Asymptotic estimates for best and stepwise approximation of convex bodies IV

Peter M. Gruber

Abstract. In this article we first prove a stability theorem for coverings in $\mathbb{E}^2$ by congruent solid circles: if the density of such a covering is close to its lower bound $2\pi/\sqrt{27}$, then most of the centers of the circles are arranged in almost regular hexagonal patterns. A version of this result then is extended to coverings by geodesic discs in two-dimensional Riemannian manifolds.

Given a sufficiently differentiable convex body $C$ in $\mathbb{E}^3$, the following two problems are closely related: (i) Approximation of $C$ with respect to the Hausdorff metric, the Banach–Mazur distance and a notion of distance due to Schneider by inscribed or circumscribed convex polytopes. (ii) Covering of the boundary of $C$ by geodesic discs with respect to suitable Riemannian metrics.

The stability result for Riemannian manifolds and the relation between approximation and covering yield rather precise information on the form of best approximating inscribed convex polytopes $P_n$ of $C$ with respect to the Hausdorff metric: if the number $n$ of vertices is large, then most of the vertices are arranged in almost regular hexagonal patterns. Consequently, the majority of facets of $P_n$ are almost regular triangles. Here ‘regular’ is meant with respect to the Riemannian metric of the second fundamental form. Similar results hold for circumscribed polytopes and also for the Banach–Mazur distance and Schneider’s notion of distance.

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1 Introduction and statement of results

1.1 The aim of this article is twofold. First, stability results for coverings in the Euclidean plane $\mathbb{E}^2$ by congruent solid circles and in two-dimensional Riemannian manifolds by geodesic discs will be proved. Second, given a convex body in $\mathbb{E}^3$, information on the form of best approximating inscribed or circumscribed convex polytopes will be obtained, where approximation is with respect to the Hausdorff metric, the Banach–Mazur distance and a notion of distance due to Schneider.

1.2 A classical result of L. Fejes Tóth [?] which refines an earlier result of Kershner [?] says the following: the density of a covering of a compact convex disc in $\mathbb{E}^2$ with
non-empty interior by two or more congruent solid circles is greater than $2\pi/\sqrt{27}$. This lower bound is best possible. (By the density of a family of sets which cover a given set we mean the sum of the areas of the sets of the family divided by the area of the given set.) If the number of circles is large and the covering has minimum density, then L. Fejes Tóth [?], p. 61, indicated that the centers of the circles, in essence, are arranged almost hexagonally.

In recent years many stability problems have been investigated in convex geometry. These either belong to the area of geometric inequalities, or are of a more geometric type, see the survey of Groemer [?] and the references in Gruber [?]. Our first result lies somewhere between these two types.

Let $\| \cdot \|$ denote the Euclidean norm on $\mathbb{E}^2$. By $C(c, \varrho)$ we denote the solid circle with center $c$ and radius $\varrho$ in $\mathbb{E}^2$. In a set $C$ in $\mathbb{E}^2$ a point $c \in C$ is the center of a regular hexagon up to $\delta > 0$ if there are a constant $\sigma > 0$, the size of the hexagon, and points $c_1, \ldots, c_6 \in C$, its vertices, such that

$$\| c_k - c \| - \sigma, \| c_{k+1} - c_k \| - \sigma \leq \sigma \delta \quad (c_7 = c_1),$$

$$C(c, \frac{3}{2} \sigma) \cap C = \{c, c_1, \ldots, c_6\}.$$  

**Theorem 1.** Let $S$ be a convex 3, 4, 5, or 6-gon and $\varepsilon > 0$ sufficiently small. Then for all coverings of $S$ by, say, $m$ congruent solid circles of sufficiently small radius and density less than

$$\frac{2\pi}{\sqrt{27}}(1 + \varepsilon)$$

holds: in the set of centers of these circles, each center, with a set of less than $50\varepsilon^{1/3}m$ exceptions, is the center of a regular hexagon up to $500\varepsilon^{1/3}$. All these hexagons have the same size.

**Remark 1.** In the proof explicit bounds for $\varepsilon$ and the radius of the circles will be given. The choice of the exponents of $\varepsilon$ and of the coefficients in the Theorem is somewhat arbitrary.

**Remark 2.** Several natural extensions of Theorem 1 suggest themselves. First, to coverings of the whole plane which is easy and left to the reader; compare the remark of L. Fejes Tóth, p. 61, for coverings of the plane of density $2\pi/\sqrt{27}$. Second, to Riemannian manifolds of dimension two. A pertinent result of this type is Theorem 2 below. Third, to higher dimensions. Since in this case not even the densities of the thinnest coverings of space by congruent balls are known, it seems at present to be out of reach, to extend Theorem 1 in a precise form to dimensions greater than two.

1.3 In recent years a series of results of Euclidean geometry have been extended to more general spaces. Our second result is a (weak) version of Theorem 1 in this context. We have chosen this version because of its applicability to approximation problems; see Theorem 3.

For exact definitions of the following notions compare section 3. Let $M$ be a two-dimensional Riemannian manifold of class $C^2$ with metric of class $C^0$ and let $\gamma_M$ and $\omega_M$
be the corresponding geodesic metric and Jordan area measure on \( M \). The geodesic disc \( D_M(c, \varrho) \) with center \( c \) and radius \( \varrho \) in \( M \) is the set \( \{ x \in M : \gamma_M(c, x) \leq \varrho \} \).

Consider a sequence of sets \( C_n \) in \( M \) such that the number \#\( C_n \) of points of \( C_n \) is \( n \). As \( n \to \infty \), the set \( C_n \) is asymptotically a regular hexagonal pattern in \( M \) if the following hold: for each \( n \) there is a constant \( \sigma_n > 0 \) such that for all points \( c \in C_n \), with a set of \( o(n) \) exceptions, there are points \( c_1, \ldots, c_6 \in C_n \) such that

\[
|\gamma_M(c, c_k) - \sigma_n|, |\gamma_M(c_{k+1}, c_k) - \sigma_n| = \sigma_n \cdot o(1) \text{ as } n \to \infty \quad (c_7 = c_1),
\]

\[
D_M(c, 1.1\sigma_n) \cap C_n = \{ c, c_1, \ldots, c_6 \}.
\]

In [?] the following result was proved: Let \( J \subset M \) be Jordan measurable with \( \omega_M(J) > 0 \) and consider for \( n = 1, 2, \ldots \), a covering of \( J \) by \( n \) geodesic discs of minimum radius. Then the densities of the coverings tend to \( 2\pi/\sqrt{27} \) as \( n \to \infty \). The following result gives, in particular, information about the distribution of the centers of such coverings.

**Theorem 2.** Let \( J \) be a Jordan measurable set in \( M \) with \( \omega_M(J) > 0 \). For each \( n = 1, 2, \ldots \), consider a covering of \( J \) by \( n \) geodesic discs of the same radius such that the densities of these coverings tend to \( 2\pi/\sqrt{27} \) as \( n \to \infty \). Then, as \( n \to \infty \), the set of centers of the \( n \)th covering is asymptotically a regular hexagonal pattern.

