Error of Asymptotic Formulae for Volume Approximation of Convex Bodies in $\mathbb{E}^3$

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Abstract. In this article the error of the asymptotic formula for volume approximation of sufficiently differentiable convex bodies in $\mathbb{E}^3$ by circumscribed convex polytopes is estimated. This then is applied to the isoperimetric problem for convex polytopes in a Minkowski 3-space.

MSC 2000. 52A15, 52A21, 52A27, 52A40, 52B11, 52B60.

Dedicated to Professor Sergej S. Ryškov on the occasion of his 70th birthday.

1 Introduction and Statement of Results

1.1 Let C be a *convex body* in Euclidean d-space $\mathbb{E}^d$, that is a compact convex subset of $\mathbb{E}^d$ with non-empty interior and let δ^V be the symmetric difference metric on the space of all convex bodies in $\mathbb{E}^d$. For $n = d+1, d+2, \cdots$, denote by $\mathcal{P}^c_{(n)}$ the family of all convex polytopes in $\mathbb{E}^d$ circumscribed to C which have at most n facets. Major approximation problems are to determine the quantity

$$\delta^V(C, \mathcal{P}^c_{(n)}) = \inf\{\delta^V(C, P) : P \in \mathcal{P}^c_{(n)}\}$$

and to describe the polytopes for which equality holds, the *best approximating polytopes* of C in $\mathcal{P}^c_{(n)}$. Disregarding trivial cases, precise solutions of these problems are out of reach. General upper estimates for $\delta^V(C, \mathcal{P}^c_{(n)})$ are easy to obtain. It is more difficult to determine the precise asymptotics of $\delta^V(C, \mathcal{P}^c_{(n)})$ as $n \to \infty$. In [6, 8] it was shown that for C (the boundary of which is a surface) of class $\mathcal{C}^2$ with Gauss curvature $\kappa_C > 0$ holds:

(1)
$$\delta^V(C, \mathcal{P}^c_{(n)}) \sim \frac{1}{2\, n^{\frac{2}{d-1}}} \,\mathrm{div}_{\mathrm{d-1}} A(C)^{\frac{d+1}{d-1}} \ \text{ as } \ n \to \infty.$$

Here $A(C)$ is the *equi-affine surface area measure* of (the boundary of) C,

(2)
$$A(C) = \int\limits_{\mathrm{bd}\, C} \kappa_C(x)^{\frac{1}{d+1}} d\sigma(x),$$

where σ is the ordinary surface area measure in $\mathbb{E}^d$. div_{d-1} is a constant depending on d. The only explicitly known values are $\mathrm{div}_1 = 1/12$ and $\mathrm{div}_2 = 5/18\sqrt{3}$. The case $d = 2$ was settled before by Fejes Tóth [4]. For more information on the voluminous literature on approximation of convex bodies by polytopes see the surveys [7, 9].

It is plausible to conjecture that for sufficiently differentiable C the formula (1) extends to an asymptotic series. For $d = 2$ the first two terms of this series were given by Ludwig [14] using tools from equi-affine differential geometry. The complete series was specified by Tabachnikov [18] in the form of a result on periodic trajectories of the dual billiard determined by C. For general d Böröczky [2] showed for convex bodies C of class $\mathcal{C}^3$ with $\kappa_C > 0$ the formula

$$\delta^V(C, \mathcal{P}^c_{(n)}) = \frac{1}{2\, n^{\frac{2}{d-1}}} \operatorname{div}_{\mathrm{d-1}} A(C)^{\frac{d+1}{d-1}} + O\Big(\frac{1}{n^{\frac{2}{d-1}+\frac{1}{8d^2}}}\Big) \text{ as } n \to \infty.$$

1.2 The aim of this article is to prove the following result for $d = 3$.

Theorem. *Let C be a convex body in $\mathbb{E}^3$ of class $\mathcal{C}^3$ with Gauss curvature $\kappa_C > 0$ and affine surface area $A(C)$. Then*

$$\delta^V(C, \mathcal{P}^c_{(n)}) = \frac{5A(C)^2}{36\sqrt{3}\, n} + O\Big(\frac{1}{n^{1+\frac{1}{4}}}\Big) \textit{ as } n \to \infty.$$

Actually, a slightly stronger result will be given, see (3) and (41). A scrutiny of the proof shows that, as in [2], it is sufficient to assume that $\operatorname{bd} C$ has a representation of class $\mathcal{C}^2$ with Lipschitz second derivatives. The proof of our result makes use of geometric tools which, so far, are available only for $d = 3$.

Böröczky [3] informs us that in case C is a Euclidean ball in $\mathbb{E}^3$, the error term in the asymptotic formula in the Theorem has a lower bound of the form

$$\frac{f(n)}{n^2}, \text{ where } f(n) \to \infty \text{ as } n \to \infty.$$

As remarked before, we conjecture that for sufficiently differentiable C, the asymptotic formula for $\delta^V(C, \mathcal{P}^c_{(n)})$ can be extended to an asymptotic series. By the remark of Böröczky we see that the second term in this series is not of the form $B(C)/n^2$, as might be expected. Possibly, it has the form $C(C)/n^{3/2}$. Here $B(C)$ and $C(C)$ are suitable quantities depending on C.

Considering the recent contributions to asymptotic approximation of convex bodies by polytopes, it is clear that results similar to our Theorem and with similar proofs hold for the mean width deviation and for L^p metrics and also for families of inscribed and general polytopes and for vertices instead of facets. In some of these cases the proofs are technically more complicated.

For a corresponding, yet weaker result for general d see [12].

