An Arithmetic Proof of John's Ellipsoid Theorem

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Abstract. Using an idea of Voronoi in the geometric theory of positive definite quadratic forms, we give a transparent proof of John's characterization of the unique ellipsoid of maximum volume contained in a convex body. The same idea applies to the 'hard part' of a generalization of John's theorem and shows the difficulties of the corresponding 'easy part'.

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Introduction and Statement of Results

The following well-known characterizations of the unique ellipsoid of maximum volume in a convex body in Euclidean d-space are due to John [11] ((i)$\Rightarrow$(ii)) and Pelczyński [13] and Ball [1] ((ii)$\Rightarrow$(i)), respectively. For references to other proofs, a generalization and to the numerous applications see [2, 6, 8, 12].

Theorem 1 *Let $C \subset \mathbb{E}^d$ be compact, convex, symmetric in the origin o, and with $B^d \subset C$. Then the following claims are equivalent:*

(i) *B^d is the unique ellipsoid of maximum volume in C.*

(ii) *There are $u_k \in B^d \cap \operatorname{bd} C$ and $\lambda_k > 0, k = 1, \dots, n$, where $d \leq n \leq \frac{1}{2}d(d+1)$, such that*

$$I = \sum_k \lambda_k \, u_k \otimes u_k.$$

Here, B^d is the solid unit ball in $\mathbb{E}^d$, I the $d \times d$ unit matrix, and for $u, v \in \mathbb{E}^d$ the $d \times d$ matrix $u \, v^T$ is denoted by $u \otimes v$. bd stands for boundary.

Theorem 2 *Let $C \subset \mathbb{E}^d$ be compact, convex, and with $B^d \subset C$. Then the following claims are equivalent:*

(i) *B^d is the unique ellipsoid of maximum volume in C.*

(ii) *There are $u_k \in B^d \cap \operatorname{bd} C$ and $\lambda_k > 0, k = 1, \dots, n$, where $d + 1 \leq n \leq \frac{1}{2}d(d+3)$, such that*

$$I = \sum_k \lambda_k \, u_k \otimes u_k, \; o = \sum_k \lambda_k \, u_k.$$

Our proof of Theorem 1 is based on the idea of Voronoi in the geometric theory of positive definite quadratic forms, to represent ellipsoids in $\mathbb{E}^d$ with center o by points in $\mathbb{E}^{\frac{1}{2}d(d+1)}$, see [4, 10, 15]. The problem on maximum volume ellipsoids in $\mathbb{E}^d$ is then transformed into a simple problem on normal cones in $\mathbb{E}^{\frac{1}{2}d(d+1)}$, which can be solved easily by Carathéodory's theorem on convex hulls. This idea has been applied before by the first author [9]. The proof of Theorem 2 is a simple extension. The proof of the latter also gives Theorem 4 of Bastero and Romance [3], where B^d is replaced by a compact connected set with positive measure.

In the context of John's theorem, it is natural to ask whether ellipsoids can be replaced by more general convex or non-convex sets. The following is a slight refinement of results of Giannopoulos, Perissinaki and Tsolomitis [7], Bastero and Romance [3] (Theorem 3) and Gordon, Litvak, Meyer and Pajor [8] (Theorems 3.5,3.8). The result of Giannopoulos et. al. was first observed by Milman in the case, where both bodies are centrally symmetric, see [17].

Theorem 3 *Let $C \subset \mathbb{E}^d$ be compact and convex, and $B \subset C$ compact with positive measure. Then* (i) *implies* (ii)*, where the claims* (i) *and* (ii) *are as follows:*

(i) *B has maximum measure amongst all its affine images contained in C.*

(ii) *There are $u_k \in B \cap \operatorname{bd} C, v_k \in N_C(u_k)$, and $\lambda_k > 0, k = 1, \dots, n$, where $d+1 \leq n \leq d\,(d+1)$, such that*

$$I = \sum_k \lambda_k \, u_k \otimes v_k, \;\; o = \sum_k \lambda_k \, v_k.$$

Here $N_C(u), u \in \operatorname{bd} C$, is the normal cone of C at u. For this concept and other required notions and results of convex geometry we refer to [16].

Note that B is not necessarily unique. A suitable modification of Voronoi's idea applies in the present context and thus leads to a proof of Theorem 3, paralleling our proofs of Theorems 1 and 2. Incidentally, the proof of Theorem 3 shows, why it is *not* clear that property (ii) implies property (i), see the Final Remarks.

