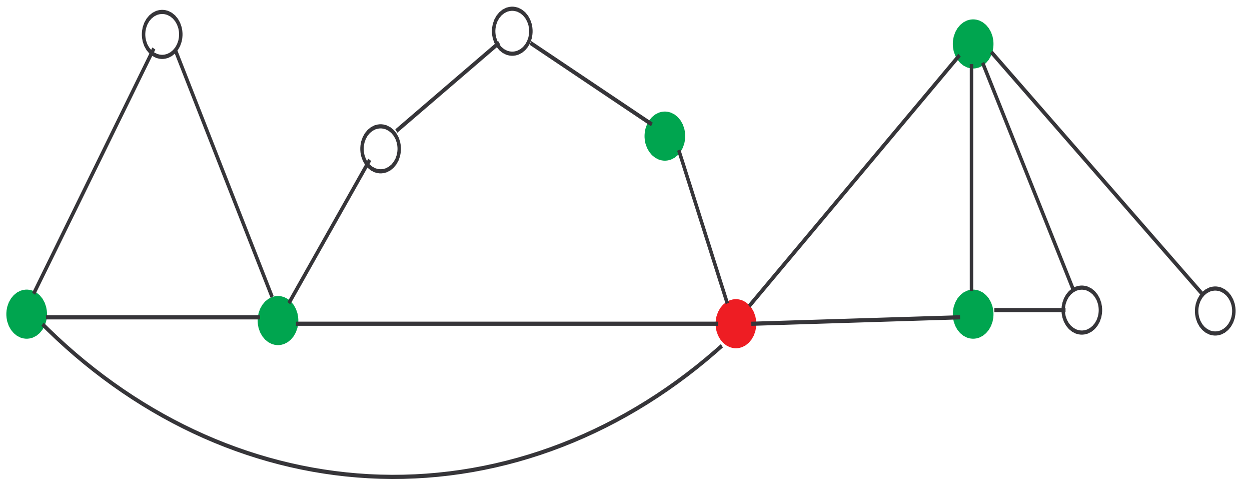




Basic Concepts of Graph Theory

A vertex v and the set of its neighbours $\Gamma(v)$

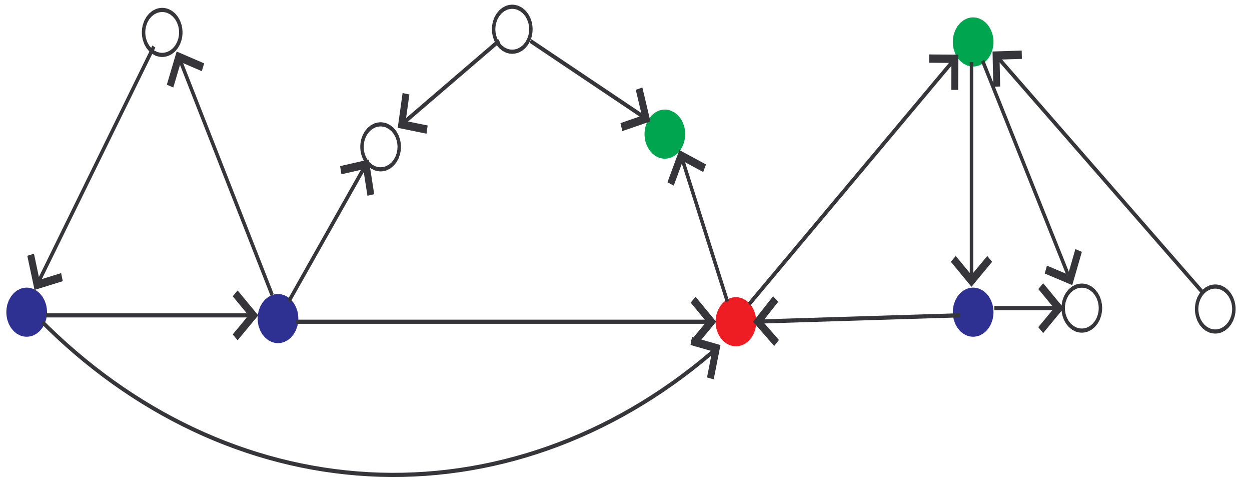
$d(v) = d_G(v) = |\Gamma(v)|$ = the degree of v



$\delta(G) = \min_{v \in V} d(v)$, $\Delta(G) = \max_{v \in V} d(v)$

Basic Concepts of Graph Theory

Directed case: successors $\Gamma^+(v)$ and predecessors $\Gamma^-(v)$
$d^+(v) = |\Gamma^+(v)|$ and $d^-(v) = |\Gamma^-(v)|$: out-degree and indegree of v, respectively.



Basic Concepts of Graph Theory

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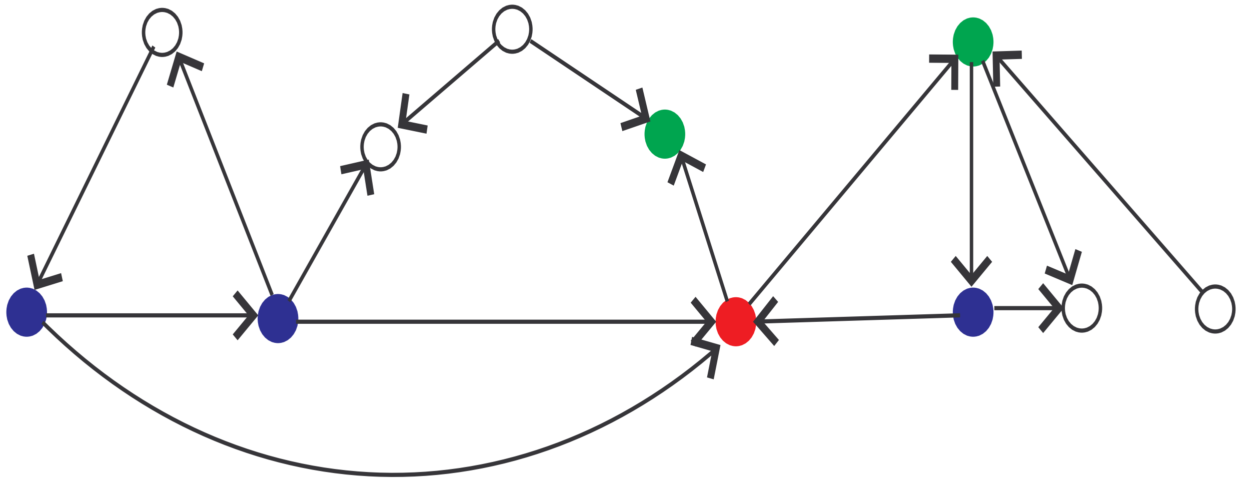