**Remark 3.** At present it seems to be difficult to extend Theorem 2 to higher dimensions, see Remark 2.

1.4 A convex body \( C \) in \( E^d \) is a compact convex subset of \( E^d \) with non-empty interior. Given \( C \), denote by \( P_n^i \) and \( P_n^o \) the spaces of inscribed and of circumscribed convex polytopes having at most \( n \) vertices, resp. facets. The Hausdorff metric \( \delta^H \) on the space of all convex bodies in \( E^d \) is defined as follows: for convex bodies \( C, D \) let \( \delta^H(C, D) \) be the maximum Euclidean distance which a point of one of the bodies can have from the other body. Call a polytope \( P_n \in P_n^i \) best approximating of \( C \) with respect to \( \delta^H \) if

\[
\delta^H(C, P_n) = \delta^H(C, P_n^i) = \inf\{ \delta^H(C, P) : P \in P_n^i \}.
\]

A basic problem is to determine or estimate \( \delta^H(C, P_n) \) and to describe \( P_n \) as \( n \to \infty \).

Let (the boundary \( \text{bd} \) \( C \) of) \( C \) be (a surface) of class \( C^2 \) with Gauss curvature \( \kappa_C > 0 \). McClure and Vitale [?] showed for \( d = 2 \) that the set \( \text{vert} P_n \) of vertices of \( P_n \) is almost equally spaced along \( \text{bd} \) \( C \) as \( n \to \infty \), if a suitable notion of length is used. Their result was substantially refined by Ludwig [?]. For general \( d \) Glasauer and Schneider [?] proved that \( \text{vert} P_n \) is uniformly distributed in \( \text{bd} C \) with respect to the density \( \sqrt{\kappa_C} \) as \( n \to \infty \) in the sense of uniform distribution theory. This result gives a rough idea about the distribution of \( \text{vert} P_n \) in \( \text{bd} C \).

For \( d = 3 \) we will describe the (local) form of \( P_n \) and of the best approximating polytopes \( Q_n \in P_n^c \) for large \( n \) in a more precise way. This will be done using the Riemannian metric of the second fundamental form on \( \text{bd} \) \( C \) which in a natural sense corresponds to \( \delta^H \). Similar results hold also for the notions of distance \( \delta^{BM} \) and \( \delta^{SCH} \).
The Banach–Mazur distance $\delta^\text{BM}$ is defined for convex bodies $C, D$ which are symmetric with respect to the origin $o$:

$$\delta^\text{BM}(C, D) = \inf\{\lambda > 1 : C \subset l(D) \subset \lambda C \text{ for suitable linear } l : \mathbb{E}^d \to \mathbb{E}^d\}.$$ 

Let $P_{s,2n}$ and $P_{s,(2n)}$ be the spaces of convex polytopes which are $o$-symmetric and have at most $2n$ vertices, resp. facets. Given a convex body $C$ which is symmetric in $o$, let $P_{s,2n}^i$ be the set of those polytopes in $P_{s,2n}$ which are inscribed into $C$; similarly for $P_{s,(2n)}^c$. It is easy to see that

$$\delta^\text{BM}(C, P_{s,2n}) = \delta^\text{BM}(C, P_{s,2n}^i), \quad \delta^\text{BM}(C, P_{s,(2n)}) = \delta^\text{BM}(C, P_{s,(2n)}^c).$$

The Riemannian metric on bd $C$ corresponding to $\delta^\text{BM}$ is that of central affine differential geometry. See Gruber [?].

Given a convex body $C$ and a convex polytope $P$ contained in $C$, Schneider’s distance $\delta^\text{SCH}(C, P)$ is the maximum volume of a cap of $C$ determined by a halfspace which contains a facet of $P$ in its boundary, but does not contain $P$. The center of a cap of $C$ determined by a facet of $P$ is the point of the cap with maximum Euclidean distance from the hyperplane containing the facet. Let capcenter$P$ denote the set of all centers of caps. The Riemannian metric on bd $C$ which corresponds to Schneider’s distance is that of equiaffine differential geometry. See Schneider [?].

**Theorem 3.** Let $C$ be a convex body in $\mathbb{E}^3$ of class $C^2$ with positive Gauss curvature. Consider sequences $(P_n), \ldots, (T_n)$ of best approximating polytopes

- $P_n \in P_{ni}^i$, $Q_n \in P_{(n)}^c$ with respect to $\delta^H$,
- $R_n \in P_{s,2n}^i$, $S_n \in P_{s,(2n)}^c$ with respect to $\delta^\text{BM}$,
- $T_n \in P_{ni}^i$ with respect to $\delta^\text{SCH}$,

where for $\delta^\text{BM}$ we assume that $C$ is symmetric in $o$. Then the sets

- $\text{vert} P_n, (\text{bd} Q_n) \cap C$ for $\delta^H$,
- $\text{vert} R_n, (\text{bd} S_n) \cap C$ for $\delta^\text{BM}$,
- $\text{capcenter} T_n$ for $\delta^\text{SCH}$

are asymptotically regular hexagonal patterns on bd $C$ with respect to the corresponding Riemannian metrics.

**Remark 4.** A consequence of this result is that for large $n$ most facets of $P_n$ and $R_n$ are almost regular triangles, approximately of the same size, and the facets of $Q_n, S_n$, and $T_n$ are almost regular hexagons, approximately of the same size; the metrics being the corresponding Riemannian metrics on bd $C$.

**Remark 5.** As may be seen from the proof, Theorem 3 holds also for the more general asymptotically best approximating polytopes, where, for example, in the case of the metric $\delta^H$ and for inscribed polytopes this means a sequence $(U_n)$ where $U_n \in P_n^c$ such that

$$\delta^H(C, U_n) \sim \delta^H(C, P_n^i)$$

as $n \to \infty$. 

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2 Proof of Theorem 1

2.1 Choose
\[(2.1) \quad 0 < \varepsilon < 10^{-9} \quad \text{and} \quad 0 < \varrho < \frac{|S|}{2p(S)} \varepsilon.\]

(Here $| \cdot |$ and $p(\cdot)$ stand for Euclidean area and perimeter in $I_E^2$, respectively.) The proof of Theorem 1 is split into several steps.

2.2 In a first step two elementary properties of convex hexagons will be proved.

2.2.1
\[(2.2) \quad \text{Let } D \text{ be a convex hexagon such that } C(\sqrt{3}/2 - \varepsilon^{1/3}) \not\subset D \subset C(1). \text{ Then}\]
\[
|D| \leq \frac{\sqrt{27}}{2} (1 - \varepsilon^{2/3}/3).
\]

(Here $C(\varrho)$ denotes the solid circle in $I_E^2$ with center $o$ and radius $\varrho$.)