Proof of Theorem 1

For (real) $d \times d$–matrices $A = (a_{ij}), B = (b_{ij})$ define $A \cdot B = \sum a_{ij} b_{ij}$. The dot $\cdot$ denotes also the inner product in $\mathbb{E}^d$. Easy arguments yield the following:

(1) Let M be a $d \times d$ matrix and $u, v, w \in \mathbb{E}^d$. Then $Mu \cdot v = M \cdot u \otimes v$ and $(u \otimes v)w = (v \cdot w)u$.

Next, we specify two tools:

(2) Each $d \times d$ matrix M with $\det M \neq 0$ can be represented in the form $M = AR$, where A is a symmetric, positive definite and R is an orthogonal $d \times d$ matrix.

(Put $A = (MM^T)^{\frac{1}{2}}, R = A^{-1}M$, see [5], p.112.) Identify a symmetric $d \times d$ matrix $A = (a_{ij})$ with the point $(a_{11}, \dots, a_{1d}, a_{22}, \dots, a_{2d}, \dots, a_{dd})^T \in \mathbb{E}^{\frac{1}{2}d(d+1)}$. The set of all symmetric, positive definite $d \times d$ matrices then is (represented by) an open convex cone $\mathcal{P} \subset \mathbb{E}^{\frac{1}{2}d(d+1)}$ with apex at the origin. The set

(3) $\mathcal{D} = \{A \in \mathcal{P} : \det A \geq 1\}$ is a closed, smooth, strictly convex set in $\mathcal{P}$ with non-empty interior.

(Use the implicit function theorem and Minkowski's inequality for symmetric, positive definite $d \times d$ matrices, see [14], p.205.)

(i)$\Rightarrow$(ii): By (2), any ellipsoid in $\mathbb{E}^d$ can be represented in the form AB^d, where $A \in \mathcal{P}$. Thus the family of all ellipsoids in C is represented by the set

$$\mathcal{E} = \{A \in \mathcal{P} : Au \cdot v = A \cdot u \otimes v \leq h_C(v) \text{ for } u, v \in S^{d-1}\},$$

see (1). Here, $h_C(\cdot)$ is the support function of C. Clearly, $\mathcal{E}$ is the intersection of the closed halfspaces

(4) $\{A \in \mathbb{E}^{\frac{1}{2}d(d+1)} : A \cdot u \otimes v \leq h_C(v)\} : u, v \in S^{d-1}$,

with the set $\mathcal{P}$. Thus, in particular, $\mathcal{E}$ is convex. By (i), $\mathcal{E} \setminus \{I\} \subset \{A \in \mathcal{P} : \det A < 1\}$. This, together with (3), shows that

(5) $\mathcal{D}$ and $\mathcal{E}$ are convex, $\mathcal{D} \cap \mathcal{E} = \{I\}$, and $\mathcal{D}$ and $\mathcal{E}$ are separated by the unique support hyperplane $\mathcal{H}$ of $\mathcal{D}$ at I in $\mathbb{E}^{\frac{1}{2}d(d+1)}$.

$\mathcal{E}$ is the intersection of the closed halfspaces in (4) with the set $\mathcal{P}$, and these halfspaces vary continuously as u, v range over S^{d-1}. Thus the support cone $\mathcal{K}$ of $\mathcal{E}$ at I can be represented as the intersection of those halfspaces, which contain I on their boundary hyperplanes, i.e. for which $I \cdot u \otimes v = u \cdot v = h_C(v)$. Since $u \cdot v \leq 1$ and $h_C(v) \geq 1$ and equality holds in both cases precisely when $u = v \in S^{d-1} \cap \operatorname{bd} C$ (note that $B^d \subset C$), we see that

(6) $$\mathcal{K} = \bigcap_{u \in B^d \cap \operatorname{bd} C} \{A \in \mathbb{E}^{\frac{1}{2}d(d+1)} : A \cdot u \otimes u \leq 1\}.$$

The normal cone $\mathcal{N}$ of ($\mathcal{E}$ or) $\mathcal{K}$ at I is generated by the exterior normals of these halfspaces,

(7) $\mathcal{N} = \operatorname{pos}\{u \otimes u : u \in B^d \cap \operatorname{bd} C\}$.