Theorem (Handshaking lemma)

$$\sum_{x \in V(G)} d(x) = 2|E(G)|, \text{ \textit{directed case:} } \sum_{x \in V(G)} d^{+}(x) = \sum_{x \in V(G)} d^{-}(x) = |E(G)|$$

Basic Concepts of Graph Theory

Example: Hypercube

$G = (\{0,1\}^n, E)$

For $v = v_1 v_2 \cdots v_n$ and $w = w_1 w_2 \cdots w_n$ we stipulate

$$vw \in E :\Longleftrightarrow \sum_{i=1}^{n} |v_i - w_i| = 1.$$

Basic Concepts of Graph Theory

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Then, for all $x \in V$ we have $d(x) = n$ and $|V| = 2^n$.

Basic Concepts of Graph Theory

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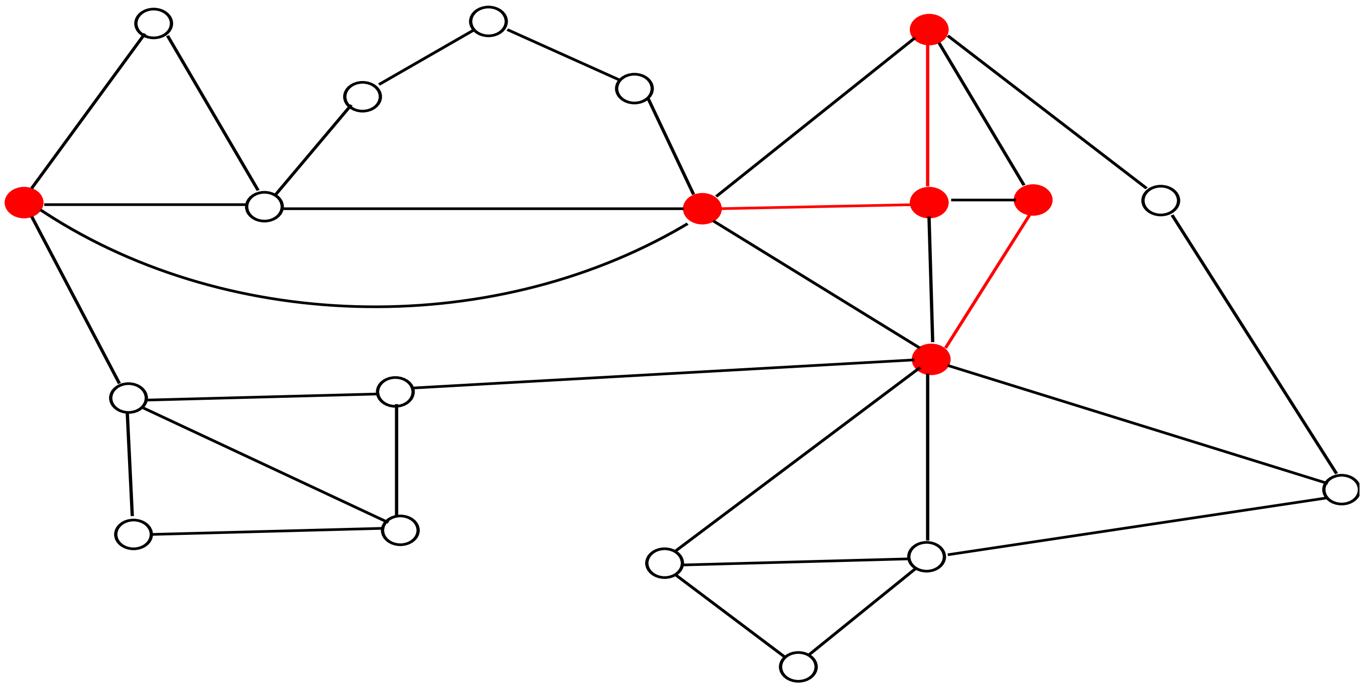
So,

$$|E| = \frac{1}{2} \sum_{x \in V} d(x) = \frac{1}{2} \sum_{x \in V} n = \frac{1}{2} \cdot n2^n = n2^{n-1}$$