Elementary arguments show that $|D|$ is bounded above by the area of a convex hexagon with the following properties: all of its vertices are on $\partial C(1)$; the hexagon contains $C(\sqrt{3}/2 - \varepsilon^{1/3})$; one of its edges is tangent to $C(\sqrt{3}/2 - \varepsilon^{1/3})$ and the five other ones have equal length. Let
\[
\frac{\pi}{3} + 2\varphi \quad \text{and} \quad \frac{\pi}{3} - \frac{2\varphi}{5}, \quad 0 < 2\varphi < \frac{\pi}{3},
\]
be the angles under which the edges of the hexagon appear from $o$ (note (2.1)). Then
\[
\frac{\sqrt{3}}{2} - \varepsilon^{1/3} = \cos\left(\frac{\pi}{6} + \varphi\right) = \cos\frac{\pi}{6} - \sin\left(\frac{\pi}{6} + \xi\varphi\right) \cdot \varphi \geq \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} \varphi \geq \frac{\sqrt{3}}{2} - \varphi
\]
by Taylor's theorem, where $\xi \in (0, 1)$ is chosen suitably. Thus
\[
\varepsilon^{1/3} \leq \varphi.
\]

Then
\[
|D| \leq \sin\left(\frac{\pi}{6} + \varphi\right) \cos\left(\frac{\pi}{6} + \varphi\right) + 5 \sin\left(\frac{\pi}{6} - \frac{\varphi}{5}\right) \cos\left(\frac{\pi}{6} - \frac{\varphi}{5}\right)
\]
\[
= \frac{1}{2} \sin\left(\frac{\pi}{3} + 2\varphi\right) + \frac{5}{2} \sin\left(\frac{\pi}{3} - \frac{2\varphi}{5}\right)
\]
\[
= \frac{1}{2} \sin\frac{\pi}{3} + \frac{1}{2} \cos\frac{\pi}{3} \cdot 2\varphi - \frac{1}{4} \sin\left(\frac{\pi}{3} + 2\eta\varphi\right) \cdot (2\varphi)^2
\]
\[
+ \frac{5}{2} \sin\frac{\pi}{3} + \frac{5}{2} \cos\frac{\pi}{3} \cdot \frac{2\varphi}{5} - \frac{5}{4} \sin\left(\frac{\pi}{3} - \frac{2\zeta\varphi}{5}\right) \cdot \left(-\frac{2\varphi}{5}\right)^2
\]
\[
\leq \frac{3\sqrt{3}}{2} - \frac{1}{4} \sin\frac{\pi}{3} \cdot (2\varphi)^2 = \frac{3\sqrt{3}}{2} - \frac{\sqrt{3}}{2} \varphi^2 \leq \frac{\sqrt{27}}{2} (1 - \varepsilon^{2/3}/3),
\]
where $\eta, \zeta \in (0, 1)$ are chosen suitably, concluding the proof of (2.2).
2.2.2

(2.3) Let $D$ be a convex hexagon such that $C(\sqrt{3}/2 - \varepsilon^{1/3}) \subset D \subset C(1)$ and let $d_1, \ldots, d_6$ be the mirror images of $o$ in the lines containing the edges of $D$ in counter clockwise ordering. Then

(i) $-2\varepsilon^{1/3} \leq \|d_k\| - \sqrt{3} \leq 22\varepsilon^{1/3},$

(ii) $|\|d_{k+1} - d_k\| - \sqrt{3}| \leq 810\varepsilon^{1/3}$

and there are points $h_1, \ldots, h_6$ forming the vertices of a regular hexagon of edgelength $\sqrt{3}$ and center $o$ such that

(iii) $\|d_k - h_k\| \leq 405\varepsilon^{1/3}$.

Let $\frac{\pi}{6} + \varphi$ and $\frac{\pi}{6} - \psi$, $0 \leq \varphi \leq \frac{\pi}{12}$, be the maximum and the minimum angle with apex $o$ between the exterior normal of an edge of $D$ and an endpoint of this edge (note (2.1)). Then

(2.4) $0 \leq \varphi \leq 2\varepsilon^{1/3}, \ 0 \leq \psi \leq 22\varepsilon^{1/3}$.

To see this note that by the assumption of (2.3) and the concavity of cos in $[0, \pi/2]$, 

$$\frac{\sqrt{3}}{2} - \varepsilon^{1/3} \leq \cos(\frac{\pi}{6} + \varphi) \leq \cos \frac{\pi}{6} - \sin \frac{\pi}{6} \cdot \varphi = \frac{\sqrt{3}}{2} - \frac{\varphi}{2},$$

which implies the first statement in (2.4). The second one follows from

$$\frac{\pi}{6} - \psi \geq 2\pi - 11(\frac{\pi}{6} + \varphi) \geq \frac{\pi}{6} - 22\varepsilon^{1/3}.$$ 

The inclusion $C(\sqrt{3}/2 - \varepsilon^{1/3}) \subset D$ yields the first inequality in (2.3)(i). To see the second inequality note that $D \subset C(1)$. The definition of $\psi$, the concavity of cos in $[0, \pi/2]$, and (2.4) then imply,

$$\|d_k\| \leq 2 \cos(\frac{\pi}{6} - \psi) \leq 2 \cos \frac{\pi}{6} - 2 \sin \frac{\pi}{6} \cdot (-\psi) = \sqrt{3} + \psi \leq \sqrt{3} + 22\varepsilon^{1/3}.$$ 

Next we prove (2.3)(iii). Let $h_k$ be chosen as follows: $h_1$ is a positive multiple of $d_1$ and $\|h_1\| = \sqrt{3}$. $h_2, \ldots, h_6$ are obtained from $h_1$ by rotations by angles $\pi/7, \ldots, 5\pi/7$ about $o$. If $\varphi_k$ is the angle with apex $o$ between $d_k$ and $h_k$, then the definitions of $\varphi$ and $\psi$ and (2.4) together yield,

$$|\varphi_k| \leq 44(k-1)\varepsilon^{1/3} \leq 220\varepsilon^{1/3}.$$ 

This combined with (2.3)(i), $\|h_k\| = \sqrt{3}$, and (2.1) shows that

$$\|d_k - h_k\| \leq \max\{\|d_k\|, \|h_k\|\} \cdot |\varphi_k| + |\|d_k\| - \|h_k\|| \leq (\sqrt{3} + 22\varepsilon^{1/3})220\varepsilon^{1/3} + 22\varepsilon^{1/3} \leq 405\varepsilon^{1/3}.$$ 


This proves (2.3)(iii) which in turn yields (2.3)(ii).

2.3 Let a covering of \( S \) by \( m \) solid circles of radius \( \varrho \) with density less than

\[
\frac{2\pi}{\sqrt{27}} (1 + \varepsilon)
\]

be given. Second, we will consider the intersections of the Dirichlet–Voronoi cells of the set \( C \) of centers with \( S \) and classify them according to their shape.

For each \( c \in C \) let

\[
D(c) = \{ x : \| x - c \| \leq \| x - d \| \text{ for each } d \in C \} \cap S.
\]

If \( D(c) \) has non-empty interior we call it a cell with center \( c \). Since the Dirichlet–Voronoi cells of \( C \) form an edge-to-edge tiling of \( \mathbb{E}^2 \) (see e.g. [?]),

(2.5) the cells \( D(c) \) form an edge-to-edge tiling of \( S \).

Our aim is to show that for most \( c \in C \) the cell \( D(c) \) is close to a regular hexagon:

(2.6) Each \( c \in C \), with a set of less than \( 4 \varepsilon^{1/3} m \) exceptions, has the following properties:

(i) \( D(c) \) is a hexagonal cell,

(ii) \( C(c, (\sqrt{3}/2 - \varepsilon^{1/3}) \varrho) \subset D(c) \subset C(c, \varrho) \subset S \).