1.3 The Theorem can be applied to the isoperimetric problem for convex polytopes in a normed space: consider besides the Euclidean norm a further norm on $\mathbb{E}^d$. A natural choice for “volume” in the normed space thus obtained is the ordinary volume V. For “surface area” several natural definitions have been proposed. These amount to the introduction of a convex body I with center at the origin o, the so-called *isoperimetrix.* The *surface area* $S_I(C)$ of a convex body C then is defined by

$$S_I(C) = \lim_{\varepsilon\to+0} \frac{V(C+\varepsilon I) - V(C)}{\varepsilon}, \text{ where } C+\varepsilon I = \{x+\varepsilon y : x \in C, y \in I\},$$

see Thompson [18]. If P is a convex polytope with minimum *isoperimetric quotient* $S_I(C)^d/V(C)^{d-1}$ amongst those with a given number of facets, a result of Diskant [5] says that after applying a suitable homothety to P, we may assume that P is circumscribed to I. The definition of $S_I(P)$ then shows that

$$\frac{S_I(P)^d}{V(P)^{d-1}} = d^d V(P) = d^d V(I) + d^d \delta^V(I, P).$$

The asymptotic formula (1) and its refinements thus yield asymptotic formulae for the minimum isoperimetric quotient. In particular, our Theorem implies the following result.

Corollary. *Let I be an isoperimetrix in $\mathbb{E}^3$ of class $\mathcal{C}^3$ with $\kappa_C > 0$ and affine surface area $A(C)$ which corresponds to a given norm on $\mathbb{E}^3$. For $n = 4, 5, \ldots$, let P_n be a convex polytope in $\mathbb{E}^3$ with minimum isoperimetric quotient $S_I(P_n)^3/V(P_n)^2$ amongst all convex polytopes in $\mathbb{E}^3$ with n facets. Assume (without restriction of generality) that P_n is circumscribed to I. Then*

$$\frac{S_I(P_n)^3}{V(P_n)^2} = 27V(I) + \frac{5\sqrt{3}A(I)^2}{4n} + O\Big(\frac{1}{n^{1+\frac{1}{4}}}\Big) \text{ as } n \to \infty.$$

1.4 In [11] we gave an asymptotic formula for the maximum error in numerical integration formulas for certain classes of continuous functions on the plane or on Riemannian 2-manifolds. Using ideas of the proof of our Theorem, it is possible to estimate the error of this asymptotic formula.

2 Proof of the Theorem

Let $\sharp$, diam, width, $|\cdot|$, $V(\cdot)$, bd, int, relbd, relint, vert and conv stand for cardinal number, diameter and minimum width (both with respect to the Euclidean norm $\|\cdot\|$ in $\mathbb{E}^3$), ordinary area (2-dimensional Hausdorff measure) and volume, boundary, interior, relative boundary and interior, set of vertices and convex hull, respectively. It will always be clear from the context with respect to which set relbd and relint are considered. B^2 and S^2 denote the solid Euclidean unit circle and the Euclidean 2-sphere. For notions not explained below see Schneider [16].

When writing const or $\alpha, \beta, \ldots$, we mean that this is a suitable positive constant depending only on C. Landau symbols such as $O(1/m)$ denote functions of the form const$/m$ (in slight contrast to the use in section 1). Constants and Landau symbols may be different, even if they are denoted alike.

2.1 Lower Estimate. We shall prove that

(3) $$\delta^V(C,\mathcal{P}^c_{(n)}) \ge \frac{5A(C)^2}{36\sqrt{3}n} - O\Big(\frac{1}{n^{1+\frac{1}{2}}}\Big) \text{ as } n\to\infty.$$

For $n=4,5,\dots$, let $P_n \in \mathcal{P}^c_{(n)}$ be best approximating for C.

2.1.1 First, some tools will be collected. A result of the author [10] says that "P_n has asymptotically regular hexagonal facets of edge length $(A(C)/3^{3/2}\,n)^{1/2}$ with respect to the Riemannian metric of equi-affine differential geometry on $\operatorname{bd} C$". Noting that two Riemannian metrics on the compact manifold $\operatorname{bd} C$ are metrically equivalent, this implies in particular that

(4) there is a facet F_0 of P_n with $\operatorname{diam} F_0 \le O(1/n^{1/2})$ and the point where F_0 touches C has distance at most $O(1/n^{1/2})$ from the points where the adjacent facets touch C.

The *metric projection* "$^\pi$" of $\mathbb{E}^3$ onto C maps each point in $\mathbb{E}^3$ onto its (unique) nearest point in C. The *rolling theorem of Blaschke* [1], pp.118 and 119, see also Leichtweiss [13], p.1055, and its dual yield the following remarks:

(5) For any $p\in \operatorname{bd} C$ and r on the support plane H_p of C at p with $\|p-r\|\le\alpha$ holds $\beta\,\|p-r\|^2 \le \|r-r^\pi\| \le \gamma\,\|p-r\|^2$.

(6) For any $r\in\mathbb{E}^3\setminus C$ with $\|r-r^\pi\|\le$const the volume of the convex cone with apex r and base $H_{r^\pi}\cap \operatorname{conv}(\{r\}\cup C)$ is at least $\delta\,\|r-r^\pi\|^2$.

Next, we state some well-known properties of $^\pi$:

(7) $\|x^\pi - y^\pi\| \le \|x-y\|$ for $x,y\in\mathbb{E}^3$.

(8) Let $z\in \operatorname{bd} C$. Then $\{y\in\mathbb{E}^3 : y^\pi = z\}$ is the exterior normal of $\operatorname{bd} C$ at z.