Basic Concepts of Graph Theory

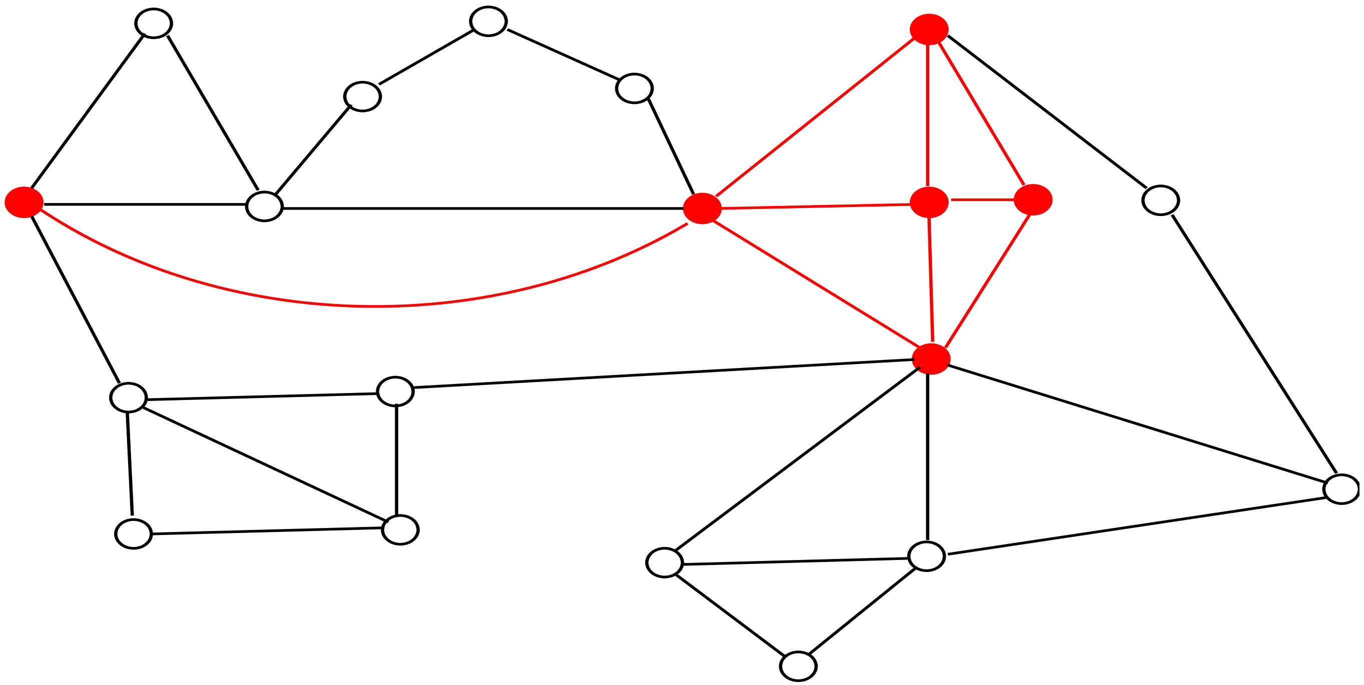
A graph G and one of its subgraphs, G'



$$G' = (V', E'), \qquad V' \subseteq V, \quad E' \subseteq E$$

Basic Concepts of Graph Theory

Induced subgraphs of $G = (V, E)$: $G[V_0]$ determined by its vertex set $V_0 \subseteq V$



edge set of $G[V_0]$ maximal w.r.t. inclusion

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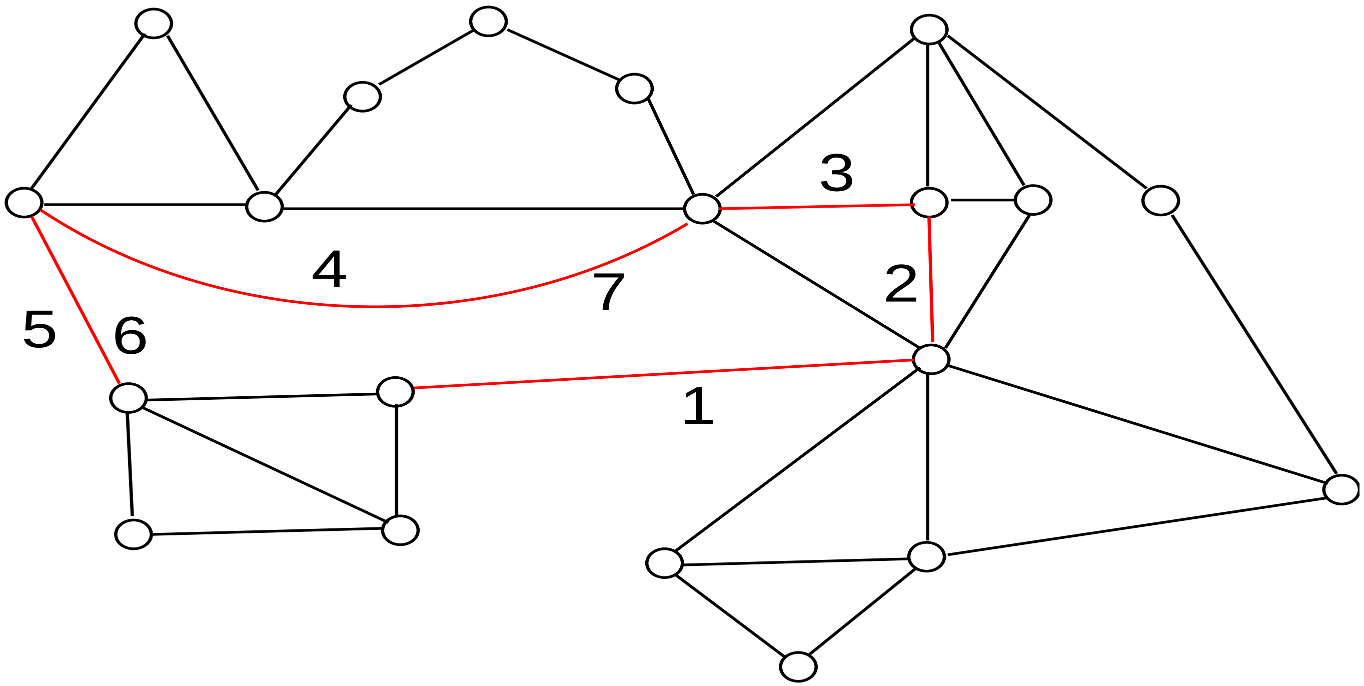
sequences of edges with "no jumps" = walks