Such centers and the corresponding cells are called good, the others bad. The proof of (2.6) will be presented in subsections 2.3.1 – 2.3.5.

2.3.1 For \( i = 3, 4, \ldots \), let \( m_i \) be the number of \( i \)-gons among the cells \( D(c) \). Since \( S \) is a 3, 4, 5, or 6-gon, proposition (2.5) and a simple consequence of Euler’s polytope formula (see e.g. [?], p. 16) together imply that

\[
3m_3 + 4m_4 + \ldots \leq 6(m_3 + m_4 + \ldots) = 6m_6, \text{ say, where } m_0 \leq m.
\]

Thus

(2.7) \( 3m_3 + \ldots + 5m_5 + 7m_7 + \ldots \leq 6(m_3 + \ldots + m_5 + m_7 + \ldots) = 6m_{\neq 6}, \text{ say.} \)

Since among all convex \( i \)-gons in \( C(\varrho) \) the regular ones have maximum area, it follows:

(2.8) if the cell \( D(c) \) is an \( i \)-gon, then

\[
\frac{|D(c)|}{\varrho^2} \leq \frac{i}{2} \sin \frac{2\pi}{i}.
\]

Proposition (2.3) shows:

(2.9) if the cell \( D(c) \) is a 6-gon and \( C(c, (\sqrt{3}/2 - \varepsilon^{1/3}) \varrho) \not\subset D \subset C(c, \varrho) \), then

\[
\frac{|D(c)|}{\varrho^2} \leq \frac{\sqrt{27}}{2} (1 - \frac{\varepsilon^{2/3}}{3}).
\]
By (2.5) and (2.8) we have,

\[(2.10) \quad \frac{|S|}{\varrho^2} = \sum_{i \geq 3} \left\{ \sum_{c \in C} \frac{|D(c)|}{\varrho^2} : c \in C \text{ the cell } D(c) \text{ is an } i\text{-gon} \right\} \]

\[\leq \frac{\sqrt{27}}{2} m_6 + \sum_{i \geq 3, \neq 6} m_i \frac{i \sin \frac{2\pi}{i}}{2}.\]

**2.3.2** See (2.7) for the definition of \( m_{\neq 6} \). We show that

\[(2.11) \quad m_{\neq 6} \leq 64\epsilon m.\]

Considering derivatives, it is easy to prove that the function \( f \) defined by

\[f(x) = \frac{x}{2} \sin \frac{2\pi}{x}, x \geq 3,\]

is concave and non-decreasing. Hence the function \( g \) whose graph is the polygon connecting the points

\[(3, f(3)), \ldots, (5, f(5)), (7, f(7)), \ldots,\]

in this order, is also concave and non-decreasing. Thus Jensen’s inequality together with (2.7) implies,

\[(2.12) \quad \frac{1}{m_{\neq 6}} \sum_{i \geq 3, \neq 6} m_i \frac{i \sin \frac{2\pi}{i}}{2} = \sum_{i \geq 3, \neq 6} \frac{m_i}{m_{\neq 6}} g(i) \leq g \left( \frac{1}{m_{\neq 6}} \sum_{i \geq 3, \neq 6} m_i \right) \leq g(6) = 2.5570257 \ldots < \frac{\sqrt{27}}{2} (1 - \frac{1}{64}) < \frac{\sqrt{27}}{2}.\]

From (2.10) and (2.12) we conclude that

\[\frac{|S|}{\varrho^2} \leq \frac{\sqrt{27}}{2} m_6 + \frac{\sqrt{27}}{2} m_{\neq 6} - \frac{\sqrt{27}}{128} m_{\neq 6} \leq \frac{\sqrt{27}}{2} m - \frac{\sqrt{27}}{128} m_{\neq 6} = \frac{\sqrt{27}}{2} (1 - \frac{m_{\neq 6}}{64m}) m\]

(note that \( m_6 + m_{\neq 6} = m_0 \leq m \)). This yields a lower bound for the density of the given covering of \( S \):

\[\frac{\varrho^2 \pi m}{|S|} \geq \frac{2\pi}{\sqrt{27}} (1 - \frac{m_{\neq 6}}{64m})^{-1} \geq \frac{2\pi}{\sqrt{27}} (1 + \frac{m_{\neq 6}}{64m}).\]

Since by assumption this density is less than

\[\frac{2\pi}{\sqrt{27}} (1 + \epsilon),\]

we obtain (2.11).

**2.3.3** Let

\[m_{nr6} = \# \{ D(c) : \text{ the cell } D(c) \text{ is a } 6\text{-gon, } C(c, (\frac{\sqrt{3}}{2} - \epsilon^{1/3}) \varrho) \not\subset D(c) \subset C(c, \varrho) \},\]

where \# means cardinal number. Then the following estimate holds:
Applying (2.10), (2.9) and (2.12) gives
\[ |S| \leq \frac{\sqrt{27}}{2} (m_6 - m_{nr6}) + \frac{\sqrt{27}}{2} (1 - \frac{\varepsilon^{2/3}}{3}) m_{nr6} + \frac{\sqrt{27}}{2} m_{\neq 6} \leq \frac{\sqrt{27}}{2} (1 - \frac{\varepsilon^{2/3} m_{nr6}}{3m}) m \]
and thus
\[ \frac{\varrho^2 \pi m}{|S|} \geq \frac{2\pi}{\sqrt{27}} \left(1 + \frac{\varepsilon^{2/3} m_{nr6}}{3m}\right). \]
Since the density of our covering is less than
\[ \frac{2\pi}{\sqrt{27}} (1 + \varepsilon), \]
this yields (2.13).

2.3.4 Finally we estimate
\[ m_{bd} = \# \{ D(c) : D(c) \text{ is a cell, } C(c, \varrho) \cap T = \emptyset \}, \]
where T is the inner parallel set of S at distance 2\varrho:
(2.14) \[ m_{bd} \leq 2 \varepsilon m. \]

Consider the \( m_0 - m_{bd} \) cells \( D(c) \) where \( C(c, \varrho) \cap T \neq \emptyset \), that is, \( C(c, \varrho) \subset S \). Since then \( D(c) \subset C(c, \varrho) \), it follows from (2.5) that the corresponding circles cover \( T(\subset S) \). Thus L. Fejes Tóth’s [?] theorem cited in 1.2 says that for the density of this family of circles we have that
\[ \frac{\varrho^2 \pi (m_0 - m_{bd})}{|T|} \geq \frac{2\pi}{\sqrt{27}}. \]
Noting that \( p(S) \) is the perimeter of \( S \), we obtain
\[ |T| \geq |S| - 2p(S). \]
By assumption, the density of the given covering of \( S \) satisfies
\[ \frac{\varrho^2 \pi m}{|S|} \leq \frac{2\pi}{\sqrt{27}} (1 + \varepsilon). \]
Combining these three inequalities and (2.1) then yields (2.14):
\[ \frac{\varrho^2 \pi (m - m_{bd})}{|S| - 2p(S)} \geq \frac{\varrho^2 \pi (m_0 - m_{bd})}{|T|} \geq \frac{2\pi}{\sqrt{27}} \geq \frac{\varrho^2 \pi m}{|S|} (1 + \varepsilon)^{-1}, \]
or
\[ \frac{m_{bd}}{m} \leq 1 - (1 - \frac{2p(S)}{|S|}) (1 - \varepsilon) \leq 1 - 1 + \frac{2p(S)}{|S|} + \varepsilon \leq 2 \varepsilon. \]