(9) $|S^\pi| \le |S|$ for any measurable piece S of a surface.

(10) Let Q be a convex polytope containing C. Then $^\pi$ maps $\operatorname{bd} Q$ homeomorphically onto $\operatorname{bd} C$.

Further needed tools are the following *standard representation* of $\operatorname{bd} C$ and some of its properties. For $p\in \operatorname{bd} C$ choose in the support plane H_p of C at p a Cartesian coordinate system with origin o equal to p. Together with the interior normal unit vector of $\operatorname{bd} C$ at p it yields a Cartesian coordinate system of $\mathbb{E}^3$. When speaking of the "lower" part of $\operatorname{bd} C$, this is meant with respect to the last coordinate. Let $\varepsilon>0$ be so small that the ε-neighborhood U_p of p in $\operatorname{bd} C$ is in the relative interior of the lower part of $\operatorname{bd} C$. Represent U_p in the form

(11) $\quad U_p = \{(s, f_p(s)) : s \in U_p'\}$,

where "$'$" denotes the orthogonal projection of $\mathbb{E}^3$ onto H_p and f_p is a convex function of class $\mathcal{C}^3$ on the relatively open set U_p'. The next proposition is a slight extension of a result of Schneider [15] and can be proved along the same lines.

(12) $\quad f_p$ is convex, of class $\mathcal{C}^3$, and
$|f_{p,k}|, |f_{p,kl}|, |f_{p,klm}|, (\mathrm{grad} f_p)^2 \le \text{const}$ on U_p,

where $f_{p,k}, f_{p,kl}, f_{p,klm}$ denote first, second and third partial derivatives of f_p. Define quadratic forms q_u on $\mathbb{E}^2$ by

$$q_u(s) = \sum_{k,l} f_{p,kl}(u) s^k s^l \ \text{ for } \ u \in U_p' \ \text{ and } \ s = (s^1, s^2) \in \mathbb{E}^2.$$

Instead of q_u, where $u = p' = p$, we also write q_p. As a consequence of *Blaschke's rolling theorem* and its dual we have the inequalities

(13) $\quad \text{const}\, \|\cdot\|^2 \le q_p(\cdot) \le \text{const}\, \|\cdot\|^2$.

For $x \in U_p$ we write instead of $\kappa_C(x)$ also $\kappa_C(u)$ where $u = x'$. Using the common expression for κ_C,

$$\kappa_C(u) = \frac{\det q_u}{(1 + (\mathrm{grad} f_p(u))^2)^2} \ \text{ for } \ u \in U_p',$$

and noting (12), it follows that

(14) $\quad \kappa_C(o) = \det q_p$ and
κ_C is of class $\mathcal{C}^1$ and has bounded first partial derivatives on U_p'.

Let D be a *convex disc*, that is a convex body in $\mathbb{E}^2$. Its *moment* $M(D, o)$ with respect to the origin o is defined by

$$M(D, o) = \int_D \|s\|^2 \, ds.$$

A related function $M(a, v)$ is defined as follows:

$$M(a, v) = \frac{a^2}{2v \tan^2 \frac{\pi}{v}} \int_0^{\frac{\pi}{v}} \frac{d\varphi}{\cos^4 \varphi} \ \text{ for } \ a > 0, v \ge 3.$$

If v is an integer, $M(a, v)$ is the moment of a regular convex v-gon with center o and area a. Elementary calculations yield the following properties:

(15) $\quad$ Let D be a convex disc in $\mathbb{E}^2$. Then $M(tD, o) = t^4 M(D, o)$ for $t \ge 0$.

(16) Let D be a convex disc in $\mathbb{E}^2$ and $q(s) = s^{\mathrm{tr}} A^{\mathrm{tr}} A s$ for $s \in \mathbb{E}^2$ a positive definite quadratic form, where A is a suitable 2×2 matrix. Then

$$\int\limits_D q(s)\, ds = M(AD, o)\, (\det q)^{-\frac{1}{2}}.$$

(17) Let H be a convex hexagon with center o which is regular with respect to the norm $q^{\frac{1}{2}}$, where q is a positive definite quadratic form on $\mathbb{E}^2$. Then

$$M(H, o) = \frac{5|H|^2 (\det q)^{\frac{1}{2}}}{18\sqrt{3}}.$$

The *moment lemma of Fejes Tóth* [4], p.198, says the following:

(18) Let D be a convex v-gon and E a regular convex v-gon with center o and $|D| = |E|$. Then $M(D, o) \geq M(E, o) = M(|E|, v)$.

The next two results are due to the author [10]:

(19) $M(a, v)$ is convex in (a, v) for $a > 0, v \geq 3$.

(20) $M(a, v)$ is non-increasing in v for any fixed $a > 0$.

The final tool is a corollary of *Euler's polytope formula*, see e.g. Fejes Tóth [4], p.15:

(21) Let P be a convex polytope in $\mathbb{E}^3$ with n facets and let $v_1, \dots, v_n$ be the number of the vertices of its facets. Then

$$\frac{1}{n}(v_1 + \cdots + v_n) < 6.$$

2.1.2 Second, we consider the form of the facets of best approximating polytopes P_n. Our first aim is to show that

(22) $\max\{\operatorname{diam} F : F \text{ facet of } P_n\} \leq O\Big(\frac{1}{n^{\frac{1}{2}}}\Big).$

For the proof of (22) note that the following weaker estimate is an immediate consequence of the fact that $\delta^V(C, P_n) \to 0$.