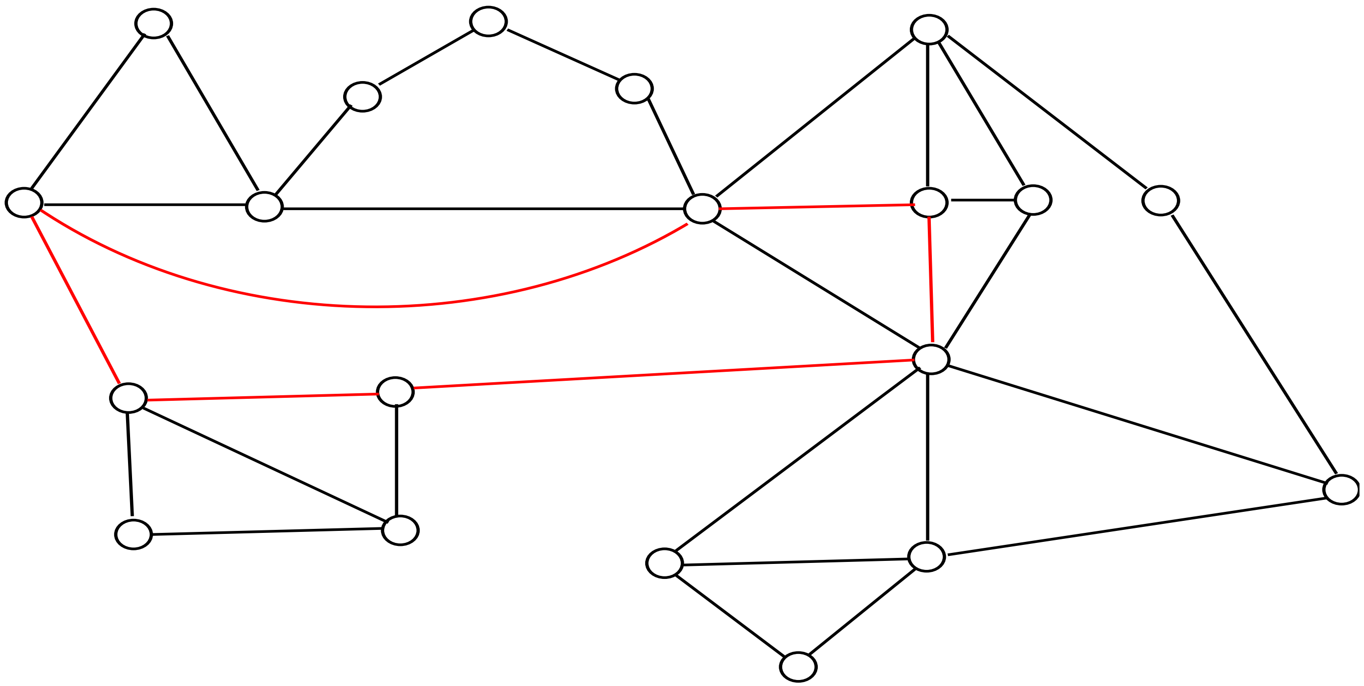


trails: no edge repetition allowed

paths: no edge and no vertex repetition allowed
(actually, paths are particular subgraphs)

Basic Concepts of Graph Theory

Particular walks: cylces (no edge and no vertex repetition allowed)
(actually, cycles are particular subgraphs)



circuits (tours): no edge repetition, but vertex repetition allowed,
(circuits are therefore closed trails, but can also be seen as subgraphs)

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Theorem *If there is a walk from v to w, then there is a path from v to w as well.*

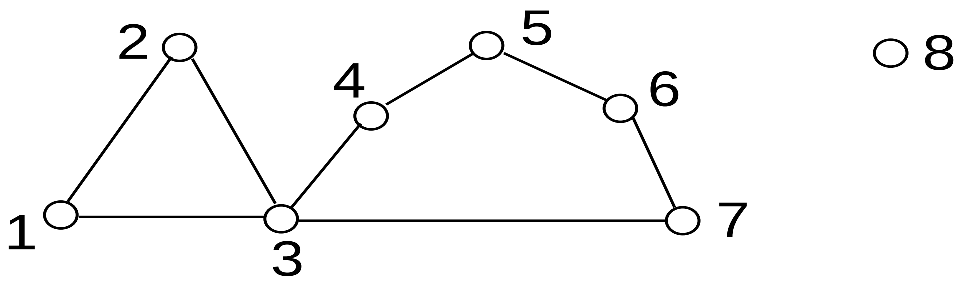
Theorem *If in an undirected graph there exist two different paths from v to w, then there is a cycle (of positive length).*

If in a directed graph there exists a closed walk, then there is a cycle (of positive length).

Basic Concepts of Graph Theory

The adjacency matrix of $G = (V, E)$:

$V = \{v_1 \ldots, v_n\}$, $A = (a_{ij})_{i,j=1,\ldots,n}$ with $a_{ij} = \begin{cases} 1, & \text{if } (v_i, v_j) \in E, \\ 0 & \text{else.} \end{cases}$



$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Basic Concepts of Graph Theory

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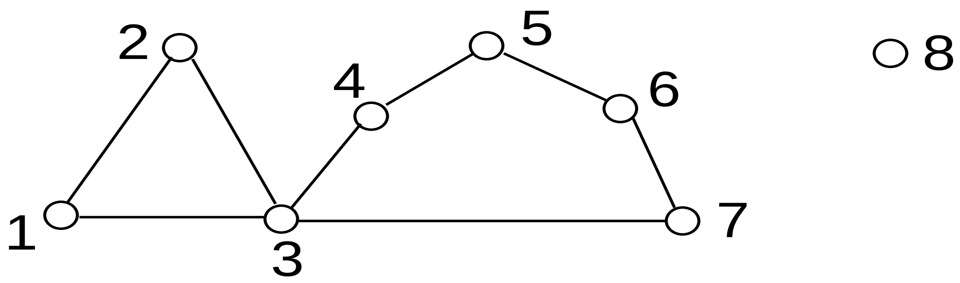


$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We have: $d(v_i) = \sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{ji}$

Basic Concepts of Graph Theory

Theorem *Let $G = (V, E)$ be an undirected graph and $v \sim w$ by definition if and only if there exists a (possibly empty) walk from v to w. Then $\sim$ is an equivalence relation.*

Theorem *Let $G = (V, E)$ be a directed graph and $v \sim w$ by definition if and only if there exists a (possibly empty) walk from v to w and likewise a (possibly empty) walk from w to v. Then $\sim$ is an equivalence relation.*

Matrix of the relation $\sim$ (undirected case): Let $G = (V, E)$ be undirected with $|V| = n$ and $|E| = m$.

