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Optimum Quantization and Its Applications

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Abstract. Minimum sums of moments or, equivalently, distorsion of optimum quantizers play an important role in several branches of mathematics. Fejes Tóth's inequality for sums of moments in the plane and Zador's asymptotic formula for minimum distortion in Euclidean d-space are the first precise pertinent results in dimension $d \geq 2$. In this article these results are generalized in the form of asymptotic formulae for minimum sums of moments, resp. distortion of optimum quantizers on Riemannian d-manifolds and normed d-spaces. In addition, we provide geometric and analytic information on the structure of optimum configurations. Our results are then used to obtain information on

(i) the minimum distortion of high resolution vector quantization and optimum quantizers,
(ii) the error of best approximation of probability measures by discrete measures and support sets of best approximating discrete measures,
(iii) the minimum error of numerical integration formulae for classes of Hölder continuous functions and optimum sets of nodes,
(iv) best volume approximation of convex bodies by circumscribed convex polytopes and the form of best approximating polytopes, and
(v) the minimum isoperimetric quotient of convex polytopes in Minkowski spaces and the form of the minimizing polytopes.

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1 Introduction

1.1 Fejes Tóth's inequality for sums of moments in the plane. Let $f : [0, +\infty) \to [0, +\infty)$ be increasing, where $f(0) = 0$, and let H be a convex 3, 4, 5, or 6-gon in the Euclidean plane $\mathbb{E}^2$. Then

$$(1.1) \qquad \inf_{\substack{S \subset \mathbb{E}^2 \\ \#S = n}} \Big\{ \int\limits_H \min_{s \in S} \{ f(\|s-u\|_2)\} du \Big\} \geq n \int\limits_{H_n} f(\|u\|_2) du,$$

where $n = \#S$ is the number of points of S and H_n a regular convex hexagon in $\mathbb{E}^2$ of area $|H|/n$ and center at the origin o. $\|\cdot\|_2$ and $|\cdot|$ stand for the Euclidean norm and the ordinary (area) measure. There are more than a dozen proofs for this result of L. Fejes Tóth [17] and its variants. Its applications range from packing and covering problems for solid circles, problems of optimal location in econometrics, quantization of data, Gauss channels and numerical integration, all in $\mathbb{E}^2$, to the isoperimetric problem for special convex polytopes and asymptotic approximation of convex bodies in $\mathbb{E}^3$. See Matérn and Persson [39] and Gruber [29] for references. Related results, including results on finite difference methods, cellular biology and territorial behaviour of animals are surveyed by Du, Faber and Gunzburger [12].

The inequality (1.1) indicates that for certain geometric and analytic problems in $\mathbb{E}^2$ or $\mathbb{E}^3$ regular hexagonal configurations are close to optimal, possibly optimal.

1.2 The corresponding planar stability problem. The objective here is to describe the sets $S = S_n$ for which equality or approximate equality holds in (1.1). Although for certain functions f no clear cut answer is possible, Gruber [28, 30] showed that in $\mathbb{E}^2$ and on Riemannian 2–manifolds under suitable restrictions for f, the sets S_n form 'asymptotically regular hexagonal patterns' as $n \to \infty$. For $\mathbb{E}^2$ a similar, slightly more general result was given by G. Fejes Tóth [15].

The stability result implies that for certain geometric and analytic problems in $\mathbb{E}^2$, $\mathbb{E}^3$ and on Riemannian 2–manifolds, the optimal or almost optimal configurations are 'asymptotically regular hexagonal pattern'. For examples see Gruber [30].

1.3 Extensions to general dimensions. From the point of view of applications the following extensions suggest themselves. First, determine or estimate the expressions

$$(1.2)\quad \inf_{\substack{S\subset M\\ \#S=n}} \Big\{ \int_J \min_{p\in S}\{f(\varrho_m(p,x))\} w(x)\,d\omega_M(x)\Big\},$$

and

$$(1.3)\quad \inf_{\substack{S\subset \mathbb{E}^d\\ \#S=n}} \Big\{ \int_J \min_{s\in S}\{f(\|s-u\|)\} w(u)\,du\Big\}.$$

Here J is a (Jordan) measurable set in a Riemannian d-manifold M with Riemannian metric ϱ_M and (area) measure ω_M, or in $\mathbb{E}^d$, where the latter is endowed with an additional norm $\|\cdot\|$, and $w : J \to \mathbb{R}^+$ a weight function. Second, describe the corresponding minimizing or almost minimizing configurations. While precise answers to these problems are out of reach, asymptotic estimates and upper and lower bounds for the expressions in (1.2) and (1.3) and information on the minimizing configurations have been given for special functions f, particular norms, and $d = 2$:

In the context of information theory Zador [48] proved that for $f(t) = t^\alpha, \alpha > 0$, and $\|\cdot\| = \|\cdot\|_2$ the expression in (1.3) is asymptotically equal to

$$\Big(\int_J w(u)^{\frac{d}{\alpha+d}}du\Big)^{\frac{\alpha+d}{d}} \frac{1}{n^{\frac{\alpha}{d}}} \text{ as } n \to \infty,$$

omitting a multiplicative constant; for related, more recent work in information theory see the survey of Gray and Neuhoff [21]. The results in information theory were used to approximate probability measures by discrete measures, see the report of Graf and Luschgy [20]. In the disguise of an asymptotic formula for the minimum error of numerical integration formulae the same result, but for general norms instead of $\|\cdot\|_2$, was proved by Chernaya [7]. A similar formula with f from a wide class of moduli of continuity including the functions of the form t^α and where the asymptotics is described by a function related to f, was given in the same year by Chernaya [8]. Finally, Gruber [26] and Glasauer and Gruber [19] proved in the context of convex geometry that the expression in (1.2) for $f(t) = t^2$ is asymptotically equal to

$$\Big(\int_J w(x)^{\frac{d}{d+2}} d\omega_M(x) \Big)^{\frac{d+2}{d}} \frac{1}{n^{\frac{2}{d}}} \text{ as } n \to \infty,$$

again omitting a multiplicative constant. It is curious to note that all these closely related results were found independently in different areas of mathematics. The proofs differ, but a crucial idea, a similarity argument, which was communicated to Zador by Hammersley, is the same in each of them. In [33] we will describe a proof for general norms which is also based on it.

If f is strictly increasing, then for $S = \{s_{n1}, \dots, s_{nn}\} \subset \mathbb{E}^d$,

$$\int_J \min_{i=1,\dots,n} \{f(\|s_{ni} - u\|_2)\} w(u) du = \sum_{i=1}^n \int_{D_{ni}} f(\|s_{ni} - u\|_2) w(u) du,$$

where the *Dirichlet–Voronoi cells* $D_{ni}, i = 1, \dots, n$, in J corresponding to S are defined by

$$D_{ni} = \{u \in J : \|s_{ni} - u\|_2 \le \|s_{nj} - u\|_2 \text{ for } j = 1, \dots, n\}.$$

(This explains why we speak of sums of moments.) A *conjecture of Gersho* [18] asserts that for $f(t) = t^2$ or, more generally, for strictly increasing f, and for $\|\cdot\| = \|\cdot\|_2$ and $w = 1$, there is a convex polytope P with $|P| = 1$ which admits a tiling of $\mathbb{E}^d$ by congruent copies such that the following holds: as $n \to \infty$, the cells D_{ni} which correspond to minimizing configurations $S = S_n$, where $\#S_n = n$, are 'asymptotically congruent' to $(|J|/n)^{1/d}P$. For $d = 2$ the author's stability result stated above readily implies the truth of this conjecture, showing also that P is a regular hexagon. This seems to be the first proof of Gersho's conjecture for $d = 2$. Also G. Fejes Tóth's [15] version of the stability theorem yields a proof of Gersho's conjecture for $d = 2$. (We point out that no proof is provided by [16] or [42], contrary to positive statements in the literature.) For $d \ge 3$ it is not clear that the conjecture is valid and if so, at present, proofs seem to be out of reach.

Assuming the truth of Gersho's conjecture, lower bounds for the expression in (1.3) have been given for certain functions f and certain norms. See [33] for simple arguments which also lead to lower bounds. For references see the survey of Gray and Neuhoff [21].

General references for results as described above are Gray and Neuhoff [21], Du, Faber and Gunzburger [12], Graf and Luschgy [20] and Gruber [33]. The latter is an easily comprehensible introduction to the present article.

1.4 Results. We will give asymptotic formulae for the expressions in (1.2) and (1.3) for a wide class of functions f. Further, geometric and analytic information on the minimizing configurations $S = S_n$ will be provided; roughly speaking, S_n is distributed over J rather uniformly and regularly. The hard part of the proof is to show that S_n is Delone (Section 2).

These results admit interpretations and applications to quantization of data, approximation of probability measures by discrete probability measures, numerical integration, approximation of convex bodies by polytopes and the isoperimetric problem for convex polytopes.

An important step in data transmission is to encode signals (in many cases vectors in $\mathbb{E}^d$) produced by a source into codewords (particular vectors) from a codebook consisting of, say, n codewords (and which are then transmitted in a channel). So-called distortions measure the quality of this encoding or quantizing process. See [21] for the pertinent literature starting with Shannon and Zador. In Shannon's theory distortion is estimated as $d \to \infty$, while in Zador's high resolution theory estimates for $n \to \infty$ are considered. Our results immediately imply asymptotic formulae for the minimum distortion of quantization as the number n of codewords tends to infinity and give information about the structure of optimum codebooks (Section 3).

Given a probability measure on a Jordan measurable subset of $\mathbb{E}^d$, it is a natural question to ask, how well can it be approximated by discrete measures. Here the space of all probability measures is endowed with the Wasserstein or Kantorovich metric, or generalizations of it. See Graf and Luschgy [20]. Our results yield asymptotic formulae for the error of the best approximation as the number of support points of the approximating discrete measures tends to infinity. In addition, information on the structure of the support of the best approximating discrete measures is provided (Section 4).

Consider a class of Riemann integrable real functions on a Jordan measurable subset of $\mathbb{E}^d$ or of a Riemannian manifold M. In general it is difficult to estimate the minimum error of numerical integration formulae with n nodes and weights. Pertinent results have been given in the context of uniform distribution theory by Koksma and Hlawka, see [34], further for Sobolev spaces of functions by Sobolev [46] and his school, including Polovinkin [43], and for classes of Lipschitz or Hölder continuous functions by Babenko [2] for $d = 2$ and Sobol' [45], Chernaya [7, 8] and others for general d. Our results yield for certain Hölder classes of functions asymptotic formulae for the minimum error of numerical integration formulae as the number n of nodes and weights tends to infinity. (Our classes are slightly less general than those of Chernaya.) Moreover, we obtain information about the structure of optimal sets of nodes (Section 5).

The special case $f(t) = t^2$ of our result for Riemannian manifolds yields asymptotic formulae for best volume approximation of convex bodies by circumscribed convex polytopes as the number of facets tends to ∞. An argument from the proof of our manifold result shows that the facets of the best approximating polytopes are all 'rather round' and 'roughly of the same size' (Section 6).

The latter results can be interpreted as asymptotic formulae for the minimum isoperimetric quotient of convex polytopes in a Minkowski space with n facets as $n \to \infty$ and as information about the form of polytopes with minimum isoperimetric quotient (Section 7).

2 Sums of Moments on Riemannian d-Manifolds and Normed d-Spaces

2.1 Preliminaries. A function $f : [0, +\infty) \to [0, +\infty)$ satisfies the *growth condition* if

(2.1) $f(0) = 0$, f is continuous and strictly increasing and, for any given $s > 1$, the quotient $f(st)/f(t)$ is decreasing and bounded above for $t > 0$.

In addition to the positive powers of t there are many other functions $f : [0, +\infty) \to [0, +\infty)$ which satisfy the growth condition. This is shown by the following result. We state it without proof.

Proposition. *Let $f : [0, +\infty) \to [0, +\infty)$ be such that $f(t) = 0$ only for $t = 0$. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(u) = \log f(e^u)$ for $u \in \mathbb{R}$. Then the following statements are equivalent:*
(i) *f satisfies the growth condition,*
(ii) *g is strictly increasing, concave and Lipschitz.*

Let M be a *Riemannian d-manifold.* By this we mean a d-dimensional manifold of class $\mathcal{C}^3$ with a metric tensorfield (which has coefficients) of class $\mathcal{C}^1$. Let ϱ_M and ω_M be the corresponding *Riemannian metric* and *(area) measure* on M. A set $J \subset M$ is *(Jordan* or *Riemann) measurable* if its closure $\mathrm{cl}\, J$ is compact and $\omega_M(\mathrm{bd}\, J) = 0$, where $\mathrm{bd}\, J$ is the boundary of J. See 2.3 for an alternative definition of neasurability. A measurable set $J \subset M$ has *positive density* if

(2.2) there are $\beta > 1, \gamma > 0$, such that $B_M(p, \varrho)$ is measurable and
$$\frac{\varrho^d}{\beta} \le \omega_M(B_M(p, \varrho) \cap J) \le \beta \varrho^d \,\text{for}\, p \in \mathrm{cl}\, J, 0 < \varrho < \gamma,$$

where the *ball* $B_M(p, \varrho)$ with center p and radius ϱ is the set $\{x \in M : \varrho_M(p, x) \le \varrho\}$. The measurability of $B_M(p, \varrho)$ for sufficiently small γ follows from an argument using the exponential mapping, compare the proof of (2.12). Let $J \subset M$ be measurable with $\omega_M(J) > 0$, let (S_n) be a sequence of sets in M with $\# S_n = n$, and let $\delta > 1$. S_n is a $(1/\delta n^{1/d}, \delta/n^{1/d})$-*Delone set* in J for $n = 1, 2, \dots$, if any two distinct points of S_n have distance at least $1/\delta n^{1/d}$ and for each point of J there is a point of S_n at distance at most $\delta/n^{1/d}$. S_n is *uniformly distributed* in J as $n \to \infty$ *with respect to* an integrable function, a *density,* $w : J \to \mathbb{R}^+$ if for any measurable set $K \subset J$ we have,

$$\#(K \cap S_n) \sim \Big(\int\limits_K w(x) d\omega_M(x) \Big/ \int\limits_J w(x) d\omega_M(x) \Big) n \text{ as } n \to \infty.$$

2.2 Asymptotic results. We will prove the following result, where $\alpha > 0$ is a constant corresponding to f, see (2.14).