(23) $\max\{\operatorname{diam} F : F \text{ facet of } P_n\} \to 0.$

This implies that for all sufficiently large n the intersection of the support halfspaces of P_n, except the facet F_0 (see (4)), is a convex polytope $Q_{n-1} \in \mathcal{P}^c_{(n-1)}$, say, and $Q_{n-1} \setminus P_n$ is a convex polytope with facet F_0 which is situated "above" F_0. It follows from (4) that $|F_0| \leq O(1/n)$ and (4) together with (5) implies that the maximum "height" of $Q_{n-1} \setminus P_n$ above F_0 is at most $O(1/n)$. Thus

$$(24) \qquad Q_{n-1} \in \mathcal{P}^c_{(n-1)},\ \ V(Q_{n-1}) - V(P_n) \le O\Big(\frac{1}{n}\Big)O\Big(\frac{1}{n}\Big) = \frac{\zeta}{n^2},$$

say. In order to prove (22), assume the contrary. Choose

$$(25) \qquad \eta > \Big(\frac{\zeta}{\beta^2\delta}\Big)^{\frac{1}{4}}.$$

Then there are infinitely many n (and from now on only these n will be considered in the proof of (22)) with the following property: P_n has a facet G_0, say, such that $\operatorname{diam} G_0 \ge 2\eta/n^{1/2}$. Thus G_0 has a vertex r, say, such that $\|p-r\| \ge \eta/n^{1/2}$, where p is the point where G_0 touches C. By (23) we see that $\|p-r\| \le \alpha$ for sufficiently large n where α is from (5). Hence (5) implies that $\|r-r^\pi\| \ge \beta\,\|p-r\|^2 \ge \beta\,\eta^2/n$. On the other hand, $\delta^V(C,P_n) \to 0$ and (24) together show that $\delta^V(C,Q_{n-1}) \to 0$, which, in turn, yields that $\|r-r^\pi\| \to 0$ as $n \to \infty$. Let R_n be the intersection of Q_{n-1} and the support halfspace of C at r^π. Then $\|r-r^\pi\| \to 0, \|r-r^\pi\| \ge \beta\,\eta^2/n$ and propositions (6) and (25) together imply that for sufficiently large n,

$$R_n \in \mathcal{P}^c_{(n)}, V(Q_{n-1}) - V(R_n) \ge \delta\,\|r-r^\pi\|^2 \ge \frac{\beta^2\,\delta\,\eta^4}{n^2} > \frac{\zeta}{n^2}.$$

Considering this and (24), we see that for sufficiently large n holds

$$R_n \in \mathcal{P}^c_{(n)}, V(R_n) < V(P_n) - \frac{\zeta}{n^2} + \frac{\zeta}{n^2} = V(P_n).$$

Since this contradicts the fact that P_n is best approximating for C in $\mathcal{P}^c_{(n)}$, the proof of (22) is complete.

A similar proof shows that

$$(26) \qquad \min\{\text{width}\,F : F \text{ facet of } P_n\} \ge O\Big(\frac{1}{n^{\frac{1}{2}}}\Big).$$

2.1.3 Third, we adapt the standard representation of $\operatorname{bd} C$ as described in 2.1.1 according to our needs.

Let $F_i, i = 1, \dots, n$, denote the facets of P_n and let p_i be the point where F_i touches C. For each p_i choose corresponding H_i, Cartesian coordinate systems for H_i, resp. $\mathbb{E}^3$ with origin o equal to p_i, and U_i, f_i, q_i and "$'$". Propositions (26) and (22) together with $o \in F_i$ imply that

$$(27) \qquad c_i + \frac{\vartheta}{n^{\frac{1}{2}}}B^2 \subseteq F_i \subseteq \frac{\iota}{n^{\frac{1}{2}}}B^2 \text{ for suitable } c_i \in H_i.$$

In the following we consider only n which are so large that

$$(28) \qquad F_i \subset U_i'.$$

(Note (22) and the fact that U_i is the ε-neighborhood of p_i in $\operatorname{bd} C$, where $\varepsilon > 0$ is a constant.) Propositions (11)–(14) now take the following form:

(29) $\quad U_i = \{(s, f_i(s)) : s \in U_i'\}$.

(30) $\quad f_i$ is convex, of class $\mathcal{C}^3$, and
$|f_{i,k}|, |f_{i,kl}|, |f_{i,klm}|, (\mathrm{grad} f_i)^2 \leq \mathrm{const}$ on U_i'.

(31) $\quad \mathrm{const}\, \|\cdot\|^2 \leq q_i(\cdot) \leq \mathrm{const}\, \|\cdot\|^2$.

(32) $\quad \kappa_C(o) = \det q_i$, κ_C is of class $\mathcal{C}^1$, and
has bounded first partial derivatives on U_i'.

Represent q_i in the form

(33) $\quad q_i(s) = s^{\mathrm{tr}} A_i^{\mathrm{tr}} A_i s$ for $s \in \mathbb{E}^2$, where A_i is a suitable 2×2 matrix.