2.3.5 Finally, (2.11), (2.12), (2.14) and (2.1) together yield (2.6):
\[ m_{\neq 6} + m_{nr6} + m_{bd} \leq (64\varepsilon + 3\varepsilon^{1/3} + 2\varepsilon)m \leq 4\varepsilon^{1/3}m. \]

2.4 Third, we choose from the set of good centers a still large set of centers having one additional property:
(2.15) Each \( c \in C \), with a set of less than \( 50\varepsilon^{1/3}m \) exceptions, has the property that all centers in \( C(c, 2\varrho) \cap C \) are good.

Such centers will be called very good.

By (2.6) the number of bad centers is less than \( 4\varepsilon^{1/3}m \). For each bad center consider all good centers at distance at most \( 2\varrho \) from it. For any of these good centers take the circle of radius \( \left( \frac{\sqrt{3}}{2} - \varepsilon^{1/3}\varrho \right) \) centered at it. Since these circles are contained in the corresponding cell (see (2.6)), they do not overlap. Comparing areas, we thus see that the number of good centers at distance at most \( 2\varrho \) from a given bad center is at most

\[
\frac{(2 + \frac{\sqrt{3}}{2} - \varepsilon^{1/3})^2 \varrho^2 \pi}{(\frac{3}{2} - \varepsilon^{1/3})^2 \varrho^2 \pi} \leq 11
\]

by (2.1). Cancelling all bad centers and for each bad center at most \( 11 \) good centers amounts to the omission of less than

\[
4\varepsilon^{1/3}m \cdot 12 \leq 50\varepsilon^{1/3}m
\]
centers. Clearly, the remaining centers all are very good. The proof of (2.15) is complete.

2.2 We come to the final step of the proof.

(2.16) Let \( c \in C \) be very good. Then there are \( c_1, \ldots, c_6 \in C \) such that

(i) \( ||c_k - c|| - \sqrt{3}\varrho| \leq 22\varepsilon^{1/3}\varrho \leq 13\varepsilon^{1/3}\sqrt{3}\varrho \),
(ii) \( ||c_{k+1} - c_k|| - \sqrt{3}\varrho| \leq 500\varepsilon^{1/3}\sqrt{3}\varrho \),
(iii) \( C(c, \frac{3}{2}\sqrt{3}\varrho) \cap C = \{c, c_1, \ldots, c_6\} \).

The definition of cells in 2.3, the fact that \( c \) is a very good and thus a good center, and propositions (2.6) and (2.3) show that there are centers \( c_1, \ldots, c_6 \), satisfying (2.16) (i), (ii). In addition we see that there are points \( h_1, \ldots, h_6 \) forming the vertices of a regular hexagon of edgelength \( \sqrt{3}\varrho \) and center at \( o \) such that

\[
(2.17) \ ||c_k - (c + h_k)|| \leq 810\varepsilon^{1/3}\varrho \leq 500\varepsilon^{1/3}\sqrt{3}\varrho.
\]

For the proof of (2.16)(iii) we proceed as follows: the definition of cells in 2.2 and (2.6), (2.3) yield the following:

(2.18) Let \( d \) be a good center. Then \( C(d, (\sqrt{3} - 2\varepsilon^{1/2})\varrho) \) contains no center except \( d \).

The centers \( c_k \) satisfy (2.16)(i) and thus are contained in \( C(c, 2\varrho) \). Since \( c \) is a very good center, each \( c_k \) is a good center, see (2.5). Clearly, \( c \) is good too. Now apply (2.18) to each of \( c, c_1, \ldots, c_6 \). Taking into account (2.17), we see that the union of the circles

\[
C(c, (\sqrt{3} - 2\varepsilon^{1/3})\varrho), C(c + h_k, (\sqrt{3} - 812\varepsilon^{1/3})\varrho), \ k = 1, \ldots, 6,
\]
contains precisely the centers \( c, c_1, \ldots, c_6 \). Noting (2.1), an elementary calculation then shows that the circle \( C(c, \frac{3}{2}\sqrt{3}\varrho) \) is contained in this union. This proves (2.16)(iii), and thus concludes the proof of (2.16).

2.6 Having shown (2.15) and (2.16), the proof of Theorem 1 is complete.
3 Proof of Theorem 2

3.1 To make this article more self-contained, we repeat the relevant parts of the definitions in [9].

Let $M$ be a two-dimensional Riemannian manifold of class $C^2$ with metric of class $C^0$. Then for any $p \in M$ there are a neighborhood $U$ of $p$ in $M$ and a homeomorphism $h = "r "$ of $U$ onto an open solid circle $U' = h(U)$ in $E^2$. To any $u \in U'$ there corresponds a positive quadratic form $q_u(s) = q_{p,u}(s)$ on $E^2$ with continuous coefficients.

A curve in $U$ is of class $C^1$ if it has a parametrization $x : [\mu, \nu] \to U$ such that $u = h \circ x$ is of class $C^1$. Its length is

$$\int_\mu^\nu q_u(\dot{u}(\tau)) d\tau.$$  

If a curve is not contained in a single neighborhood, dissect it suitably. For $x, y \in M$, let $\gamma_M(x, y)$ be the infimum of the lengths of the curves of class $C^1$ in $M$ which connect $x, y$. The metric $\gamma_M(\cdot, \cdot)$ induces the original topology on $M$. The geodesic disc $D_M(c, \varrho)$ in $M$ with center $c \in M$ and radius $\varrho$ is the set $\{x \in M : \gamma_M(c, x) \leq \varrho\}$.

A set $J \subset M$ is Jordan measurable if its closure $\text{cl} J$ is compact and for any $p, U, h$ and any neighborhood $V$ of $p$ with $\text{cl}V \subset U$ for which $V'$ is Jordan measurable in $E^2$, also $(J \cap V)'$ is Jordan measurable in $E^2$. If $M$ is compact, then it is Jordan measurable and so are geodesic discs. Finite unions, intersections and differences of Jordan measurable sets are again Jordan measurable. If $J \subset U$ is Jordan measurable, then

$$\omega_M(J) = \int_J (\det q_u)^{1/2} du \quad (du = du^1 du^2)$$

is its Jordan area; otherwise dissect $J$ suitably. Clearly, $\omega_M$ may be extended to a Borel measure on $M$. Then, equivalently, a set in $M$ may be defined Jordan measurable, if its closure is compact and its boundary has Borel measure 0.

3.2 Next, some tools are collected.

The definition of geodesic discs and the compactness of the closure of a Jordan measurable set yield the following well-known result:

(3.1) Let $K \subset M$ be Jordan measurable. Then $\omega_M(D_M(c, \varrho)) = \varrho^2 \pi (1 + o(1))$ for $c \in K$ as $\varrho \to 0$, where $o(\cdot)$ may be chosen to be independent of $c$.