Theorem 1. *Let $f : [0, +\infty) \to [0, +\infty)$ satisfy the growth condition (2.1) and let $\alpha > 0$ be the corresponding constant. Then there is a constant* $\operatorname{div} > 0$, *depending only on f and d, such that the following is true: let M be a Riemannian d-manifold, $J \subset M$ a compact measurable set with $\omega_M(J) > 0$, and $w : J \to \mathbb{R}^+$ continuous. Then*

(i) $$F_n = \inf_{\substack{S \subset M \\ \#S = n}} \Big\{ \int\limits_J \min_{p \in S} \{ f(\varrho_M(p, x)) \} w(x) d\omega_M(x) \Big\}$$
$$\sim \operatorname{div} \Big(\int\limits_J w(x)^{\frac{d}{\alpha+d}} d\omega_M(x) \Big)^{\frac{\alpha+d}{d}} f\Big(\frac{1}{n^{\frac{1}{d}}} \Big) \text{ as } n \to \infty.$$

If, in addition, J is connected and has positive density, see (2.2), then the following hold: let (S_n) be a sequence of sets $S_n \subset M$ with $\#S_n = n$ such that the infimum in (i) is attained for $S = S_n, n = 1, 2, \dots$. Then

(ii) *there is a constant $\delta > 1$ such that S_n is $(1/\delta n^{1/d}, \delta/n^{1/d})$-Delone in J for $n = 1, 2, \dots$,*

(iii) *S_n is uniformly distributed in J with density $w^{d/(\alpha+d)}$ as $n \to \infty$.*

Analogous results hold under the assumption that $S, S_n \subset J$.

A proof which is similar to that of Theorem 1, but with suitably distorted sets S_n and which makes use of a result of Ewald, Larman and Rogers [14], yields the next result.

Theorem 2. *Let $f : [0, +\infty) \to [0, +\infty)$ satisfy the growth condition (2.1), let $\alpha > 0$ be the corresponding constant, and let $\| \cdot \|$ be a norm on $\mathbb{E}^d$. Then there is a constant* $\operatorname{div} > 0$, *depending only on f and $\| \cdot \|$, such that the following holds: let $J \subset \mathbb{E}^d$ be compact and measurable with $|J| > 0$, and $w : J \to \mathbb{R}^+$ continuous. Then*

(i) $$\inf_{\substack{S \subset \mathbb{E}^d \\ \#S = n}} \Big\{ \int\limits_J \min_{s \in S} \{ f(\|s - u\|) \} w(u) du \Big\} \sim \operatorname{div} \Big(\int\limits_J w(u)^{\frac{d}{\alpha+d}} du \Big)^{\frac{\alpha+d}{d}} f\Big(\frac{1}{n^{\frac{1}{d}}} \Big)$$
as $n \to \infty$.

If, in addition, J is connected and has positive density, see (2.2), then the following assertions hold: let (S_n) be a sequence of sets $S_n \subset \mathbb{E}^d$ with $\#S_n = n$ such that the infimum in (i) is attained for $S = S_n, n = 1, 2, \dots$. Then

(ii) *there is a constant $\delta > 1$ such that S_n is $(1/\delta n^{1/d}, \delta/n^{1/d})$-Delone in J for $n = 1, 2, \dots$,*

(iii) *S_n is uniformly distributed in J with ensity $w^{d/(\alpha+d)}$ as $n \to \infty$.*

Analogous results hold under the assumption that $S, S_n \subset J$.

Remark. Theorem 2 can be generalized to Finsler spaces where all tangent norms are isometric – as in Theorem 1. For general Finsler spaces there is also such a result, but then the constant div depends on J.

The proof of Theorem 1 will be given in Sections 2.3 - 2.6. To show the Delone property of S_n is the core of the proof. This property is then used to prove the asymptotic formula. The latter is the main tool for the proof that S_n is uniformly distributed. We are aware that there are more direct proofs of the asymptotic formula; compare the sketch of the proof of the asymptotic formula in Theorem 2 in [33], or the proof of a similar formula due to Chernaya [8].

2.3 Preparations for the proof. First, let M be a Riemannian d-manifold. Let U be an open neighborhood of a point $p \in M$ with corresponding homeomorphism $h =$"$\,'\,$" which maps U onto the open set $U' = h(U) \subset \mathbb{E}^d$. To each $u \in U'$ there corresponds a positive definite quadratic form q_u on $\mathbb{E}^d$.

A curve C in U is of class $\mathcal{C}^1$ if it has a parametrization $x : [a, b] \to U$ such that $u = h \circ x$ is a parametrization of class $\mathcal{C}^1$ of a curve in U'. The *length* of C then is defined to be

$$(2.3) \qquad \int_a^b q_{u(t)}(u^{\cdot}(t))^{\frac{1}{2}} dt.$$

By means of appropriate dissection and addition one can define the length for any curve in M which is piecewise of class $\mathcal{C}^1$ and which is not contained in a single neighborhood. The *Riemannian distance* $\varrho_M(x, y)$ of two points $x, y \in M$ is the infimum of the lengths of continuous curves in M which connect x, y and are piecewise of class $\mathcal{C}^1$. A curve in M of class $\mathcal{C}^1$ which connects two points $x, y \in M$ is a *(geodesic) segment* if its length is $\varrho_M(x, y)$. A *geodesic* is a curve of class $\mathcal{C}^1$ in M which consists locally of geodesic segments. A set $N \subset M$ is *geodesically convex* if for any $x, y \in N$ there is a unique geodesic segment connecting x, y and this segment is contained in N.

A set $J \subset U$ is (*Riemann* or *Jordan*) *measurable* in M if $h(J)$ is so in $\mathbb{E}^d$. Then its *(area) measure* $\omega_M(J)$ is defined by

$$\omega_M(J) = \int_{h(J)} (\det q_u)^{\frac{1}{2}} du,$$

where $du = du_1 \cdots du_d$. Again, by dissection and addition onecan define measurability and measure for sets which are not contained in a single neighborhood. (For an alternative definition see 2.1.)

If $J \subset U$ is measurable and $w : J \to \mathbb{R}$ continuous, then the *(Riemann) integral* of w on J is defined by

$$(2.4) \qquad \int_J w(x) d\omega_M(x) = \int_{h(J)} w(h^{-1}(u))(\det q_u)^{\frac{1}{2}} du.$$

Also here one can define integrals on sets not contained in a single neighborhood.

Using the exponential map (compare the proof of (2.12)) and simple arguments involving M one can show the following:

(2.5) Let $I \subset M$ be compact. Then there are $\varepsilon > 1, \vartheta > 0$ such that $B_M(p, \varrho)$ is measurable and

$$\frac{\varrho^d}{\varepsilon} \leq \omega_M(B_M(p, \varrho)) \leq \varepsilon \varrho^d \text{ for } p \in I, 0 < \varrho \leq \vartheta.$$

Let $J, w : J \to \mathbb{R}^+$ be as in Theorem 1 and let $\lambda > 1$. For each $p \in M$ choose $U, h =$"$\,'\,$", $U' = h(U)$, $q_u, q = q_{p'}$ as above where U is so small that the following claims hold true:

$$\frac{1}{\lambda} q(x' - y')^{\frac{1}{2}} \leq \varrho_M(x, y) \leq \lambda q(x' - y') \qquad \text{for } x, y \in U,$$
$$\frac{1}{\lambda} (\det q)^{\frac{1}{2}} \leq (\det q_u)^{\frac{1}{2}} \leq \lambda (\det q)^{\frac{1}{2}} \qquad \text{for } u \in U,$$
$$\frac{1}{\lambda} |K'| (\det q)^{\frac{1}{2}} \leq \omega_M(K) \leq \lambda |K'| (\det q)^{\frac{1}{2}} \qquad \text{for measurable } K \subset U,$$
$$\lambda \inf\{w(x) : x \in J \cap U\} \geq \sup\{w(x) : x \in J \cap U\}.$$

Let V be an open measurable neighborhood of p with $\operatorname{cl} V \subset U$.

As p ranges over the compact set J, the corresponding neighborhoods V form an open covering of J. Thus there is a finite subcover. Therefore there are, say, m points in J and corresponding neighborhoods U_l, V_l, homeomorphisms $h_l =$"$\,'\,$", and positive definite quadratic forms q_l, such that

(2.6) $$\frac{1}{\lambda} q_l(x' - y')^{\frac{1}{2}} \leq \varrho_M(x, y) \leq \lambda q_l(x' - y')^{\frac{1}{2}} \quad \text{for } x, y \in U_l,$$

(2.7) $$\frac{1}{\lambda} (\det q_l)^{\frac{1}{2}} \leq (\det q_u)^{\frac{1}{2}} \leq \lambda (\det q_l)^{\frac{1}{2}} \qquad \text{for } u \in U_l',$$

(2.8) $$\frac{1}{\lambda} |K'| (\det q_l)^{\frac{1}{2}} \leq \omega_M(K) \leq \lambda |K'| (\det q_l)^{\frac{1}{2}} \quad \text{for measurable } K \subset U,$$

for $l = 1, 2, \ldots, m$. Moreover, the sets

$$J_l = J \cap (V_l \backslash (V_1 \cup \ldots \cup V_{l-1})) \subset V_l \subset U_l, l = 1, \ldots, m,$$

have the following properties:

(2.9) J is the disjoint union of the measurable sets $J_l \subset V_l \subset U_l$, where V_l is open and measurable, U_l open and $\operatorname{cl} V_l \subset U_l, l = 1, \ldots, m$.

(2.10) $w_l \leq w(x) \leq \lambda w_l, \frac{1}{\lambda} W_l \leq w(x) \leq W_l$ for $x \in J_l$, where $w_l = \inf\{w(x) : x \in J_l\}, W_l = \sup\{w(x) : x \in J_l\}, l = 1, \ldots, m$.

Since J is compact and $w : J \to \mathbb{R}^+$ continuous,

(2.11) $0 < v = \inf\{w(x) : x \in J\} \leq \sup\{w(x) : x \in J\} = W < \infty$.

The following result will be used to show that Dirichlet– Voronoi cells are measurable; here int stands for interior.

(2.12) For each point of M there is a geodesically convex compact neighborhood N such that for any $p, q \in \text{int}N, p \neq q$, the *bisector* or *Leibnizian plane* $L^N(p,q) = \{x \in N : \varrho_M(p,x) = \varrho_M(x,q)\}$ of p, q in N is measurable with measure 0.

Since M is of class $\mathcal{C}^3$ with metric tensorfield of class $\mathcal{C}^1$, each point of M has a compact neighborhood N with the following properties: (i) $N \subset V$, where V is an open measurable neighborhood; (ii) N is geodesically convex; (iii) for each $p \in N$ the geodesic segments connecting p with $\text{bd}\, N$ cover $N \backslash \{p\}$ schlicht; (iv) for each $p \in \text{int}N$ there is a diffeomorphism $E_p^{-1} : V \to \mathbb{E}^d$, the inverse of the *exponential mapping* E_p, such that for each geodesic segment G connecting p with a point of $\text{bd}\, N$ the image $E_p^{-1}(G)$ is a line segment starting at o. See [5] or [36].

Now, given a point of M, let N be the neighborhood of it just described. N is compact and it is geodesically convex by (ii). Next, let $p, q \in \text{int}N, p \neq q$. Then (ii) implies that each geodesic segment connecting p and $\text{bd}\, N$ contains at most one point of $L^N(p,q)$. Clearly, $L^N(p,q)$ is closed in the compact set N. Hence $E_p^{-1}(L^N(p,q))$ is closed in the compact set $E_p^{-1}(N)$ and each ray starting at o meets $E_p^{-1}(L^N(p,q))$ at most once by (iii) and (iv). Thus $E_p^{-1}(L^N(p,q))$ has Lebesgue measure 0. Since it is compact it is (Jordan) measurable with (Jordan) measure 0. This yields (2.12).

Secondly, f will be investigated. Clearly, (2.1) implies the following:

(2.13) Given $0 < s < 1$, the quotient $f(st)/f(t)$ is increasing and has a positive lower bound for $t > 0$.

As a consequence of (2.1) and (2.13) it will be shown that

(2.14) There is a constant $\alpha > 0$, depending on f, such that the limit $l(s) = \lim\limits_{t \to +0} (f(st)/f(t))$ exists and is equal to s^α for $s > 0$.

The existence and positivity of this limit follows from (2.1) and (2.13). Then

$$l(rs) = \lim_{t \to +0} \frac{f(rst)f(st)}{f(st)f(t)} = l(r)l(s) \text{ for } r, s > 0.$$

The function $c : \mathbb{R} \to \mathbb{R}$ defined by $c(u) = \log l(e^u)$ for $u \in \mathbb{R}$ thus satisfies *Cauchy's functional equation.* Since f is strictly increasing, (2.1) and (2.13) imply that $l(s) > 1$ for $s > 1$ and $l(s) < 1$ for $0 < s < 1$, respectively. Hence $c(u) < 0$ for $u < 0$ and $c(u) > 0$ for $u > 0$. Being a solution of Cauchy's functional equation it thus follows that $c(u) = \alpha u$ for $u \in \mathbb{R}$, with suitable $\alpha > 0$. See [37]. This yields (2.14).

The next required property of f is the following:

(2.15) Define $g : \mathbb{R}^+ \to \mathbb{R}^+$ by $g(t) = f(1/t^{1/d})$ for $t > 0$. Then, given $0 < \eta < 1$, $\vartheta,\ \iota > 0, \vartheta \neq \iota$, there is a constant $\kappa > 0$ such that

$$\eta g(\vartheta t) + (1-\eta) g(\iota t) \geq (1+\kappa) g((\eta\vartheta + (1-\eta)\iota)t) \text{ for sufficiently large } t > 0.$$

To see this note that by (2.14) and the definition of g hold:

$$g(\vartheta t) \sim \frac{1}{\vartheta^{\frac{\alpha}{d}}} g(t),\ g(\iota t) \sim \frac{1}{\iota^{\frac{\alpha}{d}}} g(t),$$
$$g((\eta\vartheta + (1-\eta)\iota)t) \sim \frac{1}{(\eta\vartheta + (1-\eta)\iota)^{\frac{\alpha}{d}}} g(t) \text{ as } t \to \infty.$$

2.4 The Delone property of $\mathbf{S_n}, \mathbf{\bar{S}_n}$. Let $M, J, w : J \to \mathbb{R}^+$ be as in Theorem 1. Later on we will assume that J is connected and has positive density.