Since $f_i(o) = 0$ and $\mathrm{grad} f_i(o) = o$, Taylor's theorem together with (28)–(31) implies that

(34) $$f_i(s) \geq \frac{1}{2} q_i(s) - O(q_i(s)^{\frac{3}{2}}) \text{ for } s \in F_i \subseteq U_i' \subseteq H_i.$$

Since $p_i = o \in F_i \cap C$ and $F_i \subseteq (\iota/n^{1/2})B^2$ by (27), proposition (7) shows that $F_i^\pi \subseteq (\iota/n^{1/2})B^3$ and thus

(35) $$F_i^{\pi\prime} \subset \frac{\iota}{n^{\frac{1}{2}}} B^2.$$

Some elementary calculations together with $f_i(o) = 0$, $\mathrm{grad} f_i(o) = o$, Taylor's theorem and propositions (27), (35), (28) and (30) imply the inequality

(36) $$\|x - x^{\pi\prime}\| < \frac{\kappa}{n^{\frac{3}{2}}} \text{ for } x \in F_i.$$

2.1.4 Fourth, we consider a dissection of $\mathrm{bd}\, C$. By (7) and (10) the metric projection $^\pi$ is non-increasing for curve length and maps $\mathrm{bd}\, P_n$ homeomorphically onto $\mathrm{bd}\, C$. Thus

(37) $\quad$ the sets $F_i^\pi, i = 1, \dots, n$, form a *dissection* of $\mathrm{bd}\, C$.

This means that the sets are closed, their boundaries are continuous curves of finite length, they have pairwise disjoint interiors and their union equals $\mathrm{bd}\, C$.

Let

(38) $$P_{ni} = \{(s,t) \in P_n : s \in (1 - \frac{\kappa}{\vartheta n})(F_i - c_i) + c_i,\, 0 \leq t \leq f_i(s)\} \subseteq P_n \setminus \mathrm{int}\, C$$

(compare (27)). Next it will be shown that

(39) $\quad$ the sets P_{ni}, $i = 1, \dots, n$, are pairwise disjoint subsets of $P_n \setminus \mathrm{int}\, C$.

By (37) it is sufficient for the proof of (39) to show that

(40) $\quad P^{\pi}_{ni} \subset \operatorname{relint} F^{\pi}_i.$

Being a continuous image of the connected set P_{ni}, the set P^{π}_{ni} is also connected. F^{π}_i and P^{π}_{ni} have a common relative interior point, namely p_i, and both are contained in $\operatorname{bd} C$. Thus, if (40) did not hold, $P^{\pi}_{ni} \cap \operatorname{relbd} F^{\pi}_i \neq \emptyset$. Then there is a point $y \in P_{ni}$ such that $y^{\pi} \in \operatorname{relbd} F^{\pi}_i$. Choose $x \in \operatorname{relbd} F_i$ with $x^{\pi} = y^{\pi}$. It follows from (8) that $y \in [x, x^{\pi}]$. Hence $\|x - y'\| \leq \|x - x^{\pi}{}'\| < \kappa/n^{3/2}$ by (36). This together with $y = (s,t) \in P_{ni}, s = y'$, (38) and (27) implies that

$$x = y' + x - y' \in (1 - \frac{\kappa}{\vartheta n})(F_i - c_i) + c_i + \frac{\kappa}{n^{\frac{3}{2}}} \ \operatorname{relint} B^2$$

$$\subseteq (1 - \frac{\kappa}{\vartheta n})(F_i - c_i) + c_i + \frac{\kappa}{\vartheta n}(\operatorname{relint} F_i - c_i) \subseteq \ \operatorname{relint} F_i.$$

Since this contradicts the fact that $x \in \operatorname{relbd} F_i$, the proof of (40) is complete which, in turn, yields (39).

2.1.5 Finally, the fact that P_n is best approximating, (39), (38) where we put $(1 - \kappa/\vartheta n)(F_i - c_i) + c_i = (1 - O(1/n))F_i + d_i$, say, (34), (31), (27), (33), (16), (15), (18) where $v_i = \sharp \ \operatorname{vert} F_i$, $\det A_i = (\det q_i)^{1/2}$ and (15), (32), (9), (20), a calculus formula for surface area, (32), (35), (15), a calculus formula for surface integrals, (19), Jensen's inequality, (37), (2), (20), (21) and (17) together imply proposition (3):

$$\begin{aligned}
\delta^V(C, \mathcal{P}^c_{(n)}) &= \delta^V(C, P_n) = V(P_n \setminus \operatorname{int} C) \geq \sum_i V(P_{ni}) \\
&= \sum_i \int\limits_{(1-O(\frac{1}{n}))F_i+d_i} f_i(s)\,ds \geq \frac{1}{2} \sum_i \int\limits_{(1-O(\frac{1}{n}))F_i+d_i} q_i(s)\,ds \ (1 - O\big(\frac{1}{n^{\frac{1}{2}}}\big)) \\
&= \frac{1}{2} \sum_i M((1 - O(\frac{1}{n}))A_i F_i + A_i d_i, o)(\det q_i)^{-\frac{1}{2}}(1 - O\big(\frac{1}{n^{\frac{1}{2}}}\big)) \\
&\geq \frac{1}{2} \sum_i M(|A_i F_i|, v_i)(\det q_i)^{-\frac{1}{2}}(1 - O\big(\frac{1}{n}\big))^4(1 - O\big(\frac{1}{n^{\frac{1}{2}}}\big)) \\
&\geq \frac{1}{2} \sum_i M((\det q_i)^{\frac{1}{4}}|F_i|, v_i)(1 - O\big(\frac{1}{n^{\frac{1}{2}}}\big)) \\
&\geq \frac{1}{2} \sum_i M(\kappa_C(o)^{\frac{1}{4}}|F^{\pi}_i|, v_i)(1 - O\big(\frac{1}{n^{\frac{1}{2}}}\big)) \\
&\geq \frac{1}{2} \sum_i M(\int\limits_{F^{\pi}_i{}'} \kappa_C(o)^{\frac{1}{4}}(1 + (\operatorname{grad} f_i(s))^2)^{\frac{1}{2}}\,ds,\ v_i)\,(1 - O\big(\frac{1}{n^{\frac{1}{2}}}\big)) \\
&\geq \frac{1}{2} \sum_i M(\int\limits_{F^{\pi}_i{}'} \kappa_C(s)^{\frac{1}{4}}(1 + (\operatorname{grad} f_i(s))^2)^{\frac{1}{2}}\,ds\,(1 - O\big(\frac{1}{n^{\frac{1}{2}}}\big)), v_i) \\
&\qquad\qquad \times (1 - O\big(\frac{1}{n^{\frac{1}{2}}}\big)) \\
&= \frac{1}{2} \sum_i M(\int\limits_{F^{\pi}_i} \kappa_C(x)^{\frac{1}{4}}\,d\sigma(x), v_i)(1 - O\big(\frac{1}{n^{\frac{1}{2}}}\big))^4(1 - O\big(\frac{1}{n^{\frac{1}{2}}}\big))
\end{aligned}$$