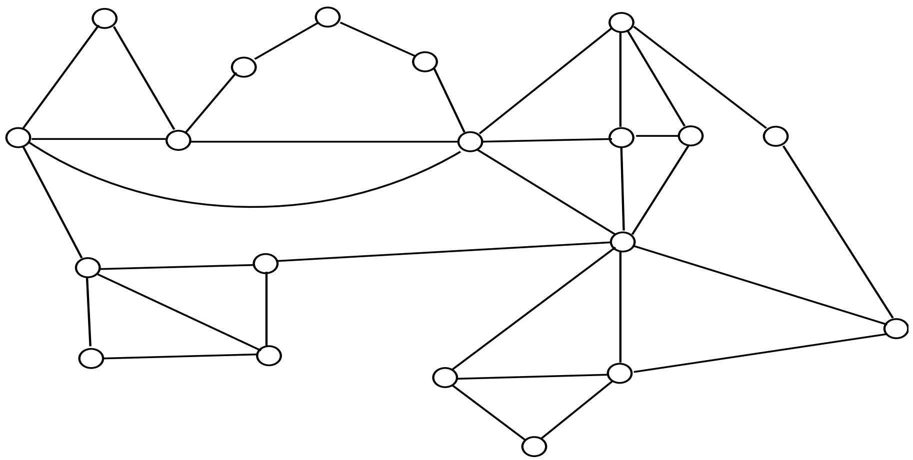
$$M = (m_{i,j})_{i,j=1,\ldots,n}, \text{ where } m_{i,j} = \mathrm{sgn}(c_{i,j})$$