At first tilings of J by Dirichlet–Voronoi cells will be investigated.Given $S = \{p_1, \ldots, p_n\} \subset M$, define the *(Dirichlet–Voronoi) cell* $D_i = D(J, S, p_i)$ of p_i with respect to S in J by

$$D_i = \{x \in J : \varrho_M(p_i, x) \leq \varrho_M(x, p_j) \text{ for } j = 1, \ldots, n\}, i = 1, \ldots, n.$$

The *circumradius* $R_i = R(D_i)$ of D_i with respect to p_i is given by

$$R_i = \inf\{R > 0 : B_M(p_i, R) \supset D_i\}, i = 1, \ldots, n.$$

With these definitions in mind the following will be shown:

(2.16) Let $S = \{p_1, \ldots, p_n\} \subset M$. If $\max\limits_{1=1,\cdots,n}\{R_i\}$ is sufficiently small, then the cells $D_1, \ldots, D_n$ are measurable and form a *tiling* of J,

i.e., they cover J and their intersections have measure 0. There are finitely many neighborhoods as described in (2.12) such that their interiors cover the compact set J. Thus there is a constant $2\mu > 0$, a *Lebesgue number* of this covering, see e.g. [13], with the following property: for each $q \in J$ the ball $B_M(q, 2\mu)$ is contained in one of these neighborhoods. Assume now that $\max\limits_{i=1,\cdots,n}\{R_i\} < \mu$. Then the following holds:

(2.17) $\omega_M(D_i \cap D_j) = 0 \quad$ for $i, j = 1, \ldots, n, i \neq j$.

If $D_i \cap D_j = \emptyset$, we are finished. Otherwise, let $q \in D_i \cap D_j \subset J$. Then $\varrho_M(p_i, q) = \varrho_M(q, p_j) \leq R_j < \mu$. Since $R_i, R_j < \mu$, we see that $D_i \cap D_j \subset B_M(q, 2\mu)$. Hence $D_i \cap D_j (\subset B_M(q, 2\mu)) \subset N$, where N is one of the neighborhoods as described in (2.12). Since $D_i \cap D_j \subset \{x \in M : \varrho_M(p_i, x) = \varrho_M(x, q_j)\}$ it follows that $D_i \cap D_j \subset L^N(p_i, p_j)$ and thus $\omega_M(D_i \cap D_j) \leq \omega_M(L^N(p_i, p_j)) = 0$, by (2.12). The proof of (2.17) is complete. Since $D_1, \ldots, D_n, J$ are closed and $D_1 \cup \ldots \cup D_n = J$,

$$\operatorname{bd} D_i \subset \Big(\bigcup_{i\neq j} D_i \cap D_j\Big) \cup \operatorname{bd} J \ \text{ for } i = 1, \dots, n.$$

This together with (2.17), the measurability of J which implies that $\omega(\operatorname{bd} J) = 0$, and the fact that $D_1, \dots, D_n$ cover J yields the measurability of the cells $D_1, \dots, D_n$ and the claim that they tile J, concluding the proof of (2.16).

The next preliminary step is to prove that

(2.18) there is a constant $\nu > 0$ such that $F_n \leq \nu f\Big(\frac{1}{n^{\frac{1}{d}}}\Big)$ for $n = 1, 2, \dots,$

where for F_n see (2.3). Let $\lambda = 2$ and choose m and $U_l, \dots, l = 1, \dots, m$, as in 2.3 such that (2.6) – (2.11) hold. Let $n \geq m$. Since the sets J_l' all are measurable in $\mathbb{E}^d$ by (2.9), we may choose $\lfloor n/m \rfloor$ points in each J_l' such that the Euclidean balls with centers at these points and radius $O(1/\lfloor n/m \rfloor^{\frac{1}{d}})$ cover J_l'. Here $O(\cdot)$ is a suitable Landau symbol, the same for each l. If n is sufficiently large, which we assume, the balls covering J_l' all are contained in U_l' by (2.9). Consider the h_l^{-1}-images of the centers of these balls and let S be the union of all these images, $l = 1, \dots, m$. Then $\#S \leq n$ and (2.6), (2.7), (2.9), (2.11), the definition of integrals on M, and (2.14) together yield (2.18).

Assume from now on in this subsection that J is connected and has positive density, i.e., satisfies (2.2), but note that these assumptions will only be needed beginning with the proof of proposition (2.26). Let $S_n = \{p_{n1}, \dots, p_{nn}\}$ and $D_{ni} = D(J, S_n, p_{ni}), R_{ni} = R(D_{ni})$ for $i = 1, \dots, n$, $n = 1, 2, \dots$. Our aim is to show the following:

(2.19) There is a constant $\delta > 1$ such that

$$\frac{1}{\delta n^{\frac{1}{d}}} \leq \min_{\substack{i,j=1,\dots,n\\ i\neq j}} \{\varrho_M(p_{ni}, p_{nj})\}, \max_{i=1,\dots,n} \{R_{ni}\} \leq \frac{\delta}{n^{\frac{1}{d}}} \ \text{ for } n = 1, 2, \dots.$$

We show the upper estimate first. Its proof is split into a series of steps. In the first step it will be shown that

(2.20) $D_{ni} \neq \emptyset$ for $i = 1, \dots, n, n = 1, 2, \dots.$

If $D_{ni} = \emptyset$, then

$$\min_{p\in S_n} \{\varrho_M(p, x)\} = \min_{p\in S_n \backslash \{p_{ni}\}} \{\varrho_M(p, x)\} \ \text{ for } x \in J.$$

Now choose $q \in (\operatorname{int} J) \backslash S_n$ and let $T_n = (S_n \backslash \{p_{ni}\}) \cup \{q\}$. Then

$$\min_{p\in S_n} \{\varrho_M(p, x)\} \geq \min_{p\in T_n} \{\varrho_M(p, x)\} \ \text{ for } x \in J,$$

where strict inequality holds for all $x \in J$ sufficiently close to q. Since f is strictly increasing by (2.1), $S = S_n$ does not minimize the left hand integral in (2.3). This contradicts our choice of S_n and thus concludes the proof of (2.20).

The next step is to show that

(2.21) $\max_{x \in J}\{\min_{p \in S_n}\{\varrho_M(p,x)\}\} \to 0$ as $n \to \infty$.

For suppose not. Then there are $\xi > 0$ and a sequence of points $x_n \in J$ where n is from an infinite subsequence of $1, 2, \ldots$, such that $\varrho_M(p, x_n) \geq 2\xi$ for each $p \in S_n$. Since J is compact, there is a point $x \in J$ such that $\varrho_M(p,x) \geq \xi$ for each $p \in S_n$ and n from a suitable infinite subsequence of the subsequence just considered. Combining this with the assumptions that J has positive density (see (2.2)) and f is strictly increasing (see (2.1)), we obtain a contradiction to (2.18). This concludes the proof of (2.21).

(2.20), (2.21) and (2.16) yield the following:

(2.22) $\max_{i=1,\ldots,n}\{R_{ni}\} \to 0$ as $n \to \infty$,

(2.23) the cells $D_{n1}, \ldots, D_{nn}$ are measurable and form a tiling of J for all sufficiently large n,

(2.24) there is a compact set $I \supset J$ such that $S_n \subset I$ for $n = 1, 2, \ldots$.

The definition of F_n in (i), our choice of $S_n = \{p_{n1}, \ldots, p_{nn}\}$ and of $D_{n1}, \ldots, D_{nn}$, (2.23), the assumptions that f is increasing and $w \geq 0$ together imply the next assertion:

(2.25) Let $T_n = \{q_{n1}, \ldots, q_{nn}\} \subset M$ and let $E_{n1}, \ldots, E_{nn} \subset J$ be measurable and form a tiling of J for $n = 1, 2, \ldots$. Then, if n is sufficiently large,

$$\begin{aligned} F_n &= \int\limits_J \min_{i=1,\ldots,n}\{f(\varrho_M(p_{ni},x))\} w(x) d\omega_M(x) \\ &= \sum_{i=1}^n \int\limits_{D_{ni}} f(\varrho_M(p_{ni},x)) w(x) d\omega_M(x) \\ &\leq \int\limits_J \min_{i=1,\ldots,n}\{f(\varrho_M(q_{ni},x))\} w(x) d\omega_M(x) \\ &\leq \sum_{i=1}^n \int\limits_{E_{ni}} f(\varrho_M(q_{ni},x)) w(x) d\omega_M(x). \end{aligned}$$

Before proving the upper estimate in (2.19), a weaker version of it will be shown in the following step:

(2.26) There is a constant $\sigma > 0$ such that $R_{nj} \leq \dfrac{\sigma}{n^{\frac{1}{d}}}$ for suitable $j \in \{1, \ldots, n\}, n = 1, 2, \ldots$.

If (2.26) did not hold, then,

(2.27) for any (arbitrarily large) $\zeta > 0$ there is an infinite set of n such that

$$\min_{i=1,\dots,n} \{R_{ni}\} \geq \frac{\zeta}{n^{\frac{1}{d}}}.$$

Take β from (2.2) and v, W from (2.11). By (2.14) one may

(2.28) choose $\zeta > 0$ so large that for suitable $\tau > 0$ holds

$$\omega_M(J) W f(3t) < \frac{\zeta^d v}{3^d \beta}\Big(f\Big(\frac{2t}{3}\Big) - f\Big(\frac{t}{3}\Big)\Big) \text{ for } 0 < t \leq \tau.$$

From now on we consider in the proof of (2.26) only such n for which

(2.29) (2.23), (2.25), and (2.27) for the chosen ζ hold, and

(2.30) $\max_{i=1,\dots,n} \{R_{ni}\} \leq \gamma, \tau.$

Here for (2.30) we have made use of (2.22) and γ, τ are as in (2.2) and (2.28), respectively. By (2.29) and (2.23) the cells $D_{n1}, \dots, D_{nn}$ are measurable and form a tiling of J. Hence there is a cell D_{nj}, say, such that

(2.31) $\omega_M(D_{nj}) \leq \dfrac{\omega_M(J)}{n}.$

Since in this subsection J is assumed to be connected and since the cells $D_{n1}, \dots, D_{nn}$ tile J by (2.29) and (2.23) and are compact, we may choose a point $p \in D_{nj} \cap D_{nk}$ for suitable $k \neq j$. Then $\varrho_M(p_{nj}, p_{nk}) \leq \varrho_M(p_{nj}, p) + \varrho_M(p, p_{nk}) = 2\varrho_M(p_{nj}, p) \leq 2R_{nj}$ and therefore,

(2.32) $k \neq j$ and $\varrho_M(p_{nk}, x) (\leq \varrho_M(p_{nk}, p_{nj}) + \varrho_M(p_{nj}, x)) \leq 3R_{nj}$ for $x \in D_{nj}$.

Choose $q \in D_{nj}$ with $\varrho_M(p_{nj}, q) = R_{nj}$. The definition of D_{nj} then shows that

(2.33) $\varrho_M(q, x) \leq \dfrac{R_{nj}}{3}, \varrho_M(p_{ni}, x) \geq \dfrac{2R_{nj}}{3}$ for $x \in B = B_M\Big(q, \dfrac{R_{nj}}{3}\Big),$
$i = 1, \dots, n.$

Define $f_n : J \to \mathbb{R}$ by

(2.34) $f_n(x) = \min_{i=1,\dots,n} \{f(\varrho_M(p_{ni}, x))\}$ for $x \in J$.

Finally, the definition of F_n in (i), our choice of $S_n = \{p_{n1}, \dots, p_{nn}\}$, (2.34), (2.29) and (2.23), the measurability of B which follows from (2.30) and (2.2), (2.25), the assumptions that f is increasing and $f, w \geq 0$, (2.32), (2.34), (2.11), (2.30), (2.2), (2.34) and (2.33), (2.11), (2.3), (2.34), (2.31), (2.29) and (2.27), (2.30), and (2.28) lead to the following contradiction and thus conclude the proof of (2.26):

$$\begin{aligned}
F_n &= \int\limits_J f_n(x)w(x)d\omega_M(x) = \sum_{i=1}^n \int\limits_{D_{ni}} f(\varrho_M(p_{ni},x))w(x)d\omega_M(x) \\
&\le \sum_{\substack{i=1\\ i\neq j}}^n \int\limits_{D_{ni}\setminus B} f(\varrho_M(p_{ni},x))w(x)d\omega_M(x) + \int\limits_{D_{nj}\setminus B} f(\varrho_M(p_{nk},x))w(x)d\omega_M(x) \\
&\quad + \int\limits_{B\cap J} f(\varrho_M(p,x))w(x)d\omega_M(x) \\
&\le \sum_{\substack{i=1\\ i\neq j}}^n \int\limits_{D_{ni}\setminus B} f(\varrho_M(p_{ni},x))w(x)d\omega_M(x) + \int\limits_{D_{nj}\setminus B} f(\varrho_M(p_{nj},x))w(x)d\omega_M(x) \\
&\quad + \int\limits_{D_{nj}\setminus B} f(3R_{nj})w(x)d\omega_M(x) + \int\limits_{B\cap J} f_n(x)w(x)d\omega_M(x) \\
&\quad - \int\limits_{B\cap J} (f_n(x) - f(\varrho_M(q,x)))w(x)d\omega_M(x) \\
&\le \int\limits_J f_n(x)w(x)d\omega_M(x) + \omega_M(D_{nj})f(3R_{nj})W \\
&\quad -\frac{R_{nj}^d}{3^d\beta}\Big(f\Big(\frac{2R_{nj}}{3}\Big) - f\Big(\frac{R_{nj}}{3}\Big)\Big)v \\
&\le F_n + \frac{1}{n}\omega_M(J)Wf(3R_{nj}) - \frac{1}{n}\frac{\zeta^d v}{3^d\beta}\Big(f\Big(\frac{2R_{nj}}{3}\Big) - f\Big(\frac{R_{nj}}{3}\Big)\Big) < F_n.
\end{aligned}$$

We come to the final step of the proof of the upper estimate in (2.19). If this estimate did not hold, then

(2.35) for any (arbitrarily large) $\nu > 0$ there is an infinite set of n such that

$$R_{nk} > \frac{\nu}{n^{\frac{1}{d}}} \text{ for suitable } k \in \{1,\dots,n\}.$$

Take β from (2.2), ε from (2.5), where I is from (2.24), σ from (2.26), and v, W from (2.11). By (2.14) we may

(2.36) choose $\nu > 0$ so large that

(i) $\nu > \sigma$,

(ii) for suitable $\varphi > 0$ hold

$$W\varepsilon\sigma^d f\Big(\frac{3\sigma}{n^{\frac{1}{d}}}\Big) - \frac{v\nu^d}{3^d\beta}\Big(f\Big(\frac{2t}{3}\Big) - f\Big(\frac{t}{3}\Big)\Big) < 0 \text{ for } \frac{\nu}{n^{\frac{1}{d}}} < t < \varphi.$$

From now on consider in the proof of the upper estimate in (2.19) only such n that