$$\geq \frac{n}{2} M\Big(\frac{1}{n} \int\limits_{\mathrm{bd}\, C} \kappa_C(x)^{\frac{1}{4}}\, d\sigma(x), \frac{1}{n}(v_1 + \cdots + v_n)\Big)\Big(1 - O\Big(\frac{1}{n^{\frac{1}{2}}}\Big)\Big)$$

$$\geq \frac{n}{2} M\Big(\frac{1}{n} A(C), 6\Big)\Big(1 - O\Big(\frac{1}{n^{\frac{1}{2}}}\Big)\Big) = \frac{5A(C)^2}{36\sqrt{3}\, n} - O\Big(\frac{1}{n^{1+\frac{1}{2}}}\Big).$$

2.2 Upper Estimate. We shall prove that

(41) $\quad \delta^V(C, \mathcal{P}^c_{(n)}) \leq \dfrac{5A(C)^2}{36\sqrt{3}\, n} + O\Big(\dfrac{1}{n^{1+\frac{1}{4}}}\Big)$ as $n \to \infty$.

2.2.1 First, a suitable representation of $\mathrm{bd}\, C$ will be introduced, different from the one in 2.1.

For a proof of the following result for general d see [12].

(42) For $m = 1, 2, \ldots$, the 2-sphere S^2 can be dissected into m convex spherical polygons $S_i, i = 1, \ldots, m$, say, such that for each i a circle of radius $O(1/m^{1/2})$ is contained in S_i and S_i is contained in a concentric circle of radius $O(1/m^{1/2})$.

Assume that $o \in \mathrm{int}\, C$. Then for each $m = 1, 2, \ldots$, dissect $\mathrm{bd}\, C$ as follows: project the convex spherical polygons $S_i \subset S^2$ in radial direction onto $\mathrm{bd}\, C$. This gives a dissection of $\mathrm{bd}\, C$ into m sets $C_i, i = 1, \ldots, m$, say. For each i consider the ray starting at o which meets S_i at the center of the two circles corresponding to S_i according to (42). Let H_i be a plane which intersects this ray orthogonally, but does not meet C. Let "$\,'\,$" denote the orthogonal projection onto H_i. Choose a Cartesian coordinate system in H_i with origin o at the point where the corresponding ray intersects H_i. Then, for sufficiently large m,

(43) the m sets $C_i\,, i = 1, \ldots, m$, form a dissection of $\mathrm{bd}\, C$, each C_i is a convex disc in H_i, and
$$\frac{\lambda}{m^{\frac{1}{2}}} B^2 \subseteq C_i' \subseteq \frac{\mu}{m^{\frac{1}{2}}} B^2.$$

The Cartesian coordinate system in H_i together with the normal unit vector of H_i which points to C forms a Cartesian coordinate system of $\mathbb{E}^3$. When we speak of the "lower" part of $\mathrm{bd}\, C$, this is with respect to the last coordinate. Represent the lower part of $\mathrm{bd}\, C$ in the form

(44) $\quad \{(s, f_i(s)) : s \in C'\}.$

Then, if m is sufficiently large, an argument of Schneider [15] shows (as in 2.1) that

(45) f_i is convex, of class $\mathcal{C}^3$, and
$$|f_{i,k}|, |f_{i,kl}|, |f_{i,klm}|, (\mathrm{grad} f_i)^2 \leq \text{const on } \frac{2\mu}{m^{\frac{1}{2}}} B^2.$$

Define quadratic forms q_u by

$$(46)\qquad q_u(s)=\sum_{k,l} f_{i,kl}(u)s^k s^l \text{ for } s\in \mathbb{E}^3, \text{ where } u\in \frac{2\mu}{m^{\frac12}}B^2,$$
$$q_i = q_o.$$

As a consequence of Blaschke's rolling theorem and the fact that the coefficient of q_u are continuous in u we have for sufficiently large m that

$$(47)\qquad \nu\,\|\cdot\|^2 \le q_i(\cdot)\le \xi\,\|\cdot\|^2,$$

$$(48)\qquad q_u\le 2q_i \text{ for any } u\in \frac{2\mu}{m^{\frac12}}B^2.$$

Finally, we note that for sufficiently large m,

$$(49)\qquad \kappa_C(u)=\frac{\det q_u}{(1+(\mathrm{grad} f_i(u))^2)^2} \text{ for } u\in \frac{2\mu}{m^{\frac12}}B^2.$$

2.2.2 Second, we shall construct "almost" best approximating polytopes Q_n for C in $\mathcal{P}^c_{(n)}$. Let n be sufficiently large and let $m=\lfloor n^{1/2}\rfloor$. Consider in H_i an edge-to-edge tiling with convex hexagons of area $|C_i'|A(C)/A(C_i)n$ which are regular with respect to the norm $q_i^{1/2}$. Here

$$(50)\qquad A(C_i)=\int\limits_{C_i}\kappa_C(x)^{\frac14}\,d\sigma(x),\, A(C)=\int\limits_{\mathrm{bd}\,C}\kappa_C(x)^{\frac14}\,d\sigma(x).$$