A consequence of Lemma 1 in [9] is the following:

(3.2) Let $K \subset M$ be Jordan measurable with $\omega_M(K) > 0$ and for $m = 1, 2, \ldots$, let $\sigma_m(K)$ be the minimum radius $\sigma > 0$ such that $m$ suitable geodesic discs of radius $\sigma$ cover $K$. Then

$$\frac{m \sigma_m(K)^2 \pi}{\omega_M(K)} \to \frac{2\pi}{\sqrt{27}} \text{ as } m \to \infty.$$  

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3.3 Let \( J \) be the Jordan measurable set of Theorem 2 and let \( \{ D_M(c, \varrho_n) : c \in C_n \} \) be the corresponding coverings of \( J \). This subsection contains two auxiliary results on \( J \).

First,

\[ \sigma_n \leq \varrho_n \leq \sigma_n(1 + o(1)) \text{ as } n \to \infty, \text{ where } \sigma_n = \sigma_n(J). \]

The left hand side inequality follows from the definition of \( \sigma_n \) (see (3.2)) and the assumption of Theorem 2. To see the right hand side inequality note that by the assumption of Theorem 2, and (3.1), (3.2), both applied for \( K = J \),

\[ \frac{n \varrho_n^2 \pi}{\omega_M(J)} \to \frac{2\pi}{\sqrt{27}}; \quad \frac{2\pi}{\sqrt{27}} \text{ as } n \to \infty. \]

Second,

\[ n(K) = \#\{ c \in C_n : D_M(c, \varrho_n) \cap K \neq \emptyset \}. \]

Then

\[ \frac{n(K) \varrho_n^2 \pi}{\omega_M(K)} \to \frac{2\pi}{\sqrt{27}} \text{ as } n \to \infty. \]

For assume not. Then the definition of \( n(K) \) together with (3.2) shows that

\[ \frac{n(K) \varrho_n^2 \pi}{\omega_M(K)} \geq (1 + \alpha) \frac{2\pi}{\sqrt{27}} \text{ for infinitely many } n, \]

where \( \alpha > 0 \) is chosen suitably. Next take a compact Jordan measurable set \( L \) in the interior \( \text{int}(J \setminus K) \) of \( J \setminus K \) such that

\[ (1 + \alpha)\omega_M(K) + \omega_M(L) > \omega_M(J). \]

By our choice of \( L \), the sets \( K \) and \( L \) have positive distance with respect to \( \gamma_M \). Since \( \varrho_n \to 0 \) as \( n \to \infty \) (see (3.3) and note that \( \sigma_n \to 0 \) by (3.2)), we thus have

\[ n(K) + n(L) \leq n \text{ for sufficiently large } n, \]

by the definition of \( n(\cdot) \) in (3.4). The definition of \( n(L) \) and (3.2) imply:

\[ \frac{n(L) \varrho_n^2 \pi}{\omega_M(L)} \geq \frac{n(L) \sigma_n(L) \omega_M(L)}{\omega_M(L)} \to \frac{2\pi}{\sqrt{27}} \text{ as } n \to \infty. \]

Finally, (3.2) and (3.3) show that

\[ \frac{n \varrho_n^2 \pi}{\omega_M(J)} \to \frac{2\pi}{\sqrt{27}} \text{ as } n \to \infty. \]

We now apply these relations:

\[ \frac{n \varrho_n^2 \pi}{\omega_M(J)} \geq \frac{(n(K) + n(L)) \varrho_n^2 \pi}{\omega_M(J)} = \frac{n(K) \varrho_n^2 \pi}{\omega_M(K)} \cdot \frac{\omega_M(K)}{\omega_M(J)} + \frac{n(L) \varrho_n^2 \pi}{\omega_M(L)} \cdot \frac{\omega_M(L)}{\omega_M(J)}, \]
and thus
\[
\frac{2\pi}{\sqrt{27}} \geq (1 + \alpha) \frac{2\pi}{\sqrt{27}} \frac{\omega_M(K)}{\omega_M(J)} + \frac{2\pi}{\sqrt{27}} \frac{\omega_M(L)}{\omega_M(J)},
\]
which is a contradiction.

3.4 Let

(3.5) \( 0 < \varepsilon < 10^{-9}, \)

(3.6) \( \lambda > 1 \) so small that \( \lambda^7 < 1 + \varepsilon, \quad 2(500\varepsilon^{1/3} + \lambda - 1)\lambda \leq \varepsilon^{1/4}, \quad \lambda - 1 < \varepsilon. \)

Given \( p \in M, \) we may choose \( U, h = \gamma^m, \) where \( U \) is so small that for \( q = q_{p,p} \) the following hold:

\[
\frac{1}{\lambda} q(x' - y')^{1/2} \leq \gamma_M(x, y) \leq \lambda q(x' - y')^{1/2} \quad \text{for} \quad x, y \in U,
\]

\[
\frac{1}{\lambda} (\det q)^{1/2} |K'| \leq \omega_M(K) \leq \lambda (\det q)^{1/2} |K'| \quad \text{for Jordan measurable} \ K \subset U
\]

(see section 2 in [?]). Let \( V \) be a Jordan measurable, open neighborhood of \( p \) with \( \cl V \subset U. \)

As \( p \) ranges over the compact set \( \cl J, \) the neighborhoods \( V \) form an open covering of \( \cl J. \) Thus there is a finite subcover. Hence we may choose points \( p_l \in \cl J, l = 1, \ldots, m, \) say, and corresponding neighborhoods \( U_l, V_l, \) homeomorphisms \( h_l = \gamma^m, \) and quadratic forms \( q_l, \) such that

(3.7) \( \frac{1}{\lambda} q_l(x' - y')^{1/2} \leq \gamma_M(x, y) \leq \lambda q_l(x' - y')^{1/2} \quad \text{for} \quad x, y \in U_l,
\]

(3.8) \( \frac{1}{\lambda} (\det q_l)^{1/2} |K'| \leq \omega_M(K) \leq \lambda (\det q_l)^{1/2} |K'| \quad \text{for Jordan measurable} \ K \subset U_l.
\]

\( V_l \) is Jordan measurable and the inclusions \( \cl V_l \subset \int U_l, \) and \( J = V_1 \cup \ldots \cup V_m \) hold. Clearly,

\[
W_l = J \cap (V_l \setminus (V_1 \cup \ldots \cup V_{l-1})) \text{ is Jordan measurable,}
\]

\[
W_l \subset \int U_l, \quad \text{and} \ J \text{ is the disjoint union of} \ W_1, \ldots, W_m.
\]

Next choose sets \( S_{li} \subset \int W_l \subset \int U_l, \) \( i = 1, \ldots, i_l, \) with the following properties:

(3.9) \( S_{li} \subset \int U_l^i \) is a compact square; thus \( S_{li} \) is Jordan measurable,

(3.10) the sets \( S_{li} \) are pairwise disjoint,

(3.11) \( \omega_M(T) < (\lambda - 1)\omega_M(J), \) where \( T = J \setminus \bigcup_{l,i} S_{li}. \)
3.5