(2.37) (2.5), where I is from (2.24), (2.23), (2.25) and (2.35) hold, and

$$(2.38)\quad \max_{i=1,\dots,n}\{R_{ni}\} \le \gamma, \vartheta, \varphi,$$

where $\gamma, \vartheta, \varphi$ are as in (2.2), (2.5) and (2.36), respectively. Given such an n, let j be as in (2.26). Then $R_{nj} \le \sigma/n^{1/d}$. Choose $p_{nl} \ne p_{nj}$ with $\varrho_M(p_{nj}, p_{nl}) \le 2R_{nj} \le 2\sigma/n^{1/d}$. Such a choice is possible since in this subsection J is connected by assumption, compare the argument that led to (2.32). Then

$$(2.39)\quad l \ne j \text{ and } \varrho_M(p_{nl}, x) \le 3R_{nj} \le \frac{3\sigma}{n^{\frac{1}{d}}} \text{ for } x \in D_{nj}.$$

By (2.37), proposition (2.35) holds for n and the chosen ν.Let k be as in (2.35). Since $R_{nj} \le \sigma/n^{1/d} < \nu/n^{1/d} < R_{nk}$ by (2.26), (2.36i) and (2.37) and (2.35),

$$(2.40)\quad k \ne j$$

follows. Choose $p \in D_{nk}$ with $\varrho_M(p_{nk}, p) = R_{nk}$. The definition of the cell D_{nk} and (2.35) then imply that $\varrho_M(p_{ni}, p) \ge \varrho_M(p_{nk}, p) = R_{nk} > \nu/n^{1/d}$ for $i = 1, \dots, n$. Hence,

$$(2.41)\quad \varrho_M(p, x) \le \frac{R_{nk}}{3}, \varrho_M(p_{ni}, x) \ge \frac{2R_{nk}}{3} \text{ for}$$
$$x \in B = B_M\Big(p, \frac{R_{nk}}{3}\Big), i = 1, \dots, n.$$

Finally, the definition of F_n in (i), our choice of $S_n = \{p_{n1}, \dots, p_{nn}\}$, (2.37) and (2.25), the measurability of B which follows from (2.38) and (2.2), (2.39), (2.40), (2.37), (2.23) and (2.25), the assumptions that f is increasing and $f, w \ge 0$, (2.39), (2.34), (2.11), (2.41), (2.38), (2.2), (2.11), (2.38), (2.26), (2.37) and (2.5), (2.26), (2.37), (2.35), (2.38) and (2.36) together yield a contradiction and thus conclude the proof of the upper estimate in (2.19):

$$\begin{aligned}
F_n &= \sum_{i=1}^n \int\limits_{D_{ni}} f(\varrho_M(p_{ni}, x))w(x)d\omega_M(x) \\
&\le \sum_{\substack{i=1\\ i\ne j,k}}^n \int\limits_{D_{ni}\setminus B} f(\varrho_M(p_{ni}, x))w(x)d\omega_M(x) + \int\limits_{D_{nj}\setminus B} f(\varrho_M(p_{nl}, x))w(x)d\omega_M(x) \\
&\quad + \int\limits_{D_{nk}\setminus B} f(\varrho_M(p_{nk}, x))w(x)d\omega_M(x) + \int\limits_{B\cap J} f(\varrho_M(p, x))w(x)d\omega_M(x) \\
&\le \sum_{\substack{i=1\\ i\ne j,k}}^n \int\limits_{D_{ni}\setminus B} f(\varrho_M(p_{ni}, x))w(x)d\omega_M(x) + \int\limits_{D_{nj}\setminus B} f(\varrho_M(p_{nj}, x))w(x)d\omega_M(x) \\
&\quad + \int\limits_{D_{nj}\setminus B} f(\frac{3\sigma}{n^{\frac{1}{d}}})w(x)d\omega_M(x) + \int\limits_{D_{nk}\setminus B} f(\varrho_M(p_{nk}, x))w(x)d\omega_M(x) \\
&\quad + \int\limits_{B\cap J} f_n(x)w(x)d\omega_M(x) - \int\limits_{B\cap J} (f_n(x) - f(\varrho_M(p, x)))w(x)d\omega_M(x)
\end{aligned}$$

$$
\begin{aligned}
&= \int_J f_n(x)w(x)d\omega_M(x) \\
&\quad +\omega_M(D_{nj})f\Big(\frac{3\sigma}{n^{\frac{1}{d}}}\Big)W - \frac{vR_{nk}^d}{3^d\beta}\Big(f\Big(\frac{2R_{nk}}{3}\Big) - f\Big(\frac{R_{nk}}{3}\Big)\Big) \\
&\le F_n + \frac{1}{n}W\varepsilon\sigma^d f\Big(\frac{3\sigma}{n^{\frac{1}{d}}}\Big) - \frac{1}{n}\frac{v\nu^d}{3^d\beta}\Big(f\Big(\frac{2R_{nk}}{3}\Big) - f\Big(\frac{R_{nk}}{3}\Big)\Big) < F_n.
\end{aligned}
$$

We proceed to the proof of the lower estimate in (2.19). The proof will be split into two steps. In the first step it will be shown that

(2.42) there is a constant $\phi > 0$ such that for $j = 1, \dots, n$, holds

$$
F_n \le \int_J g_{n-1}(x)w(x)d\omega_M(x) - \frac{\phi}{n} f\Big(\frac{1}{n^{\frac{1}{d}}}\Big),\ g_{n-1}(x) = \min_{\substack{i=1,\dots,n \\ i\neq j}} \{f(\varrho_M(p_{ni}, x))\}.
$$

It is enough to show this for sufficiently large n. The argument that led to (2.32) shows that for each $x \in J$ there is a point in $\{p_{n1}, \dots, p_{nj-1}, p_{nj+1}, \dots, p_{nn}\}$ at distance at most $3\delta/n^{\frac{1}{d}}$, where δ is from the upper estimate in (2.19). Hence the cells in J corresponding to the set $\{p_{n1}, \dots, p_{nj-1}, p_{nj+1}, \dots, p_{nn}\}$ have circumradius at most $3\delta/n^{1/d}$. Then (2.16) shows that for all sufficiently large n and any $j = 1, \dots, n$, the $n-1$ cells corresponding to $\{p_{n1}, \dots, p_{nj-1}, p_{nj+1}, \dots, p_{nn}\}$ are measurable and tile J. Thus there is a cell of measure at least $\omega_M(J)/(n-1) > \omega_M(J)/n$, say the cell corresponding to p_{nk}. Clearly, $k \neq j$. Noting (2.24) and (2.5) we see that this cell has circumradius at least $\chi/n^{1/d}$, where $\chi > 0$ is a suitable constant. Hence there is a point p of this cell with $\varrho_M(p_{nk}, p) \ge \chi/n^{1/d}$. The definition of cells then shows that $\varrho_M(p_{ni}, p) \ge \chi/n^{1/d}$ for $i = 1, \dots, d, i \neq j$. As before,

(2.43) $\varrho_M(p, x) \le \dfrac{\chi}{3n^{\frac{1}{d}}}, \varrho_M(p_{ni}, x) \ge \dfrac{2\chi}{3n^{\frac{1}{d}}}$ for $x \in B = B_M\Big(p, \dfrac{\chi}{3n^{\frac{1}{d}}}\Big)$,
$i = 1, \dots, n,\ i \neq j$,

compare (2.33) and (2.41). Now, the definitions of F_n in (i) and of g_{n-1} in (2.42), the measurability of B which follows from (2.2) for sufficiently large n, the assumption that J satisfies (2.2), (2.43), the assumptions that f is increasing and $f, w \ge 0$, and (2.11) imply, for sufficiently large n, that

$$
\begin{aligned}
F_n &\le \int_J \min\{\min_{\substack{i=1,\dots,n \\ i\neq j}} \{f(\varrho_M(p_{ni}, x))\}, f(\varrho_M(p, x))\} w(x)d\omega_M(x) \\
&= \int_J \min\{g_{n-1}(x), f(\varrho_M(p, x))\} w(x)d\omega_M(x) \\
&\le \int_{J\setminus B} g_{n-1}(x)w(x)d\omega_M(x) + \int_{B\cap J} f(\varrho_M(p, x))w(x)d\omega_M(x)
\end{aligned}
$$

$$
\begin{aligned}
&= \int\limits_J g_{n-1}(x)w(x)d\omega_M(x) - \int\limits_{B\cap J} (g_{n-1}(x) - f(\varrho_M(p,x)))w(x)d\omega_M(x) \\
&\le \int\limits_J g_{n-1}(x)w(x)d\omega_M(x) - \frac{1}{n}\frac{\chi^d v}{3^d\beta}\Big(f\Big(\frac{2\chi}{3n^{\frac{1}{d}}}\Big) - f\Big(\frac{\chi}{3n^{\frac{1}{d}}}\Big)\Big).
\end{aligned}
$$

An application of (2.14) then yields (2.42).

In the second step we assume that the lower estimate in (2.19) does not hold. Then,

(2.44) for any (arbitrarily small) $\psi > 0$ there is an infinite set of n such that

$$
\varrho_M(p_{nj}, p_{nk}) \le \frac{\psi^2}{n^{\frac{1}{d}}} \text{ for suitable } j, k \in \{1, \dots, n\}, j \ne k.
$$

Let $\delta, \varepsilon, \phi, W$ be as in the upper estimate in (2.19), (2.5), where I is from (2.24), (2.42), and (2.11), respectively. By (2.14) we may choose $0 < \omega, \psi < 1$ so small that

$$
\beta\delta^d\omega W f\Big(\frac{\delta}{n^{\frac{1}{d}}}\Big) + \beta\psi^d W f\Big(\frac{2\psi}{n^{\frac{1}{d}}}\Big) < \phi f\Big(\frac{1}{n^{\frac{1}{d}}}\Big), \tag{2.45}
$$

$$
f((1+\psi)t) \le (1+\omega)f(t) \text{ for } 0 < t \le \tau, \tag{2.46}
$$

with a suitable $\tau > 0$. In the remaining part of the proof of the lower estimate in (2.19) we consider only n for which

(2.47) (2.25) and (2.44) hold, and

$$
\max_{i=1,\dots,n}\{R_{ni}\} \le \frac{\delta}{n^{\frac{1}{2}}} \le \min\{\gamma, \tau\}, \ \frac{\psi}{n^{\frac{1}{d}}} \le \gamma, \tag{2.48}
$$

where γ is as in (2.2) and τ as in (2.46). For (2.48) we have used (2.19). Propositions (2.47) and (2.44) and our choice of $\psi(< 1)$ imply that

$$
\begin{aligned}
&\varrho_M(p_{nk}, x) \le (1+\psi)\varrho_M(p_{nj}, x) \text{ for } x \notin B = B_M\Big(p_{nj}, \frac{\psi}{n^{\frac{1}{d}}}\Big), \\
&\varrho_M(p_{nk}, x) \le \frac{2\psi}{n^{\frac{1}{d}}} \text{ for } x \in B.
\end{aligned} \tag{2.49}
$$

Now, combining (2.42), (2.47) and (2.25) applied with $n-1$ instead of n, where $E_{ni} = D_{ni}$ for $i = 1, \dots, n, i \ne j, k$ and $E_{nk} = D_{nj} \cup D_{nk}$, the measurability of B which follows from (2.24) and (2.5) for sufficiently large n, the assumptions that f is increasing and $f, w \ge 0$, (2.49), (2.46), (2.48) and (2.2), (2.11), (2.48) and (2.2), the assumptions that f is increasing and f, $w \ge 0$ and (2.45), we obtain for sufficiently large n the following contradiction which concludes the proof of the lower estimate in (2.19):

$$\begin{aligned}
F_n &\le \int\limits_J \min_{\substack{i=1,\dots,n\\ i\neq j}} \{f(\varrho_M(p_{ni},x))\} w(x)d\omega_M(x) - \frac{\phi}{n} f\Big(\frac{1}{n^{\frac{1}{d}}}\Big)\\
&\le \sum_{\substack{i=1\\ i\neq j}}^{n} \int\limits_{D_{ni}} f(\varrho_M(p_{ni},x))w(x)d\omega_M(x) + \int\limits_{D_{nj}} f(\varrho_M(p_{nk},x))w(x)d\omega_M(x)\\
&\quad -\frac{\phi}{n} f\Big(\frac{1}{n^{\frac{1}{d}}}\Big)\\
&\le \sum_{\substack{i=1\\ i\neq j}}^{n} \int\limits_{D_{ni}} f(\varrho_M(p_{ni},x))w(x)d\omega_M(x) + \int\limits_{D_{nj}\setminus B} f((1+\psi)\varrho_M(p_{nj},x))w(x)d\omega_M(x)\\
&\quad + \int\limits_{B\cap D_{nj}} f\Big(\frac{2\psi}{n^{\frac{1}{2}}}\Big) w(x)d\omega_M(x) - \frac{\phi}{n} f\Big(\frac{1}{n^{\frac{1}{d}}}\Big)\\
&\le \sum_{\substack{i=1\\ i\neq j}}^{n} \int\limits_{D_{ni}} f(\varrho_M(p_{ni},x))w(x)d\omega_M(x) + \int\limits_{D_{nj}\setminus B} f(\varrho_M(p_{nj},x))w(x)d\omega_M(x)\\
&\quad +\omega \int\limits_{D_{nj}\setminus B} f(\varrho_M(p_{nj},x))w(x)d\omega_M(x) + \frac{1}{n}\beta\psi^d W f\Big(\frac{2\psi}{n^{\frac{1}{2}}}\Big) - \frac{\phi}{n} f\Big(\frac{1}{n^{\frac{1}{2}}}\Big)\\
&\le F_n + \frac{1}{n}\beta\delta^d\omega W f\Big(\frac{\delta}{n^{\frac{1}{d}}}\Big) + \frac{1}{n}\beta\psi^d W f\Big(\frac{2\psi}{n^{\frac{1}{d}}}\Big) - \frac{\phi}{n} f\Big(\frac{1}{n^{\frac{1}{d}}}\Big) < F_n.
\end{aligned}$$

For $n = 1, 2, \dots$, choose $\bar{S}_n = \{\bar{p}_{n1,}, \dots, \bar{p}_{nn}\} \subset J$ in the following way: if in (i) the infimum is considered only for S with $\#S = n$ for which $S \subset J$ (instead of $S \subset M$), then it is attained for $S = \bar{S}_n$. If J is connected and has positive density, then a proof similar to the one which led to (2.19), but easier in some details, shows that

(2.50) there is a constant $\bar{\delta} > 1$ such that

$$\frac{1}{\bar{\delta} n^{\frac{1}{d}}} \le \min_{\substack{i,j=1,\dots,n\\ i\neq j}} \{\varrho_M(\bar{p}_{ni},\bar{p}_{nj})\},\ \max_{i=1,\dots,n}\{\bar{R}_{ni}\} \le \frac{\bar{\delta}}{n^{\frac{1}{d}}} \text{ for } n = 1, 2, \dots,$$

where the $\bar{R}_{ni}$ are the circumradii of the cells in J corresponding to $\{\bar{p}_{n1}, \dots, \bar{p}_{nn}\}$.