and

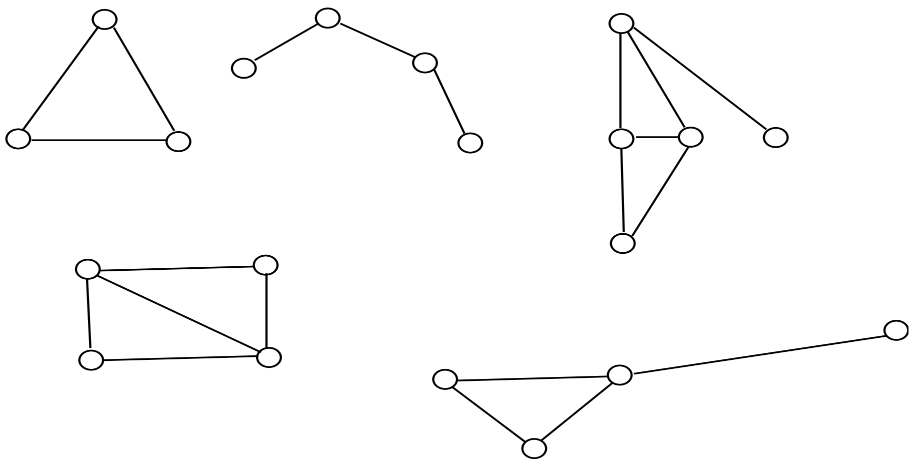
$$C = \sum_{k=0}^{\min(m,n-1)} A^k.$$

Basic Concepts of Graph Theory

connected graph

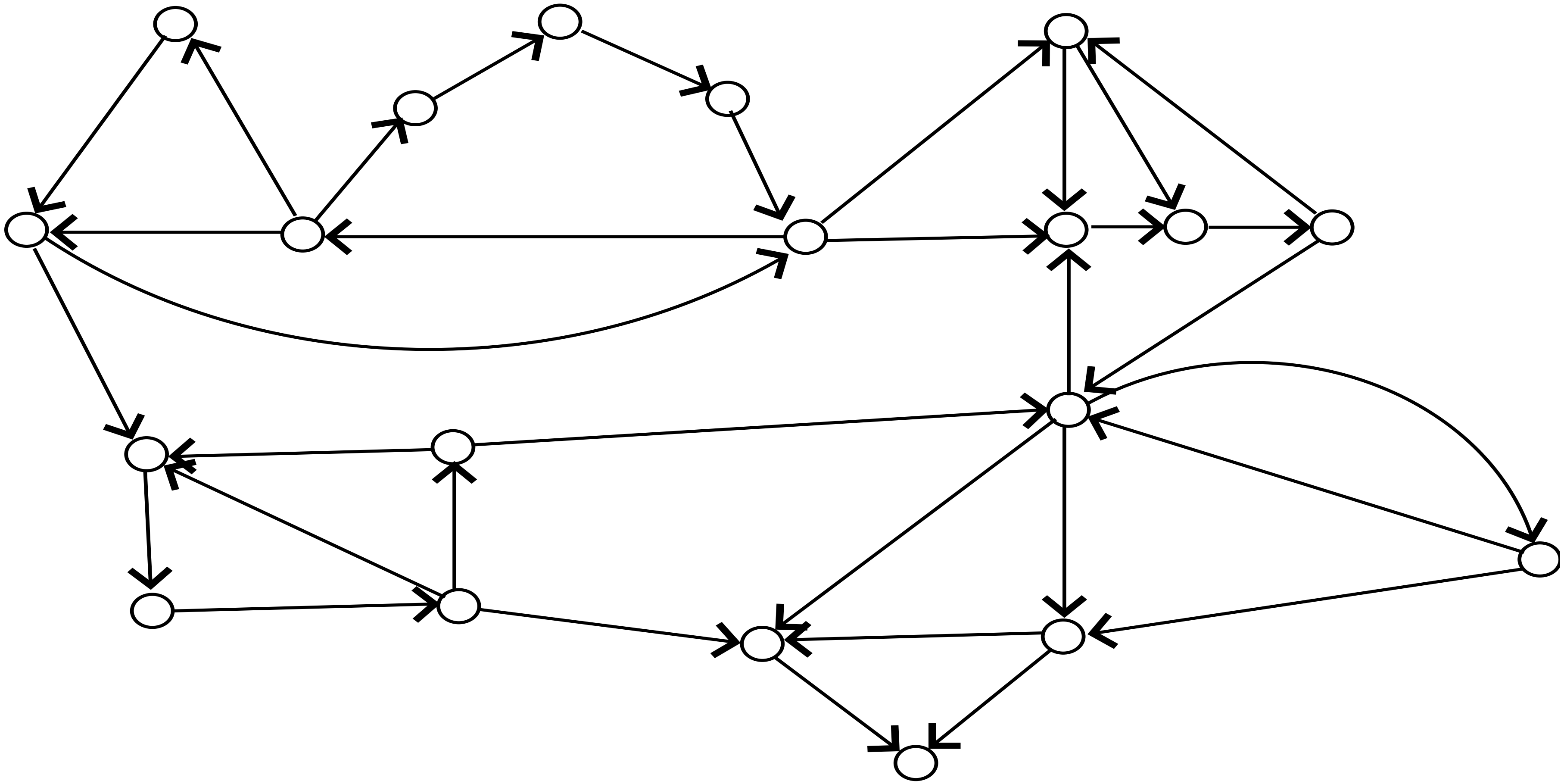


not connected graph



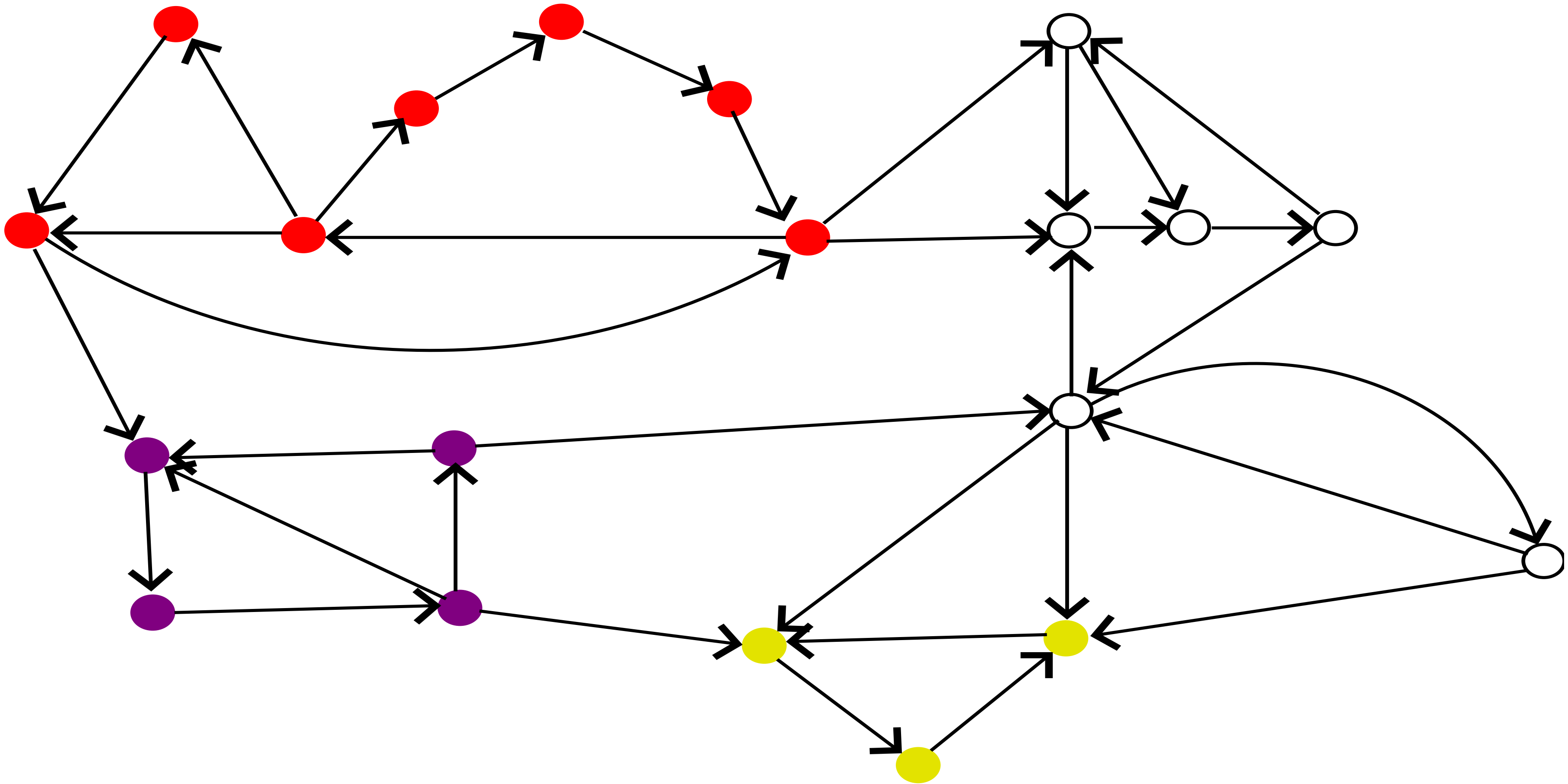
Basic Concepts of Graph Theory

A weakly, but not strongly connected graph



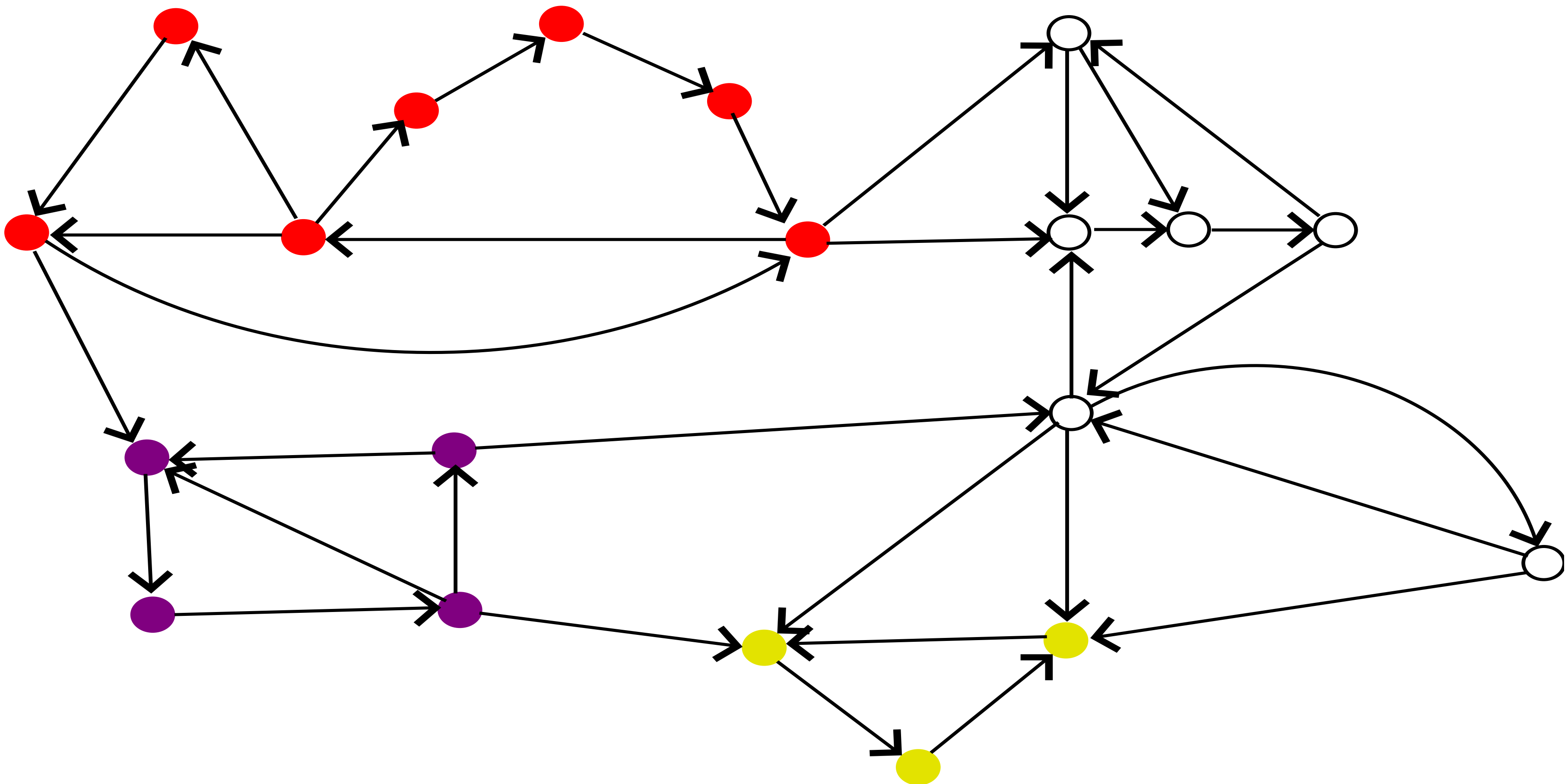
Basic Concepts of Graph Theory

The strongly connected components



Basic Concepts of Graph Theory

Reduction $G_R = (V_R, E_R)$: $V_R = \{$ strongly connected components $\}$, E_R: directions given by $G = (V, E)$.



Basic Concepts of Graph Theory

A vertex basis of a directed graph $G = (V, E)$ is a set $B \subseteq V$ such that

- For all $x \in V$ there is a vertex $y \in B$ such that there exists a (directed) path $y \leadsto x$
- B is minimal with that property, i.e. for all $B' \subsetneq B$ there is an $x \in V$ such that for all $y \in B'$ there is no (directed) path $y \leadsto x$.