Let \( C_{nli} = \{ c \in C_n : D_M(c, \varrho_n) \cap S_{li} \neq 0 \} \). Then for all sufficiently large \( n \) the \( n(S_{li}) \) geodesic discs \( D_M(c, \varrho_n) : c \in C_{nli} \) are contained in \( U_l \) and the \( n(S_{li}) \) ellipses \( \{ s : q_l(s - c')^{1/2} \leq \varrho_n \} : c \in C_{nli} \) form a covering of \( S_{li}' \) of density less than

\[
\frac{2\pi}{\sqrt{27}}(1 + \varepsilon).
\]

Since \( S_{li} \) is compact, \( S_{li} \subset \text{int} U_l \), and \( \varrho_n \to 0 \) as \( n \to \infty \), it follows from (3.7) that

\[
\{ s : q_l(s - c')^{1/2} \leq \frac{\varrho_n}{\lambda} \} \subset D_M(c, \varrho_n) \subset \{ s : q_l(s - c')^{1/2} \leq \lambda \varrho_n \} \subset U'_l
\]

for all sufficiently large \( n \).

The geodesic discs \( D_M(c, \varrho_n) : c \in S_{nli} \) cover \( S_{nli} \). Thus the ellipses \( \{ s : q_l(s - c')^{1/2} \leq \lambda \varrho_n \} : c \in C_{nli} \) cover \( S_{nli} \) by (3.13). We determine the density of this covering. By (3.7) and (3.13) we have:

\[
\frac{1}{\lambda} (\det q_l)^{1/2} |\{ s : q_l(s - c')^{1/2} \leq \frac{\varrho_n}{\lambda} \}| \leq \frac{1}{\lambda} (\det q_l)^{1/2} |D_M(c, \varrho_n)'| \leq \omega_M(D_M(c, \varrho_n))
\]

for \( c \in C_{nli} \) and all sufficiently large \( n \)

and

\[
\omega_M(S_{li}) \leq \lambda (\det q_l)^{1/2} |S_{li}'|.
\]

Propositions (3.1), (3.4) yield,

\[
\frac{\sum \{ \omega_M(D_M(c, \varrho_n)) : c \in C_{nli} \}}{\omega_M(S_{nli})} < \frac{2\pi}{\sqrt{27}} \lambda
\]

for all sufficiently large \( n \).

(3.13) together with these inequalities and (3.6) gives the desired bound for the density of our covering of \( S_{li}' \) by ellipses:

\[
\frac{n(S_{li}) |\{ s : q_l(s - c')^{1/2} \leq \lambda \varrho_n \}|}{|S_{li}'|} = \frac{n(S_{nli}) \frac{1}{\lambda} (\det q_l)^{1/2} |\{ s : q_l(s - c')^{1/2} \leq \frac{\varrho_n}{\lambda} \}|}{\lambda (\det q_l)^{1/2} |S_{li}'|} \leq \frac{\sum \{ \omega_M(D_M(c, \varrho_n)) : c \in C_{nli} \}}{\omega_M(S_{li})} \cdot \lambda^6 < \frac{2\pi}{\sqrt{27}} \lambda^\varepsilon \leq (1 + \varepsilon) \frac{2\pi}{\sqrt{27}}
\]

for all sufficiently large \( n \), concluding the proof of (3.12).

3.6 Next we show that with an obvious extension of terminology from \( E^2 \) to \( M \),

(3.14) for all sufficiently large \( n \) the following hold: each of the \( n(S_{li}) \) centers \( c \in C_{nli} \), with a set of at most \( 50 \varepsilon^{1/3} n(S_{li}) \) exceptions, is contained in \( S_{li} \) and is the center of a regular hexagon with vertices \( c_1, \ldots, c_6 \), say, in \( C_{nli} \) which is regular up to \( \varepsilon^{1/4} \) and has size \( \sqrt{3} \varrho_n \). In addition, \( D_M(c, 1.1 \sqrt{3} \varrho_n) \cap C_n = \{ c, c_1, \ldots, c_6 \} \).
Consider the Euclidean norm $q_i^{1/2}$ in the plane. By (3.12) the $n(S_{it})$ circles (in the sense of the norm $q_i^{1/2}$) \{ $s : q_i(s-c)^{1/2} \leq \lambda \varrho_n$ \} $c \in C_{nli}$ form a covering of the square $S_{it}$ of density less than $\frac{\pi n}{\sqrt{27}} (1 + \varepsilon)$ for all sufficiently large $n$. Thus by Theorem 1 and its proof we conclude that for all sufficiently large $n$ all $n(S_{it})$ centers in $C_{nli}'$, with a set of less than $\lambda_{nli}$ exceptions, are very good. Let $c \in C_{nli}$ be such that $c'$ is very good and thus good.

By (2.6)(ii) and (3.13) $c' \in D_M(c, \varrho_n) \subset \{ s : q_i(s-c)^{1/2} \leq \lambda \varrho_n \} \subset S_{it}$ and thus

$$(3.15) \quad c \in D_M(c, \varrho_n) \subset S_{it}.$$

Note (2.16). Let \(c_1, \ldots, c_6 \in C_{nli}\) be such that $c'_1, \ldots, c'_6$ form a hexagon with center $c'$ which is regular up to $500\varepsilon^{1/3}$ of size $\sqrt{3}\lambda \varrho_n$ and such that

$$(3.16) \quad \{ s : q_i(s-c)^{1/2} \leq 2\lambda \varrho_n \} \cap C_{nli}' = \{ c', c'_1, \ldots, c'_6 \}.$$ Then

$$(3.17) \quad |\gamma_M(c_k, c) - \sqrt{3}\varrho_n|$$

$$\leq |\gamma_M(c_k, c) - q_i(c'_k - c')^{1/2}| + |q_i(c'_k - c')^{1/2} - \sqrt{3}\lambda \varrho_n| + \sqrt{3}(\lambda - 1)\varrho_n$$

$$\leq (\lambda - 1)q_i(c'_k - c')^{1/2} + 500\varepsilon^{1/3}\sqrt{3}\lambda \varrho_n + \sqrt{3}(\lambda - 1)\varrho_n$$

$$\leq (\lambda - 1)(500\varepsilon^{1/3} + 1)\sqrt{3}\lambda \varrho_n + 500\varepsilon^{1/3}\sqrt{3}\lambda \varrho_n + \sqrt{3}(\lambda - 1)\varrho_n$$

$$\leq ((\lambda - 1)(500\varepsilon^{1/3} + 1) + 500\varepsilon^{1/3} + \lambda - 1)\sqrt{3}\varrho_n \leq \varepsilon^{1/3} \sqrt{3}\varrho_n, \quad |\gamma_M(c_{k+1}, c_n) - \sqrt{3}\varrho_n| \leq \ldots \leq \varepsilon^{1/3} \sqrt{3}\varrho_n$$

for all sufficiently large $n$.