2.5 The constant div. Let $M = \mathbb{E}^d, J = [0,1]^d, w = 1$. If the infimum in (i) is attained for, say, $T_n = \{t_{n1}, \dots, t_{nn}\}$, then $T_n \subset [0,1]^d$. Thus an application of (2.19) or (2.50), and (2.14) together with the assumptions that f is increasing and $f \ge 0$ yields the following: if

$$G_n = \inf_{\substack{T\subset\mathbb{E}^d\\ \#T=n}} \Big\{ \int\limits_{[0,1]^d} \min_{t\in T}\{f(\|t-u\|_2)\}du \Big\} = \int\limits_{[0,1]^d} \min_{i=1,\dots,n}\{f(\|t_{ni}-u\|_2)\}du,$$

then $G_n/f(1/n^{\frac{1}{d}})$ is bounded between positive constants. Thus, defining

(2.51) $\operatorname{div} = \liminf_{n\to\infty} \dfrac{G_n}{f\left(\frac{1}{n^{\frac{1}{d}}}\right)}$,

it follows that

(2.52) $\operatorname{div} \in \mathbb{R}^+$.

We will show that

(2.53) $\operatorname{div} = \lim_{n\to\infty} \dfrac{G_n}{f\left(\frac{1}{n^{\frac{1}{d}}}\right)}$.

Let $\delta > 1$ be as in (2.19). Choose $\mu > 1$ (arbitrarily close to 1) and let m be such that

(2.54) $$\frac{G_m}{f\left(\frac{1}{m^{\frac{1}{d}}}\right)} \leq \mu \operatorname{div}, \quad \frac{f\left(\frac{\delta}{km^{\frac{1}{d}}}\right)}{f\left(\frac{1}{km^{\frac{1}{d}}}\right)} \leq \mu \frac{f\left(\frac{\delta}{m^{\frac{1}{d}}}\right)}{f\left(\frac{1}{m^{\frac{1}{d}}}\right)} \quad \text{for } k = 1, 2, \dots .$$

This is possible by the definition of div in (2.51), (2.52) and (2.1). An application of (2.13) shows that

(2.55) $$\frac{f\left(\frac{\delta}{km^{\frac{1}{d}}}\right)}{f\left(\frac{\delta}{m^{\frac{1}{d}}}\right)} f(t) \geq f\left(\frac{t}{k}\right) \quad \text{for } 0 < t \leq \frac{\delta}{m^{\frac{1}{d}}}, k = 1, 2, \dots .$$

For $k = 1, 2, \dots$, dissect the cube $[0,1]^d$ in the usual way into k^d cubelets, each of edge length $1/k$. The k^d affine transformations

$$A_a : u_1 \to \frac{u_1 + a_1}{k}, \dots, u_d \to \frac{u_d + a_d}{k}, z \to \frac{f\left(\frac{\delta}{km^{\frac{1}{d}}}\right)}{f\left(\frac{\delta}{m^{\frac{1}{d}}}\right)} z \text{ for } (u, z) \in \mathbb{E}^{d+1},$$

where $a = (a_1, \dots, a_d)^T \in \{0, \dots, k-1\}^d$,

map the set

$$\{(u, z) \in \mathbb{E}^{d+1} : u \in [0,1]^d, 0 \leq z \leq \min_{i=1,\dots,m} \{f(\|t_{mi} - u\|_2)\}\}.$$

onto k^d non-overlapping sets. Together, these sets contain the set

$$\{(u, z) \in \mathbb{E}^{d+1} : u \in [0,1]^d, 0 \leq z \leq \min_{\substack{i=1,\dots,m \\ a\in\{0,\dots,k-1\}^d}} \{f(\|A_a t_{mi} - u\|_2)\}.$$

This follows from (2.55) and the fact that by (2.19) the cells in $[0,1]^d$ corresponding to $T_m = \{t_{m1}, \dots, t_{mm}\}$ have circumradius at most $\delta/m^{\frac{1}{d}}$. Hence the definition of $G_{k^d m}$ and (2.54) show that

$$G_{k^d m} \leq k^d \cdot \frac{1}{k^d} \frac{f\left(\frac{\delta}{km^{\frac{1}{d}}}\right)}{f\left(\frac{\delta}{m^{\frac{1}{d}}}\right)} G_m \leq \mu \frac{f\left(\frac{1}{km^{\frac{1}{d}}}\right)}{f\left(\frac{1}{m^{\frac{1}{d}}}\right)} G_m,$$

and thus, again by (2.54),

$$\frac{G_{k^d m}}{f\left(\frac{1}{km^{\frac{1}{d}}}\right)} \le \mu \frac{G_m}{f\left(\frac{1}{m^{\frac{1}{d}}}\right)} \le \mu^2 \mathrm{div}.$$

By (2.14) there is k_0 such that

$$\frac{f\left(\frac{1}{km^{\frac{1}{d}}}\right)}{f\left(\frac{1}{(k+1)m^{\frac{1}{d}}}\right)} \le \mu \text{ for } k \ge k_0.$$

Finally, given $n \ge k_0^d m$, let $k \ge k_0$ be such that $k^d m \le n < (k+1)^d m$. Thus,

$$\frac{G_n}{f\left(\frac{1}{n^{\frac{1}{d}}}\right)} \Bigg(\le \frac{G_{k^d m}}{f\left(\frac{1}{km^{\frac{1}{d}}}\right)} \frac{f\left(\frac{1}{km^{\frac{1}{d}}}\right)}{f\left(\frac{1}{(k+1)m^{\frac{1}{d}}}\right)} \Bigg) \le \mu^3 \mathrm{div} \text{ for all sufficiently large } n.$$

Since $\mu > 1$ was arbitrary, this together with (2.51) implies (2.53).

Next, (2.53) will be extended to arbitrary cubes:

(2.56) Let C be a cube in $\mathbb{E}^d$. Then

$$H_m = \inf_{\substack{T \subset \mathbb{E}^d \\ \#T = m}} \Big\{ \int_C \min_{t \in T} \{ f(\|t-u\|_2)\} du \Big\} \sim \mathrm{div} |C|^{\frac{\alpha+d}{d}} f\left(\frac{1}{m^{\frac{1}{d}}}\right) \text{ as } m \to \infty,$$

where α is as in (2.14). We may assume that $C = \nu[0,1]^d = [0,\nu]^d$ for suitable $\nu > 0$. By (2.19) the circumradii of the cells corresponding to $T_m \subset [0,1]^d$ are bounded above by $\delta/m^{1/d}$. Thus by (2.14),

$$\begin{aligned} G_m &= \int_{[0,1]^d} \min_{t \in T_m} \{ f(\|t-u\|_2)\} du \sim \int_{[0,1]^d} \min_{\nu t \in \nu T_m} \{ f(\|\nu t - \nu u\|_2)\} du \frac{1}{\nu^\alpha} \\ &= \int_{[0,\nu]^d} \min_{v \in \nu T_m} \{ f(\|v-w\|_2)\} dw \frac{1}{\nu^{\alpha+d}} \ge \frac{H_m}{\nu^{\alpha+d}} = \frac{H_m}{|C|^{\frac{\alpha+d}{d}}}, \end{aligned}$$

or

$$G_m |C|^{\frac{\alpha+d}{d}} \gtrsim H_m \text{ as } m \to \infty.$$

The same argument with $[0,1]^d$ and C exchanged shows that

$$H_m \gtrsim G_m |C|^{\frac{\alpha+d}{d}} \text{ as } m \to \infty,$$

and therefore,

$$H_m \sim G_m |C|^{\frac{\alpha+d}{d}} \text{ as } m \to \infty.$$

This together with (2.53) yields (2.56).

A suitable affine transformation applied to (2.56) finally gives the following:

(2.57) Let $q(\cdot)$ be a positive definite quadratic form on $\mathbb{E}^d$ and C' a cube in $\mathbb{E}^d$ with respect to the norm $q(\cdot)^{1/2}$. Then

$$\inf_{\substack{T'\subset\mathbb{E}^d\\ \#T'=m}} \Big\{ \int\limits_{C'} \min_{t\in T'}\{f(q(t-u)^{\frac{1}{2}})\}du \Big\} \sim \operatorname{div}(\det q)^{\frac{\alpha}{2d}}|C'|^{\frac{\alpha+d}{d}} f\left(\frac{1}{m^{\frac{1}{d}}}\right) \text{ as } m\to\infty.$$

2.6 The asymptotic formula. Choose $S_n = \{p_{n1},\dots,p_{nn}\}$ and $D_{n1},\dots,D_{nn}$ for $n = 1, 2, \dots$, as in 2.4. Define

$$S_n(K) = \{q_{ni}\in S_n : D_{ni}\cap K \neq \phi\}, n(K) = \#S_n(K) \text{ for measurable } K\subset J.$$

If in the following an integral is written omitting the integrand, the integrand is to be $w(x)^{d/(\alpha+d)}$ and by $\int^\alpha$ we mean the αth power of $\int$.

In the first part of the proof of the asymptotic formula (i) it will be shown that

(2.58) $$F_n \gtrsim \operatorname{div}\Big(\int\limits_J w(x)^{\frac{d}{\alpha+d}} d\omega_M(x)\Big)^{\frac{\alpha+d}{d}} f\left(\frac{1}{n^{\frac{1}{d}}}\right) \text{ as } n\to\infty.$$

Let $\lambda > 1$ and choose $U_l, V_l, \dots$, for $l = 1,\dots,m$, as in 2.3 such that

(2.59) (2.6) – (2.11) hold.

The first step of the proof of (2.58) is to show that

(2.60) there is a constant $\nu > 1$ such that $\frac{1}{\nu} f\left(\frac{1}{n^{\frac{1}{d}}}\right) \le F_n \le \nu f\left(\frac{1}{n^{\frac{1}{d}}}\right)$ for $n = 1, 2, \dots$

The upper estimate was proved in (2.18). For the proof of the lower estimate choose a set C in J_l, say, such that C' is a cube with respect to the norm $q_l(\cdot)^{\frac{1}{2}}$. Then C is measurable and the definition of F_n in (i), (2.11), the assumptions that f is increasing and $f, w \ge 0$, (2.59) and (2.6), the definition of integrals in M, (2.59) and (2.7) show that

$$\begin{aligned} F_n &\ge \int\limits_C \min_{p\in S_n}\{f(\varrho_M(p,x))\}w(x)d\omega_M(x)\\ &\ge \frac{1}{\lambda} v\,(\det q_l)^{\frac{1}{2}} \int\limits_{C'} \min_{s\in S_n(C)'} \Big\{f\Big(\frac{1}{\lambda}q_l(s-u)^{\frac{1}{2}}\Big)\Big\}du\\ &\ge \frac{1}{\lambda} v\,(\det q_l)^{\frac{1}{2}} \inf_{\substack{T'\subset\mathbb{E}^d\\ \#T'=n(C)}} \Big\{\int\limits_{C'} \min_{t\in T'}\Big\{f\Big(\frac{1}{\lambda}q_l(t-u)^{\frac{1}{2}}\Big)\Big\}du\Big\}. \end{aligned}$$

If $n(C)$ did not tend to ∞ as $n\to\infty$, the latter expression would not tend to 0, in contradiction to the upper estimate for F_n in (2.60). Hence $n(C)\to\infty$ as

$n \to \infty$. Thus, using (2.14), (2.57), and the assumption that f is increasing, we conclude further that

$$\begin{aligned} F_n &\gtrsim \frac{1}{\lambda^{1+\alpha}}\, v\,(\det q_l)^{\frac{\alpha+d}{2d}} \operatorname{div}|C'|^{\frac{\alpha+d}{d}} f\Big(\frac{1}{n(C)^{\frac{1}{d}}}\Big) \\ &\geq \frac{1}{\lambda^{1+\alpha}}\, v\,(\det q_l)^{\frac{\alpha+d}{2d}} \operatorname{div}|C'|^{\frac{\alpha+d}{d}} f\Big(\frac{1}{n^{\frac{1}{d}}}\Big) \text{ as } n \to \infty. \end{aligned}$$

This readily implies the lower estimate in (2.60), concluding the proof of (2.60).

Noting that $F_n \leq \nu f(1/n^{1/d})$ by (2.60), the assumption that f is strictly increasing, and (2.14), we obtain the following proposition as a byproduct of the last part of the proof of (2.60):

(2.61) Let $C \subset J_l$, say, such that C' is a cube with respect to the norm $q_l(\cdot)^{\frac{1}{2}}$. Then there is a constant ξ with $0 < \xi < 1$ such that $\xi n \leq n(C) \leq n$ for sufficiently large n.

We come to the second step of the proof of (2.58). Noticing (2.9), we may choose sets $C_{li} \subset J_l, i = 1, \dots, i_l, l = 1, \dots, m$, with the following properties:

(2.62) The sets $C_{li} \subset J, i = 1, \dots, i_l,\ l = 1, \dots, m$, are compact and pairwise disjoint.

(2.63) C'_{li} is a cube with respect to the norm $q_l(\cdot)^{\frac{1}{2}}$. Thus, in particular, C_{li} is measurable.

(2.64) $\displaystyle\sum_{l,i} \int_{C_{li}} d\omega_M(x) \geq \frac{1}{\lambda} \int_J d\omega_M(x).$

By the definition of $S_n(C_{li})$, (2.62) and (2.22) (which has been proved without the assumptions that J is connected and has positive density),

(2.65) the sets $S_n(C_{li})$ are pairwise disjoint for sufficiently large n. Thus,

(2.66) $\displaystyle\sum_{l,i} n(C_{li}) \leq n$ for sufficiently large n.

Finally, it follows from (2.63) and (2.61) that

(2.67) there is a constant σ with $0 < \sigma < 1$ such that $\sigma n \leq n(C_{li}) \leq n$ for $i = 1, \dots, i_l, l = 1, \dots, m$, and sufficiently large n.

In the third step note that (2.60), (2.67) and (2.66) together yield the following propositions, where

$$L = \liminf_{n\to\infty} \frac{F_n}{f\Big(\frac{1}{n^{\frac{1}{d}}}\Big)} \quad (> 0):$$

(2.68) There is an infinite subsequence of $1, 2, \ldots$, such that $F_n/f(1/n^{\frac{1}{d}}) \to L$ as $n \to \infty$ in this subsequence.