The cone $\mathcal{K}$ has apex I and, by (5), is separated from the convex set $\mathcal{D}$ by the hyperplane $\mathcal{H}$, where $\mathcal{H}$ is the unique support hyperplane of $\mathcal{D}$ at I. By considering the gradient of the function $A \to \det A$, we see that I is an interior normal vector of $\mathcal{D}$ at I and thus a normal vector of $\mathcal{H}$ pointing away from $\mathcal{K}$. Hence $I \in \mathcal{N}$. (7) and Carathéodory's theorem for convex cones then yield the following: there are $u_k \otimes u_k \in \mathcal{N}$, i.e. $u_k \in B^d \cap \operatorname{bd} C$, and $\lambda_k > 0$ for $k = 1, \dots, n$, where $n \leq \frac{1}{2}d(d+1)$, such that

(8) $\qquad I = \sum_k \lambda_k \, u_k \otimes u_k.$

For the proof that $n \geq d$, it is sufficient to show that $\mathrm{lin}\,\{u_1, \dots, u_n\} = \mathbb{E}^d$. If this were not true, we could choose $u \neq o, u \perp u_1, \dots, u_n$, and then (1) yields the contradiction

$$0 \neq u^2 = Iu \cdot u = (\sum_k \lambda_k \, (u_k \otimes u_k) \, u) \cdot u = (\sum_k \lambda_k \, (u_k \cdot u) \, u_k) \cdot u = 0.$$

(ii)⇒(i): Let $\mathcal{E}$ be as above. $\mathcal{E}$ is convex. $B^d \subset C$ implies that I satisfies all defining inequalities of $\mathcal{E}$, in particular those corresponding to $u = v = u_k, k = 1, \dots, n$. Since $h_C(u_k) = 1$, these inequalities are satisfied even with the equality sign. Thus $I \in \mathrm{bd}\,\mathcal{E}$. Define $\mathcal{K}, \mathcal{N}$ and $\mathcal{H}$ as before. (ii) implies that $I \in \mathcal{N}$. Hence $\mathcal{K}$ is contained in the closed halfspace with boundary hyperplane $\mathcal{H}$ through I and exterior normal vector I. Clearly, $\mathcal{H}$ separates $\mathcal{K}$ and $\mathcal{D}$ and thus, a fortiori, $\mathcal{E}(\subset \mathcal{K})$ and $\mathcal{D}$. Since $\mathcal{D}$ is strictly convex by (3), $\mathcal{D} \cap \mathcal{E} = \{I\}$. Hence B^d is the unique ellipsoid of maximum volume in C.

Outline of the Proof of Theorem 2

The proof of Theorem 2 is almost identical with that of Theorem 1: an ellipsoid now has the form $AB^d + a$ and is represented by $(A, a) \in \mathcal{P} \times \mathbb{E}^d \subset \mathbb{E}^{\frac{1}{2}d(d+3)}$. $\mathcal{E}$ is the set

$$\{(A, a) \in \mathcal{P} \times \mathbb{E}^d : A \cdot u \otimes v + a \cdot v \leq h_C(v) \text{ for } u, v \in S^{d-1}\}$$

and instead of (5) we have

> $\mathcal{D} \times \mathbb{E}^d$ and $\mathcal{E}$ are convex, $(\mathcal{D} \times \mathbb{E}^d) \cap \mathcal{E} = \{(I, o)\}$ and $\mathcal{D} \times \mathbb{E}^d$ and $\mathcal{E}$ are separated by the hyperplane $\mathcal{H} \times \mathbb{E}^d$, where $\mathcal{H}$ is the unique support hyperplane of $\mathcal{D}$ at I (in $\mathbb{E}^{\frac{1}{2}d(d+1)}$).

$\mathcal{K}$ and $\mathcal{N}$ are the cones

$$\mathcal{K} = \bigcap_{u \in B^d \cap \mathrm{bd}\, C} \{(A, a) \in \mathbb{E}^{\frac{1}{2}d(d+3)} : A \cdot u \otimes u + a \cdot u \leq 1\},$$

$$\mathcal{N} = \mathrm{pos}\,\{(u \otimes u, u) : u \in B^d \cap \mathrm{bd}\, C\}.$$

As before, $(I, o) \in \mathcal{N}$. Carathéodory's theorem for cones in $E^{\frac{1}{2}d(d+3)}$ then shows the following: there are $(u_k \otimes u_k, u_k) \in \mathcal{N}$ or, equivalently, $u_k \in B^d \cap \mathrm{bd}\, C$ and $\lambda_k > 0,\ k = 1, \dots, n$, where $n \leq \frac{1}{2}d(d+3)$, such that instead of (8) we have

$$(I, o) = (\sum_k \lambda_k \, u_k \otimes u_k, \sum_k \lambda_k u_k).$$

Since $o = \sum \lambda_k u_k$ and $\lambda_k > 0$, the proof that $n \geq d + 1$ is the same as that for $n \geq d$ above. This concludes the proof that (i)⇒(ii). The proof of (ii)⇒(i) is almost the same as that of the corresponding part of the proof of Theorem 1.