Basic Concepts of Graph Theory

A vertex basis of a directed graph $G = (V, E)$ is a set $B \subseteq V$ such that

- For all $x \in V$ there is a vertex $y \in B$ such that there exists a (directed) path $y \rightsquigarrow x$
- B is minimal with that property, i.e. for all $B' \subsetneq B$ there is an $x \in V$ such that for all $y \in B'$ there is no (directed) path $y \rightsquigarrow x$.

Remarks:

- The reduction G_R of a graph $G = (V, E)$ is always acyclic.
- The reduction G_R of a graph $G = (V, E)$ has a unique vertex basis, namely

$$B = \{x \in G_R \mid d^-(x) = 0\}$$

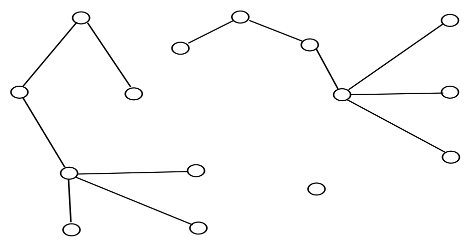
- The vertex bases of G constructible from the vertex basis of G_R.

Basic Concepts of Graph Theory

Trees and Forests

A simple undirected graph without cycles of positive length is called *forest*.

A connected forest is called *tree*.