(2.69) There are constants $\sigma_{li} > 0$ such that $n(C_{li}) \sim \sigma_{li} n$ for $i = 1, \ldots, i_l, l = 1, \ldots, m$, as $n \to \infty$ in the same subsequence as in (2.68). Furthermore,

$$\sum_{l,i} \sigma_{li} \leq 1.$$

In the final step of the proof of (2.58) assume that n is from the subsequence considered in (2.68) and (2.69) and that it is so large that (2.65) and (2.66) hold. Then the definition of F_n in (i), our choice of S_n, (2.62), (2.63), the definition of $S_n(C_{li})$, the definition of integrals on M, the assumptions that f is increasing and $f, w \geq 0$, (2.59) and (2.6), (2.10), the inclusion $C_{li} \subset J_l$, (2.14), (2.57), (2.69), the definition of integrals on M, (2.59) and (2.6), (2.10), (2.14), (2.64), (2.15), the assumption that f is increasing, (2.64), (2.69), and (2.14) imply the following:

$$\begin{aligned}
F_n &= \int\limits_J \min_{p \in S_n} \{f(\varrho_M(p,x))\} w(x) d\omega_M(x) \\
&\geq \sum_{l,i} \int\limits_{C_{li}} \min_{p \in S_n(C_{li})} \{f(\varrho_M(p,x))\} w(x) d\omega_M(x) \\
&\geq \frac{1}{\lambda} \sum_{l,i} (\det q_l)^{\frac{1}{2}} \int\limits_{C'_{li}} \min_{s \in S_n(C_{li})'} \Big\{ f\Big(\frac{1}{\lambda} q_l(s-u)^{\frac{1}{2}}\Big)\Big\} w_l du \\
&\sim \frac{1}{\lambda^{1+\alpha}} \operatorname{div} \sum_{l,i} (\det q_l)^{\frac{\alpha+d}{2d}} |C'_{li}|^{\frac{\alpha+d}{d}} w_l f\Big(\frac{1}{(\sigma_{li} n)^{\frac{1}{d}}}\Big) \\
&\geq \frac{1}{\lambda^{1+\alpha+1+\frac{\alpha+d}{d}}} \operatorname{div} \sum_{l,i} \Big(\int\limits_{C_{li}} w(x)^{\frac{d}{\alpha+d}} d\omega_M(x)\Big)^{\frac{\alpha+d}{d}} f\Big(\frac{1}{(\sigma_{li} n)^{\frac{1}{d}}}\Big) \\
&\sim \frac{1}{\lambda^{3+\alpha+\frac{\alpha}{d}}} \operatorname{div} \sum_{l,i} \int\limits_{C_{li}} f\Big(\Big(\frac{\int_{C_{li}}}{\sigma_{li} n}\Big)^{\frac{1}{d}}\Big) \\
&= \frac{1}{\lambda^{3+\alpha+\frac{\alpha}{d}}} \operatorname{div} \sum_{l,i} \int\limits_{C_{li}} \sum_{l,i} \frac{\int_{C_{li}}}{\sum\limits_{l,i} \int_{C_{li}}} f\Bigg(\frac{1}{\Big(\frac{\sigma_{li} n}{\int_{C_{li}}}\Big)^{\frac{1}{d}}}\Bigg) \\
&\gtrsim \frac{1}{\lambda^{4+\alpha+\frac{\alpha}{d}}} \operatorname{div} \int\limits_J f\Bigg(\frac{1}{\Big(\sum\limits_{l,i} \frac{\int_{C_{li}} \sigma_{li} n}{\int_{C_{li}} \sum\limits_{l,i} \int_{C_{li}}}\Big)^{\frac{1}{d}}}\Bigg) \\
&= \frac{1}{\lambda^{4+\alpha+\frac{\alpha}{d}}} \operatorname{div} \int_J f\Bigg(\Bigg(\frac{\sum\limits_{l,i} \int_{C_{li}}}{\sum\limits_{l,i} \sigma_{li} n}\Bigg)^{\frac{1}{d}}\Bigg) \\
&\geq \frac{1}{\lambda^{4+\alpha+\frac{\alpha}{d}}} \operatorname{div} \int\limits_J f\Big(\Big(\frac{1}{\lambda} \int\limits_J \frac{1}{n}\Big)^{\frac{1}{d}}\Big) \sim \frac{1}{\lambda^{4+\alpha+\frac{2\alpha}{d}}} \operatorname{div} \int\limits_J^{\frac{\alpha+d}{d}} f\Big(\frac{1}{n^{\frac{1}{d}}}\Big)
\end{aligned}$$

as $n \to \infty$ in the subsequence considered in (2.68) and (2.69). Since $\lambda > 1$ was arbitrary, we have thus shown that

$$\liminf_{n\to\infty} \frac{F_n}{f\Big(\frac{1}{n^{\frac{1}{d}}}\Big)} \geq \operatorname{div} \Big(\int\limits_J w(x)^{\frac{d}{\alpha+d}} d\omega_M(x) \Big)^{\frac{\alpha+d}{d}},$$

which is equivalent to (2.58).

In the second part of the proof of the asymptotic formula (i) we consider, in addition to F_n, the quantity

$$\bar{F_n} = \inf_{\substack{\bar{S}\subset J\\ \#\bar{S}=n}} \Big\{ \int\limits_J \min_{\bar{p}\in\bar{S}} \{ f(\varrho_M(\bar{p},x))\} w(x) d\omega_M(x) \Big\}.$$

Our aim is to show that

$$\text{(2.70)}\quad F_n \leq \bar{F_n} \lesssim \operatorname{div} \Big(\int\limits_J w(x)^{\frac{d}{\alpha+d}} d\omega_M(x) \Big)^{\frac{\alpha+d}{d}} f\Big(\frac{1}{n^{\frac{1}{d}}}\Big) \text{ as } n \to \infty.$$

Let $\lambda > 1$ and choose $U_l, V_l, \dots, l = 1, \dots, m$, as in 2.3 such that

(2.71) (2.6) – (2.11) hold.

In the first step of the proof of (2.70) we choose sets $C_{li}, i = 1, \dots, i_l, l = 1, \dots, m$, with the following properties, where $\sigma > 0$ (is small) and W_l is as in (2.10):

(2.72) For each $l = 1, \dots, m$, the sets $C_{li}, i = 1, \dots, i_l$, are compact, non-overlapping and such that

$$J_l \subset C_{l1} \cup \ldots \cup C_{li_l} \subset V_l, C_{li} \cap J_l \neq \emptyset.$$

(2.73) C'_{li} is a cube with respect to the norm $q_l(\cdot)^{\frac{1}{2}}$ (thus, in particular, C_{li} is measurable) and such that

$$(\det q_l)^{\frac{1}{2}} |C'_{li}| W_l^{\frac{d}{\alpha+d}} = \sigma \int\limits_J w(x)^{\frac{d}{\alpha+d}} d\omega_M(x).$$

$$\text{(2.74)}\quad \sum_{l,i} \int\limits_{C_{li}} \leq \lambda \int\limits_J .$$

Let $i_0 = i_1 + \ldots + i_m$. Since

$$\sum_{l,i} (\det q_l)^{\frac{1}{2}} |C'_{li}| W_l^{\frac{d}{\alpha+d}} \leq \lambda^{\frac{d}{\alpha+d}+1} \sum_{l,i} \int\limits_{C_{li}} w(x)^{\frac{d}{\alpha+d}} d\omega_M(x) \leq \lambda^3 \int\limits_J w(x)^{\frac{d}{\alpha+d}} d\omega_M(x)$$

by (2.71) and (2.10), (2.7), the definition of integrals on M, and (2.74), it follows from (2.73) that

(2.75) $\sigma i_0 \leq \lambda^3$.

By (2.14) we may chooose k_0 so large that

(2.76) $\frac{1}{i_0^{\frac{\alpha}{d}}} f\Big(\frac{1}{k^{\frac{1}{d}}}\Big) \leq \lambda f\Big(\frac{1}{(i_0 k)^{\frac{1}{d}}}\Big)$ for $k \geq k_0$,

(2.77) $f\Big(\frac{1}{(i_0 k)^{\frac{1}{d}}}\Big) \leq \lambda f\Big(\frac{1}{(i_0(k+1))^{\frac{1}{d}}}\Big)$ for $k \geq k_0$.

In the next step of the proof of (2.70) we consider first the case where n is of the form $n = i_0 k$ for some $k \geq k_0$. Assume that

(2.78) $\inf\limits_{\substack{T' \subset \mathbb{E}^d \\ \# T' = k}} \Big\{ \int\limits_{C'_{li}} \min\limits_{t \in T'} \{ f(q_l(t-u)^{\frac{1}{2}})\} du \Big\}$

is attained for $T' = T'_{lik}$ where $T'_{lik} \subset C'_{li}, \# T'_{lik} = k$.

Applying the result (2.19) for $M = \mathbb{E}^d$ and $J = C'_{li}, i = 1, \dots, i_l, l = 1, \dots, m$, we obtain the following, where, omitting indices, $T'_{lik} = \{t_1, \dots, t_k\}$ and $D'_1, \dots, D'_k$ are the cells in C'_{li} corresponding to the set T'_{lik} and the norm $q_l(\cdot)^{1/2}$.

(2.79) There is a $\tau > 1$, independent of l, i, k, such that any two distinct points from $T'_{lik} = \{t_1, \dots, t_k\}$ have distance at least $1/\tau k^{1/d}$ and each of the cells $D'_1, \dots, D'_k$ has circumradius at most $\tau/k^{1/d}$ (with respect to the norm $q_l(\cdot)^{1/2}$).

Since J is (Jordan) measurable, $\operatorname{bd} J$ is measurable and $\omega_M(\operatorname{bd} J) = 0$. hence $(C_{li} \cap \operatorname{bd} J)'$ is measurable in $\mathbb{E}^d$ with $|(C_{li} \cap \operatorname{bd} J)'| = 0$. A result on Jordan measurable sets of measure 0 in $\mathbb{E}^d$ then says that the *parallel set*

$$\bigcup_{u \in (C_{li} \cap \operatorname{bd} J)'} (u + \nu B^d)$$

has arbitrarily small measure if $\nu > 0$ is sufficiently small. Here B^d is the solid Euclidean unit ball in $\mathbb{E}^d$. This remark together with (2.79) yields the next proposition, where $D_j = h_l^{-1}(D'_j)$:

(2.80) There are at most $o(k)$ cells D'_j in C'_{li} with $(D_j \cap \operatorname{bd} J)' \neq \emptyset$, where we may choose the same Landau symbol $o(k)$ for all l, i.

If $D'_j \subset (C_{li} \cap \operatorname{int} J)'$, let $\bar{t}_j = t_j$, if $D'_j \cap (C_{li} \cap \operatorname{bd} J)' \neq \emptyset$, choose for $\bar{t}_j$ an arbitrary point of $D'_j \cap (C_{li} \cap \operatorname{bd} J)'$. Let $\bar{T}'_{lik}$ be the set of all points $\bar{t}_j$ which can be obtained in this way and let $\bar{T}_{lik} = h_l^{-1}(\bar{T}'_{lik}) \subset C_{li}$. Then the definition of integrals on M in (2.4), the assumptions that f is increasing and $f, w \geq 0$, (2.72), (2.71) and (2.6), (2.7), (2.10), (2.14), (2.2), (2.79), (2.80), (2.57), and (2.73) yield the following:

(2.81)

$$
\begin{aligned}
&\int\limits_{C_{li}\cap J} \min_{\bar p\in \bar T_{lik}} \{f(\varrho_M(\bar p, x))\} w(x) d\omega_M(x)\\
&\leq \lambda \int\limits_{(C_{li}\cap J)'} \min_{\bar t\in \bar T'_{lik}} \{f(\lambda q_l(\bar t-u)^{\frac12})\} du\ W_l\ (\det q_l)^{\frac12}\\
&\lesssim \lambda^{1+\alpha} \int\limits_{C'_{li}} \min_{t\in T_{lik}} \{f(q_l(t-u)^{\frac12})\} du\ W_l\ (\det q_l)^{\frac12} + o(k)\beta\frac{2^d\tau^d}{k} f\Big(\frac{2\tau}{k^{\frac1d}}\Big)\\
&\sim \lambda^{1+\alpha}\ \mathrm{div}\ (\det q_l)^{\frac{\alpha+d}{2d}} |C'_{li}|^{\frac{\alpha+d}{d}}\ W_l\ f\Big(\frac{1}{k^{\frac1d}}\Big) + o(1) f\Big(\frac{1}{k^{\frac1d}}\Big)\\
&\lesssim \lambda^{1+\alpha+1+\frac{\alpha+d}{d}}\ \mathrm{div}\ \Big(\sigma \int\limits_J w(x)^{\frac{d}{\alpha+d}} d\omega_M(x)\Big)^{\frac{\alpha+d}{d}} f\Big(\frac{1}{k^{\frac1d}}\Big) \text{ as } k\to\infty,
\end{aligned}
$$

where the symbols $o(k), o(1)$ are independent of l, i.

For $n = i_o k$ let $\bar T_n = \bigcup_{l,i} \bar T_{lik}$. Then $\#\bar T_n \leq n$ and $\bar T_n \subset J$. The definitions of F_n and $\bar F_n$, (2.72), (2.71) and (2.9), $i_0 = i_1 + \ldots + i_m$, (2.81), (2.75) and (2.14) together imply that

$$
\begin{aligned}
F_n \leq \bar F_n &\leq \int\limits_J \min_{\bar p\in \bar T_n} \{f(\varrho_M(\bar p, x))\} w(x) d\omega_M(x)\\
&\leq \sum_{l,i} \int\limits_{C_{li}\cap J} \min_{\bar p\in \bar T_{lik}} \{f(\varrho_M(\bar p, x))\} w(x) d\omega_M(x)\\
&\lesssim \lambda^{3+2\alpha} i_0\ \mathrm{div}\ \Big(\sigma \int\limits_J w(x)^{\frac{d}{\alpha+d}} d\omega_M(x)\Big)^{\frac{\alpha+d}{d}} f\Big(\frac{1}{k^{\frac1d}}\Big)\\
&= \lambda^{3+2\alpha}\ \mathrm{div}\ \Big(\int\limits_J w(x)^{\frac{d}{\alpha+d}} d\omega_M(x)\Big)^{\frac{\alpha+d}{d}} \frac{(i_0\sigma\Big)^{\frac{\alpha+d}{d}}}{i_0^{\frac{\alpha}{d}}} f\Big(\frac{1}{k^{\frac1d}}\Big)\\
&\lesssim \lambda^{3+2\alpha+\frac{3\alpha+3d}{d}}\ \mathrm{div}\ \Big(\int\limits_J w(x)^{\frac{d}{\alpha+d}} d\omega_M(x)\Big)^{\frac{\alpha+d}{d}} f\Big(\frac{1}{n^{\frac1d}}\Big) \text{ as } n = i_o k\to\infty.
\end{aligned}
$$

Since $\lambda > 1$ was arbitrary, this proves (2.70) for n of the form $n = i_o k$. This, together with (2.77) yields (2.70) for general n; compare the argument in the last part of the proof of (2.53).