By (3.7) and (3.6). By (3.7) again and (3.13) we have,

$$(3.18) \quad D_M(c, 2\varrho_n) \subset \{ s : q_i(s-c)^{1/2} \leq 2\lambda \varrho_n \}.$$ (3.15), (3.17), (3.5), the definition of $C_{nli}$ in (3.12), (3.18) and (3.16) together yield

$$\{ c, c_1, \ldots, c_6 \} \subset D_M(c, 1.1\sqrt{3}\varrho_n) \cap C_n \subset D_M(c, 2\varrho_n) \cap C_{nli} \subset \{ c, c_1, \ldots, c_6 \}$$

for all sufficiently large $n$.

The proof of (3.14) is complete.

3.7 Since $\varrho_n \to 0$ as $n \to \infty$ and the compact sets $S_{it}$ are disjoint by (3.10), the definition of the sets $C_{nli}$ in (3.12) shows that the sets $C_{nli}$ are disjoint if $n$ is sufficiently large. Since $n(S_{it}) = \#C_{nli}$, we thus see that

$$(3.19) \quad \sum_{t,i} n(S_{it}) \leq n \quad \text{for all sufficiently large } n.$$

In order to prove that

$$(3.20) \quad n(T) \leq \varepsilon n \quad \text{for all sufficiently large } n$$

note that

$$\frac{n(T)}{n} = \frac{n(T)\varrho_n^2\pi}{\omega_M(J)} \cdot \frac{\omega_M(J)}{n\varrho_n^2\pi} \cdot \frac{\omega_M(T)}{\omega_M(J)} < \lambda - 1 \quad \text{for all sufficiently large } n$$

by (3.4) and (3.11). Now apply (3.6).

3.8 Combining (3.14), (3.19), (3.20) and noting (3.6), the following is obtained:
(3.21) Let \( \varepsilon > 0 \) be sufficiently small. Then for all sufficiently large \( n \) hold: each 
\( c \in C_n \), with a set of at most \( 51 \varepsilon^{1/3} n \) exceptions, is the center of a hexagon in 
\( C_n \) which is regular up to \( \varepsilon^{1/4} \), has size \( \sqrt{3} \varrho_n \) and \( D_M(c, 1.1 \sqrt{3} \varrho_n) \cap C_n \) consists 
precisely of \( c \) and the vertices of the hexagon.

Let \( k \) be so large that for \( \varepsilon = j^{-12}, j = k, k + 1, \ldots \), proposition (3.21) holds 
for \( n \geq n(j) \), say. We clearly may assume that \( n(j) \rightarrow \infty \) as \( j \rightarrow \infty \). Now define 
\( f, g : \mathbb{N} \rightarrow \mathbb{R}^+ \) by

\[
 f(n) = n, \quad g(n) = 1 \text{ for } 1 \leq n \leq n(k), \\
 f(n) = \frac{51}{j^3} n, \quad g(n) = \frac{1}{j^3} \text{ for } n(j) \leq n < n(j + 1), j = k, \ldots .
\]

Then

each \( c \in C_n \), with a set of at most \( f(n) \) exceptions, is the center of a hexagon in 
\( C_n \) which is regular up to \( g(n) \), has size \( \sqrt{3} \varrho_n \) and \( D_M(c, 1.1 \sqrt{3} \varrho_n) \cap C_n \) consists 
precisely of \( c \) and the vertices of the hexagon.

Since \( f(n) = o(n), g(n) = o(1) \) as \( n \rightarrow \infty \), this concludes the proof of Theorem 2.

4 Proof of Theorem 3

4.1 The following is the proof for \( P_n \).

Let \( M = \text{bd} C \) be endowed with the Riemannian metric \( \gamma_C \) of the second fundamental form 
and let \( \omega_C \) be the corresponding Jordan area measure. Then \( \text{bd} C \) is a 
Riemannian manifold of dimension two of class \( C^2 \) with metric of class \( C^0 \).

The following proposition was first proved by Schneider \[?\]. Under the present 
assumptions it is due to the author \[?\], (5.2).

\[
(4.1) \quad \text{For } n = 4, 5, \ldots , \text{ let } \varrho_n \text{ be the minimum radius such that the geodesic discs } \text{ } D_C(v, \varrho_n) : v \in \text{vert } P_n \text{ cover } \text{bd} C. \text{ Then } \\
\varrho_n = (2 \delta^H(C, P_n))^{1/2}(1 + o(1)) \text{ as } n \rightarrow \infty.
\]

Conversely, if for \( n = 4, 5, \ldots , \sigma_n \) is the minimum radius such that \( n \) suitable 
geodesic discs \( D_C(w_1, \sigma_n), \ldots , D_C(w_n, \sigma_n) \), say, cover \( \text{bd} C \), then

\[
\delta^H(C, \text{conv}\{w_1, \ldots , w_n\}) \leq \frac{\sigma_n^2}{2}(1 + o(1)) \text{ as } n \rightarrow \infty.
\]

(By conv the convex hull is meant.) As a consequence of (4.1) we have,

\[
(4.2) \quad \sigma_n \leq \varrho_n \leq \sigma_n(1 + o(1)) \text{ as } n \rightarrow \infty.
\]

Applying propositions (3.1) and (3.2) with \( K = M = \text{bd} C \), yields the following:

\[
(4.3) \quad \omega_C(D_C(v, \varrho_n)) = \varrho_n^2 \pi (1 + o(1)) \text{ as } n \rightarrow \infty, \text{ where } o(1) \text{ is independent of } v,
\]
Combining (4.1) – (4.4) shows that

\[
\frac{n \sigma_n^2 \pi}{\omega_C(\text{bd} C)} \rightarrow \frac{2\pi}{\sqrt{27}} \text{ as } n \rightarrow \infty.
\]

the density of the covering of bd \( C \) by the geodesic discs \( D_C(v, \rho_n) : v \in \text{vert} P_n \) converges to \( 2\pi/\sqrt{27} \) as \( n \rightarrow \infty \).

An application of Theorem 2 then yields Theorem 3 for \( P_n \).

4.2 The proofs for \( Q_n, \ldots, T_n \), in essence, are the same, where in place of (4.1) use is made of analogous propositions; see [?] (5.4), [?] (3.5), (3.6), [?] (3.11), (3.13) and [?].

Final remark

In a subsequent article best approximating polytopes with respect to the symmetric difference metric will be considered. The result is based on a stability version of L. Fejes Tóth’s [?], p. 81, so-called moment lemma which has also other applications.

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In the course of the proof of Theorem 3 I was looking for a stability version of L. Fejes Tóth’s circle covering result cited in 1.2 and asked his son G. Fejes Tóth for references. The latter informed me that, surprisingly, no such results were known and suggested to prove a stability theorem using ideas from the classical proofs of L. Fejes Tóth. This led to Theorem 1. Theorem 2 was the natural extension needed for the proof of Theorem 3. I am obliged to G. Fejes Tóth for his advice and to F. J. Schnitzer for his many helpful remarks. Part of this article was written while the author visited the Mathematical Sciences Research Institute at Berkeley in April 1996. Thanks for this invitation are due to Professors Milman and Thurston.

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Peter M. Gruber, Abteilung für Analysis, Technische Universität Wien, Wiedner Hauptstraße 8–10/1142, A-1040 Vienna, Austria, pmgruber@pop.tuwien.ac.at