Since $|C_i'|\le |C_i|\le \mathrm{const}\,|C_i'|$ by (43) and (46) and since κ_C is continuous and positive on the compact surface $\mathrm{bd}\,C$, we see that $|C_i'|A(C)/A(C_i)$ is bounded between positive constants. This combined with (47) shows that the diameter of each of these hexagons is at most $\rho/n^{1/2}$. From (43) it follows that

$$(1-\frac{\rho}{\lambda}\Big(\frac{m}{n}\Big)^{\frac12})C_i'+\frac{\rho}{n^{\frac12}}B^2\subseteq (1-\frac{\rho}{\lambda}\Big(\frac{m}{n}\Big)^{\frac12}+\frac{\rho}{\lambda}\Big(\frac{m}{n}\Big)^{\frac12})C_i' = C_i'.$$

Thus $D_i'=(1-(\rho/\lambda)(m/n)^{1/2})C_i'$ is contained in $\mathrm{relint}\,C_i'$ and has distance at least $\rho/n^{1/2}$ from $\mathrm{relbd}\,C_i'$. Consider the hexagons of our tiling which meet D_i'. Since each hexagon has diameter at most $\rho/n^{1/2}$, all these hexagons are contained in C_i'. Hence there are

$$(51)\qquad n_i\le \frac{A(C_i)}{A(C)}n$$

such hexagons. Multiplying each of these hexagons with the factor

$$\frac{1+\frac{\rho}{\lambda}\Big(\frac{m}{n}\Big)^{\frac12}}{1-\frac{\rho}{\lambda}\Big(\frac{m}{n}\Big)^{\frac12}} \quad (\le 1+O\Big(\frac{1}{n^{\frac14}}\Big)\le 2 \text{ for sufficiently large } n),$$

gives a system of n_i hexagons $H_{ij}, j=1,\dots,n_i$, in the plane H_i. For sufficiently large n this system of hexagons has the following properties:

(52) $\quad \mathrm{diam} H_{ij} \le \dfrac{2\rho}{n^{\frac{1}{2}}}$,

(53) $\quad |H_{ij}| \le \dfrac{|C_i'|A(C)(1+O\big(\frac{1}{n^{\frac{1}{4}}}\big))^2}{A(C_i)n} \le \dfrac{|C_i'|A(C)}{A(C_i)n}(1+O\Big(\dfrac{1}{n^{\frac{1}{4}}}\Big)) \le O\Big(\dfrac{1}{n}\Big)$,

(54) $\quad H_{ij} \subset (1+O\Big(\dfrac{1}{n^{\frac{1}{4}}}\Big))C_i' \subseteq 2C_i' \subseteq \dfrac{2\mu}{m^{\frac{1}{2}}}B^2 \subset C'$,

(55)
$$H_{i1}\cup\dots\cup H_{in_i} \supseteq \frac{1+\frac{\rho}{\lambda}\big(\frac{m}{n}\big)^{\frac{1}{2}}}{1-\frac{\rho}{\lambda}\big(\frac{m}{n}\big)^{\frac{1}{2}}}(1-\frac{\rho}{\lambda}\Big(\frac{m}{n}\Big)^{\frac{1}{2}})C_i'$$
$$= C_i' + \frac{\rho}{\lambda}\Big(\frac{m}{n}\Big)^{\frac{1}{2}}C_i' \supseteq C_i' + \frac{\rho}{n^{\frac{1}{2}}}B^2 \supseteq C_i'.$$

This follows from the definition of these hexagons, the above remarks and propositions (43) and (50). Choose points p_{ij} on the lower side of $\mathrm{bd}\, C$ such that p_{ij}' is the center of H_{ij} and define corresponding quadratic forms q_{ij} by

(56) $\quad q_{ij}(s) = q_{p_{ij}'}(s) = \sum_{k,l} f_{i,kl}(p_{ij}')s^k s^l$ for $s \in \mathbb{E}^2$.

Noticing (46), (48) and (54), we have that

(57) $\quad q_{ij} \le 2q_i$.

Finally, define Q_n to be the intersection of the support halfspaces of C at the points p_{ij}. (If n is sufficiently large, these points are distributed rather densely over $\mathrm{bd}\, C$ and thus) Q_n is a convex polytope with at most $n_1+\dots+n_m \le n$ facets (see (51) and (43)) which is circumscribed to C. Hence

(58) $\quad Q_n \in \mathcal{P}^c_{(n)}$.

2.2.3 Third, we cover $Q_n \setminus \mathrm{int}\, C$ by sets whose volume may be easily calculated. Since the sets $C_i, i=1,\dots,m$, form a dissection of $\mathrm{bd}\, C$ by (43), we have that

(59) $\quad Q_n \setminus \mathrm{int}\, C \subset Q_{n1}\cup\dots\cup Q_{nm}$, where $Q_{ni} = \{x \in Q_n \setminus \mathrm{int}\, C : x^\pi \in C_i\}$.

Next, the following will be shown:

(60) $\quad Q_{ni} \subset R_{ni} = \{(s,t) \in Q_n : s \in C_i' + \dfrac{\rho}{n^{\frac{1}{2}}}B^2, t \le f_i(s)\}$
for sufficiently large n.