Having proved (2.58) and (2.70), the proof of the asymptotic formula (i) with $S\subset M$ and also with $\bar S\subset J$ is complete.

2.7 Uniform distribution of $\mathbf{S_n}, \mathbf{\bar S_n}$. Assume that J is connected and has positive density, i.e. satisfies (2.2). We will prove that

(2.82) S_n is uniformly distributed in J with respect to the density $w^{d/(\alpha+d)}$ as $n\to\infty$.

A proof which is similar to that of (2.80), but slightly more complicated in several details since it deals with M instead of $\mathbb{E}^d$, and which makes use of (2.19), (2.24), and (2.5) yields the following:

(2.83) Let $Z \subset J$ be measurable with $\omega_M(Z) = 0$.
Then $n(Z)$ $(= \#\{i : D_{ni} \cap Z \neq \emptyset\}) = o(n)$ as $n \to \infty$.

More difficult is the proof of

(2.84) Let $K \subset J$ be measurable with $0 < \omega_M(K) \leq \omega_M(J)$. Then
$n(K) \sim \eta(K)n$ as $n \to \infty$, where $\eta(K) = \int\limits_K \Big/ \int\limits_J$.

If $\omega_M(K) = \omega_M(J)$, then (2.84) follows from (2.20), which shows that $n(J) = n$, and from (2.83). Suppose now that $\omega_M(K) < \omega_M(J)$ and assume that (2.84) does not hold:

(2.85) $n(K) \not\sim \eta(K)n$ as $n \to \infty$.

¿From (2.18) and the definitions of $S_n(K), n(K)$ in 2.6 it follows that

$$\begin{aligned}
\nu f\Big(\frac{1}{n^{\frac{1}{d}}}\Big) \geq F_n &\geq \int\limits_K \min_{p \in S_n}\{f(\varrho_M(p,x))\}w(x)d\omega_M(x) \\
&= \int\limits_K \min_{p \in S_n(K)}\{f(\varrho_M(p,x))\}w(x)d\omega_M(x) \\
&\geq \inf_{\substack{T \subset M \\ \#T = n(K)}} \Big\{ \int\limits_K \min_{p \in T}\{f(\varrho_M(p,x))\}w(x)d\omega_M(x)\Big\}.
\end{aligned}$$

This can hold only if $n(K) \to \infty$ as $n \to \infty$. Now, applying the asymptotic formula (i) to K instead of J, we conclude further that

$$\nu f\Big(\frac{1}{n^{\frac{1}{d}}}\Big) \gtrsim \operatorname{div} \int\limits_K^{\frac{\alpha+d}{d}} f\Big(\frac{1}{n(K)^{\frac{1}{d}}}\Big) \text{ as } n \to \infty.$$

Using the assumption that f is strictly increasing and (2.14) it follows that $n(K)/n$ is bounded below by a positive constant. An analogous statement holds for the set $L = J \backslash K$. By (2.20) and (2.83),

(2.86) $n(K) + n(L) = n + o(n)$ as $n \to \infty$.

(2.85), (2.86) and the above statements on $n(K)/n$ and $n(L)/n$ show that

(2.87) there are $\vartheta, \iota > 0$ such that $\vartheta < 1 < \iota$ or $\iota < 1 < \vartheta$ and $n(K) \sim \vartheta\eta(K)n, n(L) \sim \iota\eta(L)n$ as $n \to \infty$, where n is from a suitable subsequence of $1, 2, \ldots$.

¿From now on consider in the proof of (2.82) only n from the subsequence in (2.87). We consider only the case that $\vartheta < 1 < \iota$. By (i), the definition of $S_n, J = K \cup L, K \cap L = \emptyset$, the definitions of $S_n(K), n(K), S_n(L), n(L)$ and $n(K), n(L) \to \infty$ as $n \to \infty$ (see (2.87)), the asymptotic formula (i) for K and L instead of J, (2.87), (2.14), $\eta(L) = 1 - \eta(K)$ (see (2.84)), (2.15), $\vartheta < 1 < \iota$, (2.87) and (2.86), and (2.3) we arrive at the following contradiction, where $\kappa > 0$, and thus conclude the proof of (2.84):

$$\begin{aligned}
F_n &= \int_K \min_{p \in S_n(K)} \{f(\varrho_M(p,x))\} w(x) d\omega_M(x) \\
&\quad + \int_L \min_{p \in S_n(L)} \{f(\varrho_M(p,x))\} w(x) d\omega_M(x) \\
&\gtrsim \operatorname{div} \Big(\int_K^{\frac{\alpha+d}{d}} f\Big(\frac{1}{n(K)^{\frac{1}{d}}}\Big) + \int_L^{\frac{\alpha+d}{d}} f\Big(\frac{1}{n(L)^{\frac{1}{d}}}\Big)\Big) \\
&\sim \operatorname{div} \int_J^{\frac{\alpha+d}{d}} \Big(\eta(K)^{\frac{\alpha+d}{d}} f\Big(\frac{1}{(\vartheta\eta(K)n)^{\frac{1}{d}}}\Big) + \eta(L)^{\frac{\alpha+d}{d}} f\Big(\frac{1}{(\iota\eta(L)n)^{\frac{1}{d}}}\Big)\Big) \\
&\sim \operatorname{div} \int_J^{\frac{\alpha+d}{d}} \Big(\eta(K) f\Big(\frac{1}{(\vartheta n)^{\frac{1}{d}}}\Big) + (1-\eta(K)) f\Big(\frac{1}{(\iota n)^{\frac{1}{d}}}\Big)\Big) \\
&\geq (1+\kappa) \operatorname{div} \int_J^{\frac{\alpha+d}{d}} f\Big(\frac{1}{(\eta(K)\vartheta n + (1-\eta(K))\iota n)^{\frac{1}{d}}}\Big) \\
&\sim (1+\kappa) \operatorname{div} \int_J^{\frac{\alpha+d}{d}} f\Big(\frac{1}{(n(K)+n(L))^{\frac{1}{d}}}\Big) \sim (1+\kappa) \operatorname{div} \int_J^{\frac{\alpha+d}{d}} f\Big(\frac{1}{n^{\frac{1}{d}}}\Big) \\
&\sim (1+\kappa)\, F_n \text{ as } n \to \infty \text{ in the subsequence from (2.87).}
\end{aligned}$$

Applying (2.83) and (2.84) to measurable $K \subset J$ yield the following:

$$\begin{aligned}
&\#(K \cap S_n) \leq n(K) = o(n) && \text{as } n \to \infty \quad \text{for } \omega_M(K) = 0, \\
&\#(K \cap S_n) = n(K) - o(n) \sim \eta(K)n \text{ as } n \to \infty && \text{for } 0 < \omega_M(K) \leq \omega_M(J).
\end{aligned}$$

This concludes the proof of (2.82). A similar, even simpler proof shows that

(2.88) $\bar{S}_n$ is uniformly distributed in J with respect to the density $w^{d/(\alpha+d)}$ as $n \to \infty$.

2.8 Conclusion. Having proved (2.58) and (2.70), (2.19) and (2.50), and (2.82) and (2.88), the proof of Theorem 1 is complete. □

3 Distortion of High Resolution Vector Quantization

3.1 Distortion of vector quantization. Let $C_1, \dots, C_n$ be n measurable sets which tile a measurable set C in $\mathbb{E}^d$ and let $c_1, \dots, c_n \in \mathbb{E}^d$. To each *signal* $x \in C$

assign the *codevector* or *codeword* c_i from the *codebook* $\{c_1, \dots, c_n\}$, where $x \in C_i$. (In case of ambiguity, which can occur only for x in a set of measure 0, choose for c_i any codevector where $x \in C_i$.) Common measures for the quality of the thus defined *encoder* or (*vector*) *quantizer* on C can be described as follows: let $f : [0, +\infty) \to [0, +\infty)$ with $f(0) = 0$ be increasing, $\| \cdot \|$ a norm on $\mathbb{E}^d$, and $w : C \to \mathbb{R}^+$ continuous. Up to normalization, w is the *density* of the *source* which generates the signals x. Then the corresponding (*average*) *distortion* of our quantizer is defined to be

$$\sum_{i=1}^{n} \int_{C_i} f(\|c_i - u\|) w(u) du.$$

How should the *cells* C_i and the codevectors c_i be chosen in order to minimize the distortion? Given $c_1, \dots, c_n$, the distortion is minimized if $C_1 \subset D_1, \dots, C_n \subset D_n$ where $D_1, \dots, D_n$ are the Dirichlet–Voronoi cells in C corresponding to $c_1, \dots, c_n$ and using $\| \cdot \|$. Then the distortion is

$$\int_C \min_{i=1,\dots,n} \{f(\|c_i - u\|)\} w(u) du.$$

Thus the minimum distortion is given by

$$\inf_{\substack{\bar{S} \subset C \\ \#\bar{S} = n}} \Big\{ \int_C \min_{c \in \bar{S}} \{f(\|c_i - u\|)\} w(u) du \Big\}.$$

3.2 Asymptotic results. Clearly, Theorem 2 translates into the following result.

Theorem 3. *Let $f : [0, +\infty) \to [0, +\infty)$ satisfy the growth condition (2.1), let $\alpha > 0$ be the corresponding constant, and let $\| \cdot \|$ be a norm on $\mathbb{E}^d$. Then there is a constant* $\operatorname{div} > 0$, *depending only on f and $\| \cdot \|$, such that the following statement holds: let $C \subset \mathbb{E}^d$ be compact and measurable with $|C| > 0$ and $w : C \to \mathbb{R}^+$ continuous. Then*

(i) *the minimum distortion in the above sense of a quantizer on C with source density w with n codewords is asymptotically equal to*
$$\operatorname{div}\Big(\int_C w(u)^{\frac{d}{\alpha+d}} du \Big)^{\frac{\alpha+d}{d}} f\Big(\frac{1}{n^{\frac{1}{d}}} \Big) \quad \text{as } n \to \infty.$$

If, in addition, C is connected and has positive density, i.e. satisfies (2.2), then the following claims hold: let (S_n) be a sequence of codebooks in C, where $\#S_n = n$ for $n = 1, 2, \dots$, and such that the infimum in (i) is attained for S_n. Then

(ii) *there is a constant $\delta > 1$ such that S_n is $(1/\delta n^{1/d}, \delta/n^{1/d})$-Delone in J for $n = 1, 2, \dots$,*

(iii) *S_n is uniformly distributed in C with density $w^{d/(\alpha+d)}$ as $n \to \infty$.*

Remark. Theorem 1 admits an analogous interpretation.

4 Error of Approximation of Probability Measures by Discrete Measures

4.1 Quantization of probability measures. Let $f : [0, \infty) \to [0, \infty)$ be monotone increasing, such that $f(t) = 0$ only for $t = 0$ and let $\|\cdot\|$ be a norm on $\mathbb{E}^d$. Then we can define on the space of all probability measures on $\mathbb{E}^d$ the following notion of distance:

$$\rho^f(P, Q) = \inf\Big\{ \int\limits_{\mathbb{E}^d \times \mathbb{E}^d} f(\|x - y\|) d\mu(x, y) \Big\} \text{ for probability measures } P, Q,$$

where the infimum is extended over all Borel probability measures μ on $\mathbb{E}^d \times \mathbb{E}^d$ which satisfy the following condition:

$$P(B) = \mu(B \times \mathbb{E}^d),\ Q(C) = \mu(\mathbb{E}^d \times C) \text{ for all Borel sets } B, C \subset \mathbb{E}^d.$$

ρ^f is a slight extension of the *Wasserstein-* or *Kantorovich metric* on the space of all probability measures on $\mathbb{E}^d$. The question arises as to how well can probability measures be approximated by discrete measures and to describe the best approximating discrete measures. Let $\mathcal{D}_n$ denote the space of all discrete probabiliy measures on $\mathbb{E}^d$ the support of which consists of at most n points, $n = 1, 2, \dots$. Then for the *error of best approximation* of a probability measure P on $\mathbb{E}^d$ by probability measures from $\mathcal{D}_n$ the following representation holds:

$$\rho^f(P, \mathcal{D}_n) = \inf_{D \in \mathcal{D}_n} \{\rho^f(P, D)\} = \inf_{\substack{S \subset \mathbb{E}^d \\ \# S = n}} \Big\{ \int\limits_{\mathbb{E}^d} \min_{s \in S} \{f(\|u - s\|)\} dP(u) \Big\}.$$

For this and for references compare Graf and Luschgy [20].

4.2 Asymptotic approximation results. Considering the above, it is clear that Theorem 2 yields the following result on approximation of probability measures.

Theorem 4. *Let $f : [0, +\infty) \to [0, +\infty)$ satisfy the growth condition (2.1), let $\alpha > 0$ be the corresponding constant, and let $\|\cdot\|$ be a norm on $\mathbb{E}^d$. Then there is a constant* div > 0*, depending only on f and $\|\cdot\|$, such that the following statement holds: let $J \subset \mathbb{E}^d$ be compact and measurable with $|J| > 0$ and P a probability measure on J with continuous density $p : J \to \mathbb{R}^+$. Then*

(i) *the error of best approximation of P by probability measures from $\mathcal{D}_n$ is asymptotically equal to*
$$\operatorname{div}\Big(\int\limits_C p(u)^{\frac{d}{\alpha+d}} du \Big)^{\frac{\alpha+d}{d}} f\Big(\frac{1}{n^{\frac{1}{d}}}\Big) \text{ as } n \to \infty.$$

If, in addition, J is connected and has positive density, i.e. satisfies (2.2), then the following claims hold: let (S_n) be a sequence of supports in J, where $\# S_n = n$ for $n = 1, 2, \dots$, and such that for suitable corresponding discrete probability measures in $\mathcal{D}_n$ the minimum error is attained. Then

(ii) *there is a constant* $\delta > 1$ *such that* S_n *is* $(1/\delta n^{1/d}, \delta/n^{1/d})$*-Delone in* J *for* $n = 1, 2, \dots,$

(iii) S_n *is uniformly distributed in* J *with density* $w^{d/(\alpha+d)}$ *as* $n \to \infty$.

Remark. Obviously, Theorem 1 yields a similar result for the approximation of probability measures by discrete probability measures on Riemannian manifolds.