Proof of Theorem 3

Identify a $d \times d$ matrix $M = (m_{ij})$ with the point $(m_{11}, \dots, m_{1d}, m_{21}, \dots, m_{2d}, \dots, m_{dd})^T \in \mathbb{E}^{d^2}$. The set $\mathcal{P}'$ of all non-singular $d \times d$ matrices then is (represented by) an open cone in $\mathbb{E}^{d^2}$ with apex at the origin. The set

> $\mathcal{D}' = \{M \in \mathcal{P}' : |\det M| \geq 1\}$ is a closed body in $\mathcal{P}'$, i.e. it is the closure of its interior, with a smooth boundary surface.

The set of all affine images of B in C is represented by the set

$$\mathcal{E}' = \{(M, a) \subset \mathcal{P}' \times \mathbb{E}^d : Mu \cdot v + a \cdot v = M \cdot u \otimes v + a \cdot v \leq h_C(v) \text{ for } u \in B, v \in S^{d-1}\}.$$

This set is the intersection of the closed halfspaces

$$(9) \qquad \{(M, a) \in \mathbb{E}^{d(d+1)} : M \cdot u \otimes v + a \cdot v \leq h_C(v)\} : u \in B, v \in S^{d-1},$$

and thus of a convex set, with the set $\mathcal{P}' \times \mathbb{E}^d$. Choose a convex neighborhood $\mathcal{U}'$ of (I, o) $(\in \mathcal{E}' \cap (\mathcal{P}' \times \mathbb{E}^d))$ which is so small that it is contained in the open set $\mathcal{P}' \times \mathbb{E}^d$. By (i),

> the convex set $\mathcal{E}' \cap \mathcal{U}'$ and the smooth body $\mathcal{D}' \times \mathbb{E}^d$ only have boundary points in common, one being (I, o).

Hence $\mathcal{E}' \cap \mathcal{U}'$ and thus the support cone $\mathcal{K}'$ of $\mathcal{E}' \cap \mathcal{U}'$ at (I, o) is contained in the closed halfspace whose boundary hyperplane is the tangent hyperplane of the smooth body $\mathcal{D}' \times \mathbb{E}^d$ at (I, o) and with exterior normal pointing into $\mathcal{D}' \times \mathbb{E}^d$. This normal is (I, o). The normal cone $\mathcal{N}'$ of $\mathcal{K}'$ thus contains (I, o).

The support cone $\mathcal{K}'$ is the intersection of those halfspaces in (9), which contain the apex (I, o) on their boundary hyperplanes. Thus $I \cdot u \otimes v + o \cdot v = h_C(v)$, which is equivalent to $u \in B \cap \operatorname{bd} C, v \in N_C(u)$. Hence, these halfspaces have the form

$$\{(M, a) \in \mathbb{E}^{d(d+1)} : M \cdot u \otimes v + a \cdot v \leq h_C(v)\} : u \in B \cap \operatorname{bd} C,\ v \in N_C(u),$$

where $N_C(u)$ is the normal cone of C at the boundary point u. Thus, being the normal cone of $\mathcal{K}'$,

$$\mathcal{N}' = \operatorname{pos}\{(u \otimes v, v) : u \in B \cap \operatorname{bd} C,\ v \in N_C(u)\}.$$

Since $(I, o) \in \mathcal{N}'$, Carathéodory's theorem for convex cones in $\mathbb{E}^{d(d+1)}$ yields the following: there are $(u_k \otimes v_k, v_k) \in \mathcal{N}'$ or, equivalently, $u_k \in B \cap \operatorname{bd} C, v_k \in N_C(u_k)$, and $\lambda_k > 0,\ k = 1, \dots, n$, where $n \leq d(d+1)$, such that

$$(I, o) = (\sum_k \lambda_k \, u_k \otimes v_k, \sum_k \lambda_k \, v_k).$$

For the proof that $n \geq d + 1$ we show by contradiction that $\operatorname{lin}\{v_1, \dots, v_n\} = \mathbb{E}^d$ as in the proof of Theorem 1.

Final Remarks

In different versions of the proofs of Theorems 1 and 2, which are closer to Voronoi's idea, ellipsoids are represented in the form $x^T Ax \leq 1$ and $(x-a)^T A(x-a) \leq 1$, respectively.

If in Theorem 3 claim (ii) holds, then the support cone $\mathcal{K}'$ of $\mathcal{E}' \cap \mathcal{U}'$ at (I, o) is contained in the halfspace whose boundary is the tangent hyperplane of $\mathcal{D}' \times \mathbb{E}^d$ at (I, o) and with exterior normal pointing into $\mathcal{D}' \times \mathbb{E}^d$. Since $\mathcal{D}' \times \mathbb{E}^d$ is not convex, this does *not* guarantee that $\mathcal{D}' \times \mathbb{E}^d$ and $\mathcal{E}'$ do not overlap , i.e. that (i) holds.

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