For the proof of (60) it is sufficient to show the following: let $x \in Q_n \setminus \operatorname{int} C$ with $x^\pi \in C_i$ be given, then $\|x' - x^{\pi\prime}\| < \rho/n^{1/2}$. Since $x^\pi \in C_i$ and thus $x^{\pi\prime} \in C_i'$, proposition (55) implies that $x^{\pi\prime} \in H_{ij}$ for suitable j. Elementary calculations, Taylor's theorem, (52), (45), (47) and (48) together show that the distance of x^π from the intersection point y of the exterior normal of $\operatorname{bd} C$ at x^π with the support plane of C at p_{ij} is at most $O(1/n)$. Since $x \in [x^\pi, y]$ by (8),

$$\|x' - x^{\pi\prime}\| \le \|y' - x^{\pi\prime}\| \le \|y - x^\pi\| \le O\Big(\frac{1}{n}\Big).$$

Since $O(1/n) < \rho/n^{1/2}$ for sufficiently large n, the proof of (60) is complete.

2.2.4 Finally, (59), (60), (55), the definition of Q_{ni}, Taylor's theorem applied to $f_i|H_{ij}$, (56) and $\|s - p_{ij}'\| \le O(1/n^{1/2})$ (see (52)), (55) and (45), Taylor's theorem applied to the coefficients of q_i and the fact that $\|p_{ij}' - o\| \le O(1/m^{1/2})$ (see (54)) and $\|s - p_{ij}{}'\| \le O(1/n^{1/2})$ (see (52)), (17), (53), (51), (53), (50), (49), (46), Taylor's theorem applied to $\kappa_C(s)^{1/4}(1 + (\operatorname{grad} f_i(s))^2)^{1/2}|C_i'$ which has bounded partial derivatives by (45) combined with $\|s - o\| \le O(1/m^{1/2})$ (see (43)), a calculus formula for surface integrals, the definition of $A(C_i)$ and (43) together imply proposition (41):

$$\begin{aligned}
\delta^V(C, Q_n) &= V(Q_n \setminus \operatorname{int} C) \le \sum_i V(Q_{ni}) \le \sum_i V(R_{ni}) \\
&\le \sum_{i,j} \int_{H_{ij}} \{f_i(s) - f_i(p_{ij}{}') - \operatorname{grad} f_i(p_{ij}{}')(s - p_{ij}{}')\}\, ds \\
&\le \frac{1}{2} \sum_{i,j} \int_{H_{ij}} q_{ij}(s - p_{ij}{}')\, ds + |H_{ij}|\, O\Big(\frac{1}{n^{\frac{3}{2}}}\Big)\} \\
&\le \frac{1}{2} \sum_{i,j} \{ \int_{H_{ij}} q_i(s - p_{ij}{}')\, ds + |H_{ij}|\, O\Big(\frac{1}{n^{\frac{3}{2}}}\Big) + |H_{ij}|\, O\Big(\frac{1}{n\, m^{\frac{1}{2}}}\Big)\} \\
&\le \frac{1}{2} \sum_i n_i \frac{5\, |H_{ij}|^2 (\det q_i)^{\frac{1}{2}}}{18\sqrt{3}} \Big(1 + O\Big(\frac{1}{n^{\frac{1}{4}}}\Big)\Big)^4 + O\Big(\frac{1}{n^{1+\frac{1}{4}}}\Big) \\
&\le \frac{5}{36\sqrt{3}\, n} \sum_i A(C_i)^{1-2} A(C)^{2-1} |C_i'|^2 \kappa_C(o)^{\frac{1}{2}} (1 + (\operatorname{grad} f_i(o))^2) \\
&\qquad\qquad \times \Big(1 + O\Big(\frac{1}{n^{\frac{1}{4}}}\Big)\Big) + O\Big(\frac{1}{n^{1+\frac{1}{4}}}\Big) \\
&= \frac{5A(C)}{36\sqrt{3}\, n} \sum_i A(C_i)^{-1} \Big(\int_{C_i'} \kappa_C(o)^{\frac{1}{4}} (1 + (\operatorname{grad} f_i(o))^2)^{\frac{1}{2}}\, ds\Big)^2 \\
&\qquad\qquad \times \Big(1 + O\Big(\frac{1}{n^{\frac{1}{4}}}\Big)\Big) + O\Big(\frac{1}{n^{1+\frac{1}{4}}}\Big) \\
&\le \frac{5A(C)}{36\sqrt{3}\, n} \sum_i A(C_i)^{-1} \Big(\int_{C_i'} \kappa_C(s)^{\frac{1}{4}} (1 + (\operatorname{grad} f_i(s))^2)^{\frac{1}{2}}\, ds\Big)^2 \\
&\qquad\qquad \times \Big(1 + O\Big(\frac{1}{m^{\frac{1}{2}}}\Big)\Big)^2 \Big(1 + O\Big(\frac{1}{n^{\frac{1}{4}}}\Big)\Big) + O\Big(\frac{1}{n^{1+\frac{1}{4}}}\Big) \\
&= \frac{5A(C)}{36\sqrt{3}\, n} \sum_i A(C_i) \Big(1 + O\Big(\frac{1}{n^{\frac{1}{4}}}\Big)\Big) + O\Big(\frac{1}{n^{1+\frac{1}{4}}}\Big) \\
&= \frac{5A(C)^2}{36\sqrt{3}\, n} + O\Big(\frac{1}{n^{1+\frac{1}{4}}}\Big).
\end{aligned}$$

2.3 Conclusion. The Theorem now follows from (3) and (41).

3 Acknowledgement

I am obliged to Professors Goodey, Ludwig and Schnitzer for their advice and references to the literature.

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