5 Error of Numerical Integration Formulae

5.1 Error of numerical integration formulae. Let $J \subset \mathbb{E}^d$ be a compact measurable set in $\mathbb{E}^d$ with $|J| > 0$, let $\mathcal{F}$ be a class of Riemann integrable functions $g : J \to \mathbb{R}$ and $w : J \to \mathbb{R}^+$ a continuous *(weight) function.* For given sets of n *nodes* $N = \{p_1, \dots, p_n\} \subset J$ and n *weights* $W = \{w_1, \dots, w_n\} \subset \mathbb{R}$ the *error* of the numerical integration formula

$$\int_J g(u)w(u)du \approx \sum_{i=1}^n g(p_i)w_i \text{ for } g \in \mathcal{F},$$

is defined by

$$(5.1) \quad E(\mathcal{F}, w, N, W) = \sup_{g\in\mathcal{F}} \Big\{ \Big| \int_J g(u)w(u)du - \sum_{i=1}^n g(x_i)w_i \Big| \Big\}.$$

The *minimum error* is then

$$(5.2) \quad E(\mathcal{F}, w, n) = \inf_{\substack{N\subset J, \#N=n \\ W\subset\mathbb{R}, \#W=n}} \{E(\mathcal{F}, w, N, W)\}.$$

Now the problems arise to determine $E(\mathcal{F}, w, n)$ and to describe the optimal choices of nodes and weights. While for general $\mathcal{F}$ not much can be said, we consider a space of functions for which rather precise information can be given.

5.2 Asymptotic results. A *modulus of continuity* is an increasing continuous function $f : [0, +\infty) \to [0, +\infty)$ with $f(0) = 0$ such that

$$(5.3) \quad f(s + t) \le f(s) + f(t) \text{ for } s, t \ge 0.$$

Given a modulus of continuity f and a norm $\|\cdot\|$ on $\mathbb{E}^d$, define the corresponding *Hölder class* of functions for $J \subset \mathbb{E}^d$ by

$$(5.4) \quad \mathcal{H}^f(\|\cdot\|, J) = \{g : J \to \mathbb{R} : |g(u) - g(v)| \le f(\|u - v\|) \text{ for } u, v \in J\}.$$

Theorem 5. *Let* $f : [0, +\infty) \to [0, +\infty)$ *be a modulus of continuity which satisfies the growth condition (2.1), let* $\alpha > 0$ *be the corresponding constant, and let* $\|\cdot\|$ *be a norm on* $\mathbb{E}^d$*. Then there is a constant* $\operatorname{div} > 0$*, depending only on* f *and* $\|\cdot\|$*, such that we have the following: let* $J \subset \mathbb{E}^d$ *be compact and measurable with* $|J| > 0$ *and* $w : J \to \mathbb{R}^+$ *continuous. Then*

(i) *for the minimum error of a numerical integration formula with weight function* w *for the Hölder class* $\mathcal{H}^f = \mathcal{H}^f(J, \|\cdot\|)$ *holds the asymptotic formula,*

$$E(\mathcal{H}^f, w, n) \sim \operatorname{div}\Big(\int_J w(u)^{\frac{d}{\alpha+d}} du\Big)^{\frac{\alpha+d}{d}} f\Big(\frac{1}{n^{\frac{1}{d}}}\Big) \text{ as } n \to \infty.$$

If, in addition, J is connected and has positive density, i.e. satisfies (2.2), then the following assertions hold: let (N_n) be a sequence of sets of nodes in J, where $\# N_n = n$ for $n = 1, 2, \ldots$, and such that the infimum in (4.2) is attained for N_n and suitable corresponding W_n. Then

(ii) *there is a constant $\delta > 1$ such that N_n is $(1/\delta n^{1/d}, \delta/n^{1/d})$-Delone in J for $n = 1, 2, \ldots$,*

(iii) *N_n is uniformly distributed in J with density $w^{d/(\alpha+d)}$ as $n \to \infty$.*

Remark. An analogous result holds for Riemannian d-manifolds instead of $\mathbb{E}^d$ and $\|\cdot\|$. The asymptotic formula (5.3) for $d = 2, \|\cdot\| = \|\cdot\|_2$ and $w = 1$ is due to Babenko [2]. For general d and arbitrary norm a related yet different asymptotic formula was given by Chernaya [8]. The fact that in Chernaya's result w is assumed to be Lebesgue integrable can be reduced to the continuous case by Lusin's theorem. A pertinent idea is described also by Sobol' [45].

5.3 Proof of Theorem 5. First, the following will be shown:

(5.5) Let $N = \{p_1, \ldots, p_n\} \subset J$ and let $h : J \to \mathbb{R}$ be defined by $h(u) = \min\limits_{p \in N}\{f(\|p - u\|)\}$. Then $h \in \mathcal{H}^f$.

To see this, let $u, v \in J$. By exchanging u and v, if necessary, we may assume that $h(u) \geq h(v)$. Then

$$\begin{aligned}
0 &\leq h(u) - h(v) = \min_{p \in N}\{f(\|p-u\|)\} - \min_{q \in N}\{f(\|q-v\|)\} \\
&= f(\|p-u\|) - f(\|q-v\|) \text{ for suitable } p, q \in N \\
&\leq f(\|q-u\|) - f(\|q-v\|) \\
&= f(\|q-v+v-u\|) - f(\|q-v\|) \\
&\leq f(\|q-v\| + \|v-u\|) - f(\|q-v\|) \\
&\leq f(\|q-v\|) + f(\|v-u\|) - f(\|q-v\|) = f(\|u-v\|),
\end{aligned}$$

where we have used the assumptions that f is increasing and that is a modulus of continuity.

Secondly,

(5.6) let $N = \{p_1, \ldots, p_n\} \subset J$ and $W = \{w_1, \ldots, w_n\} \subset \mathbb{R}$. Define

$$\begin{aligned}
D_i &= \{u \in J : \|p_i - u\| \leq \|p_j - u\| \text{ for } j = 1, \ldots, n\} \\
E_i &= D_i \backslash (D_1 \cup \ldots \cup D_{i-1}), \bar{w}_i = \int\limits_{E_i} w(u) du, i = 1, \ldots, n \\
\bar{W} &= \{\bar{w}_, \ldots, \bar{w}_n\}
\end{aligned}$$

Then

$$E(\mathcal{H}^f, w, N, W) \geq E(\mathcal{H}^f, w, N, \bar{W}) = \int\limits_J \min_{p \in N}\{f(\|p-u\|)\} w(u) du.$$

Since J is the disjoint union of $E_1, \dots, E_n, g \in \mathcal{H}^f, E_i \subset D_i$, the assumptions that f is increasing and $w \geq 0$, the definition of h, $h \in \mathcal{H}^f$ and $h(p_i) = 0$ (see (5.5)) together imply (5.6):

$$\begin{aligned}
E(\mathcal{H}^f, w, N, \bar{W}) &= \sup_{g \in \mathcal{H}^f} \Big\{ \Big| \int_J g(u)w(u)du - \sum_{i=1}^n g(p_i) \int_{E_i} w(u)du \Big| \Big\} \\
&\leq \sup_{g \in \mathcal{H}^f} \Big\{ \sum_{i=1}^n \int_{E_i} |g(u) - g(p_i)| w(u)du \Big\} \\
&\leq \sum_{i=1}^n \int_{E_i} f(\|p_i - u\|) w(u)du = \sum_{i=1}^n \int_{E_i} \min_{p_j \in N} \{ f(\|p_j - u\|)\} w(u)du \\
&= \int_J \min_{p \in N} \{ f(\|p - u\|)\} w(u)du = \int_J h(u)w(u)du \\
&= \int_J h(u)w(u)du - \sum_{i=1}^n h(p_i)w_i \leq E(\mathcal{H}^f, w, N, W).
\end{aligned}$$

Thirdly, it follows from (5.6) that

$$\begin{aligned}
E(\mathcal{H}^f, w, n) &= \inf_{\substack{N \subset J, \# N = n \\ W \subset \mathbb{R}, \# W = n}} \{ E(\mathcal{H}^f, w, N, W)\} = \inf_{\substack{N \subset J \\ \# N = n}} \{ E(\mathcal{H}^f, w, N, \bar{W})\} \\
&= \inf_{\substack{N \subset J \\ \# N = n}} \Big\{ \int_J \min_{p \in N} \{ f(\|p - u\|)\} w(u)du \Big\}.
\end{aligned}$$

Now apply Theorem 2 to obtain Theorem 5.

6 Volume Approximation of Convex Bodies

6.1 Best volume approximation of convex bodies. Let C be a *convex body* in $\mathbb{E}^d$, i.e. a compact convex subset of $\mathbb{E}^d$ with non-empty interior and let $\mathcal{P}^c_{(n)}$ be the set of all convex polytopes with n facets which are circumscribed to C. Let $\delta^V(\cdot, \cdot)$ denote the symmetric difference metric on the space of all convex bodies in $\mathbb{E}^d$. The problems arise to determine or estimate the quantity

$$\delta^V(C, \mathcal{P}^c_{(n)}) = \inf\{\delta^V(C, P) : P \in \mathcal{P}^c_{(n)}\}$$

and to describe the polytopes $P \in \mathcal{P}^c_{(n)}$ for which the infimum is attained, the *best approximating polytopes* of C in $\mathcal{P}^c_{(n)}$ with respect to δ^V. See [25] and [27] for the literature on this and related problems.

6.2 Asymptotic approximation results. Refining a result of Böröczky [4] (asymptotic formula), the following result has been proved by Gruber [32] (asymptotic formula and Delone property) and Glasauer and Gruber [19] (uniform distribution property). A weaker version of it (without the error term in the asymptotic formula) can be obtained as a simple consequence of Theorem 1 above and its proof; compare the corresponding argument in [30].

Theorem 6. *There is a constant* $\mathrm{div}_{d-1} > 0$ *with the following property: let* C *be a convex body in* $\mathbb{E}^d$ *(with boundary) of class* $\mathcal{C}^3$ *and with Gauss curvature* $\kappa_C > 0$. *Let* $(P_n), P_n \in \mathcal{P}^c_{(n)}$, *be a sequence of best approximating polytopes of* C. *Then we have the following statements:*

(i) $\delta^V(C, P^c_{(n)}) \sim \frac{1}{2}\mathrm{div}_{d-1}\Big(\int\limits_{\mathrm{bd}\, C} \kappa_C(x)^{\frac{1}{d+1}}\, d\sigma(x)\Big)^{\frac{d+1}{d-1}} \frac{1}{n^{\frac{2}{d-1}}} + O\Big(\frac{1}{n^{\frac{2}{d-1}+\frac{1}{3(d-1)}-\varepsilon}}\Big)$ *as* $n \to \infty$, *for any* $\varepsilon > 0$,

(ii) *there is a constant* $\delta > 1$ *such that the inradius of each facet of* P_n *is at least* $1/\delta n^{1/(d-1)}$ *and the circumradius at most* $\delta / n^{1/(d-1)}$ *for* $n = d+1, d+2, \ldots$,

(iii) *the sets* $C \cap \mathrm{bd}\, P_n$ *are uniformly distributed on* $\mathrm{bd}\, C$, *endowed with the ordinary surface area measure* σ, *with respect to the density* $\kappa_C^{1/(d+1)}$ *as* $n \to \infty$.

Remark. For $d = 3$, a sharper result was given in [31].

7 The Isoperimetric Problem for Convex Polytopes in Minkowski Spaces

7.1 The isoperimetric problem in Minkowski spaces. Let $\mathbb{E}^d$ be endowed with an additional norm. While in this new normed space the natural notion of volume is the ordinary volume $V(\cdot)$ or $|\cdot|$ *(Haar measure)*, several different proposals for surface area have been made by Busemann [6], Holmes and Thompson [35] and Benson [3]. These amount to the introduction of an o-symmetric convex body I, the *isoperimetrix*, depending in a suitable way on the solid unit ball of this norm. The *surface area* $S_I(C)$ of a convex body C then is defined to be

$$S_I(C) = \lim_{\delta \to +0} \frac{V(C+\delta I) - V(C)}{\delta}, \text{ where } C + \delta I = \{x + \delta y : x \in C, y \in I\}.$$

See the book [47]. For $n = d+1, d+2, \ldots$, let P_n be a convex polytope which has minimum *isoperimetric quotient*

$$\frac{S_I(P_n)^d}{V(P_n)^{d-1}}$$

among all convex polytopes in $\mathbb{E}^d$ with n facets. The problem arises to determine or estimate the minimum isoperimetric quotient and to describe the form of the minimizing polytopes P_n.

7.2 Asymptotic isoperimetric results. A result of Diskant [10, 11], which generalizes a theorem of Lindelöf [38] for the Euclidean case, says that after a suitable homothety has been applied to P_n, we may assume that P_n is circumscribed to I, i.e. $P_n \in \mathcal{P}^c_{(n)}(I)$. Then it is easy to see that

$$\frac{S_I(P_n)^d}{V(P_n)^{d-1}} = d^d V(P_n).$$

Since the isoperimetric quotient is minimal for P_n, the polytope P_n thus is best approximating of I in $\mathcal{P}^c_{(n)}(I)$. Hence Theorem 5 yields the following result.

Theorem 7. *There is a constant* $\operatorname{div}_{d-1} > 0$ *such that the following hold: let* I *be an isoperimetrix in* $\mathbb{E}^d$ *of class* $\mathcal{C}^3$ *with* $\kappa_I > 0$. *For* $n = d+1, d+2, \dots$, *let* P_n *be a convex polytope with* n *facets circumscribed to* I *and minimum isoperimetric quotient* $S_I(P_n)^d / V(P_n)^{d-1}$. *Then the following assertions hold:*

(i) $$\frac{S_I(P_n)^d}{V(P_n)^{d-1}} \sim d^d V(I) + \frac{d^d}{2}\operatorname{div}_{d-1}\Big(\int\limits_{\operatorname{bd} I} \kappa_C(x)^{\frac{1}{d+1}} d\sigma(x)\Big)^{\frac{d+1}{d-1}} \frac{1}{n^{\frac{2}{d-1}}}$$
$$+ O\Big(\frac{1}{n^{\frac{2}{d-1}+\frac{1}{3(d-1)}-\varepsilon}}\Big) \textit{ as } n \to \infty, \textit{ for any } \varepsilon > 0,$$

(ii) *there is a constant* $\delta > 1$ *such that the inradius of each facet of* P_n *is at least* $1/\delta n^{1/(d-1)}$ *and the circumradius at most* $\delta / n^{1/(d-1)}$ *for* $n = d+1, d+2, \dots$,

(iii) *the sets* $C \cap \operatorname{bd} P_n$ *are uniformly distributed on* $\operatorname{bd} C$, *endowed with the ordinary surface area measure* σ, *with respect to the density* $\kappa_C^{1/(d+1)}$ *as* $n \to \infty$.

Remark. For $d = 3$, a sharper result was given in [31].

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