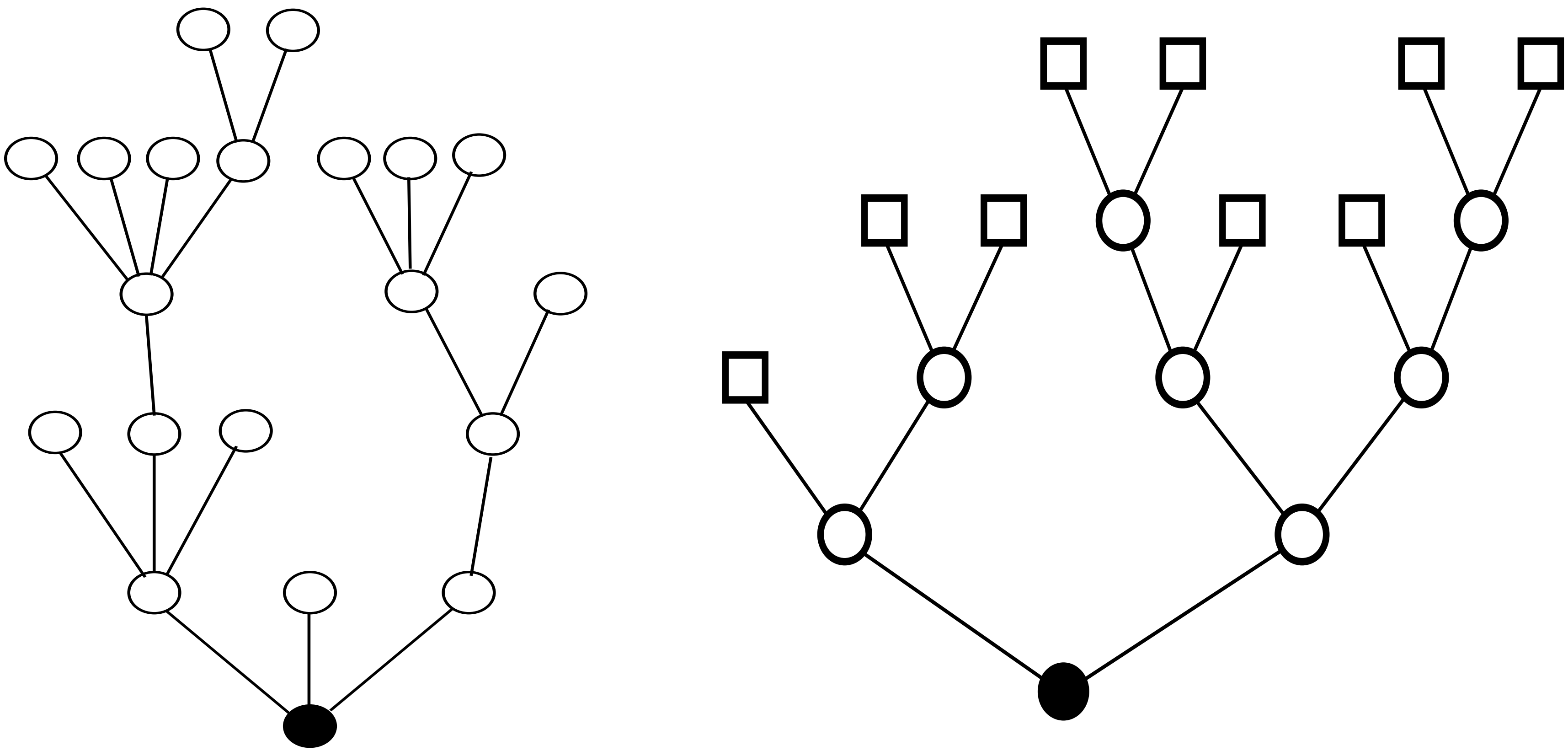


Theorem *In a tree $T = (V, E)$ any two vertices $v, w \in V$ are connected by a unique path $W(v, w)$.*

The length of $W(v, w)$ is denoted by $d_T(v, w)$ and called the *distance* between v and w

Basic Concepts of Graph Theory

Particular classes of trees: rooted trees, plane rooted trees, binary trees, . . .



Basic Concepts of Graph Theory

A vertex v with $d(v) = 1$ is called a *leaf*.

Theorem *A tree with at least two vertices has at least two leaves.*

Proof: Consider a path

$$v - v_1 - v_2 - v_3 - \cdots\cdots - v_k - w$$

of maximal length. Then v and w must be leaves.

Basic Concepts of Graph Theory

Theorem *Let $T = (V, E)$. Then the following statements are equivalent:*

1. *T is a tree.*

2. *For all $v, w \in V(T)$ there is a unique path from v to w.*

3. *T is connected and $|V| = |E| + 1$.*

4. *T is a minimal connected graph (every edge is a bridge)*

5. *T is a maximal acyclic graph.*

Basic Concepts of Graph Theory

Proof: (1)$\Longrightarrow$(3), that is

"If $T = (V, E)$ is a tree, then it is connected and satisfies $|V| = |E| + 1$."

We prove the state by induction on $n = \alpha_0(T) = |V(T)|$.

Induction start: $\alpha_0(T) = 1$, $\alpha_1(T) = |E(T)| = 0$.

Now consider a tree $T = (V, E)$ with $n + 1$ vertices. Then there is a leaf v and let e be the edge incident to v. Let $T' = (V \setminus \{v\}, E \setminus \{e\})$.

As $\alpha_0(T') = n$, we can apply the induction hypothesis to T'.

Basic Concepts of Graph Theory

(3)$\Longrightarrow$(1), that is

"$T = (V, E)$ connected and $|V| = |E| + 1 \Longrightarrow T$ is a tree."

Set $n = |V|$.
If T has no cycle, we are done.

If T has a cycle, then remove an edge from the cycle. $\longrightarrow$ graph T'.

T' is connected and cycle-free and has $n - 2$ edges.

But every connected graph on n vertices has at least $n - 1$ edges ↯
(proof by induction)

Basic Concepts of Graph Theory

A *spanning tree* of a connected graph $G = (V, E)$ is a subgraph T of G such that T is a tree, $V(T) = V(G)$, $E(T) \subseteq E(G)$.

A *spanning forest* of a graph $G = (V, E)$ is a subgraph F of G such that F is a forest, $V(F) = V(G)$, $E(F) \subseteq E(G)$, each connected component of F is a spanning tree of a connected component of G.

Remark: If a subgraph H of G satisfies $V(H) = V(G)$, then it is called a spanning subgraph.

Theorem *Every connected graph contains a spanning tree.*