Trudy Mat. Inst. Steklov
Proc. Steklov Math. Inst.

Optimal Arrangement of Finite Point Sets in Riemannian 2-Manifolds

Peter M.Gruber

Dedicated to Academician S. Novikov on the occasion of his 60th birthday

Abstract. First a stability version of a theorem of L. Fejes Tóth on sums of moments is given: a large finite point set in a 2-dimensional Riemannian manifold, for which a certain sum of moments is minimal, must be approximately a regular hexagonal pattern. This result is then applied to show the following: (i) The nodes of optimal numerical integration formulae for Hoelder continuous functions on such manifolds form approximately regular hexagonal patterns if the number of nodes is large. (ii) Given a smooth convex body in $\mathbb{E}^3$, most facets of the circumscribed convex polytopes of minimum volume in essence are affine regular hexagons if the number of facets is large. A similar result holds with volume replaced by mean width. (iii) A convex polytope in $\mathbb{E}^3$ of minimal surface area, amongst those of given volume and given number of facets, has the property that most of its facets are almost regular hexagons assuming the number of facets is large.

MSC 1991: 53C20, 52A15, 52A27, 52A38, 65D30.

1 Introduction

One of the few general results in discrete geometry is the following *theorem of L. Fejes Tóth [?, ?] on sums of moments*: let $f : [0, +\infty) \to [0, +\infty)$ be non-decreasing and let H be a convex $3, 4, 5$, or 6-gon in the Euclidean plane $\mathbb{E}^2$. Then, for any set S of n points in $\mathbb{E}^2$,

$$(1) \qquad \int\limits_H \min\{f(\|x-p\|) : p \in S\} dx \geq n \int\limits_{H_n} f(\|x\|) dx,$$

where H_n is a regular hexagon in $\mathbb{E}^2$ of area $|H|/n$ and center at the origin; $\|\cdot\|$ and $|\cdot|$ denote the Euclidean norm and the ordinary area measure in $\mathbb{E}^2$.

Fejes Tóth [?] proved his result first for the 2-sphere S^2 instead of $\mathbb{E}^2$ and only slightly later for $\mathbb{E}^2$, see [?]. For alternative proofs, in some cases for surfaces of constant curvature and for weight functions, see L. Fejes Tóth [?], Imre [?], G. Fejes Tóth [?], and Florian [?]. The theorem on sums of moments has a series of applications. Among these are:

(i) Packing and covering problems for solid circles in $\mathbb{E}^2$, in S^2, and in the hyperbolic plane and also for convex discs in $\mathbb{E}^2$. See L. Fejes Tóth [?, ?], Imre [?], G. Fejes Tóth [?, ?], and Florian [?].

(ii) Problems of optimal location, of errors of quantization of data, and of Gauss channels, all in $\mathbb{E}^2$. See Matérn and Person [?] for the origin of the location problem and Conway and Sloane [?] for surveys of quantization of data and Gauss channels and the articles of L. Fejes Tóth [?, ?], Bollobás [?], Gersho [?], and Newman [?].

(iii) The isoperimetric problem for three dimensional convex polytopes of given combinatorial type such as tetrahedra, hexahedra, and dodecahedra and other extremum problems for convex polytopes in $\mathbb{E}^3$. See L. Fejes Tóth [?, ?, ?] and Coxeter and L. Fejes Tóth [?], L. Fejes Tóth [?, ?, ?], Florian [?] and Linhart [?]. Compare also the survey of Florian [?].

(iv) Optimal choice of nodes in numerical integration formulae for Hoelder continuous functions of two variables. See Babenko [?, ?], and for general information, Stein [?].

(v) Asymptotically best approximation of smooth convex bodies in $\mathbb{E}^3$ by circumscribed convex polytopes of minimum volume as the number of facets tends to infinity and similarly for the mean width deviation. See Gruber [?] and Glasauer and Gruber [?].

In most of these applications f is of the form $f(t) = t^a$ where $0 < a \leq 2$. In view of the applications, two extensions of the theorem on the sum of moments suggest themselves, namely extensions to higher dimensions and to Riemannian manifolds. While the first extension is out of reach, it is not too difficult to give a version of the theorem for a 2-dimensional Riemannian manifold M. It is more complicated to prove a stability version of the second extension. These two results can be applied to obtain information on the distribution of nodes in optimal numerical integration formulae for M, the form of convex polytopes of minimum volume circumscribed to a smooth convex body in $\mathbb{E}^3$, and the form of convex polytopes in $\mathbb{E}^3$ with minimum isoperimetric quotient. The proofs of these results, except the last one, are lengthy and involved and will be given elsewhere. There is little hope to extend them to higher dimensions. The results in this article belong to a group of geometric stability problems of recent origin in convex geometry. For some references see Gruber [?, ?].

2 A stability version of Fejes Tóth's theorem for Riemannian 2-manifolds

Let M be a 2-*dimensional Riemannian manifold of class* $\mathcal{C}^k, k \geq 2$. By this we mean a 2-dimensional differentiable manifold of class $\mathcal{C}^k$ with metric tensorfield of class $\mathcal{C}^{k-2}$. Let ϱ_M and ω_M be the corresponding Riemannian metric and area measure on M. A set J in M is *Jordan measurable* if it has compact closure $\operatorname{cl} J$ and $\omega_M(\operatorname{bd} J) = 0$, where bd stands for boundary. Let (S_n) be a sequence of sets in M such that $\# S_n = n$. Then S_n is said to be *asymptotically a regular hexagonal pattern in* M if there are Landau symbols $o(n)$ and $o(1)$ and a positive sequence (σ_n) such that the following hold: for each point p in S_n, with a set of at most $o(n)$ exceptions, the six points $p_1, \dots, p_6 \in S_n$ closest to p are unique and

$$\varrho_M(p, p_i), \varrho_M(p_i, p_{i+1}) = (1 \pm o(1))\sigma_n \text{ for } i = 1, \dots, 6, \; p_7 = p_1.$$

(A quantity is equal to $(1 \pm o(1))\sigma_n$ if it is between $(1 - o(1))\sigma_n$ and $(1 + o(1))\sigma_n$.) σ_n is called the *edgelength of the hexagon* $\{p_1, \dots, p_6\}$. For $0 < a \leq 2$ and $b = 1 + \frac{a}{2}$ let

$$\alpha(a) = \frac{6 \cdot 3^{\frac{b}{2}}}{b} \int_0^{\frac{\pi}{6}} \frac{d\varphi}{(\cos\varphi)^{2b}}.$$

Theorem 1. *Let M be a 2-dimensional Riemannian manifold of class $\mathcal{C}^2$, let J be a Jordan measurable set in M with $\omega_M(J) > 0$, and let $0 < a \leq 2$ and $b = 1 + \frac{a}{2}$. Then the following statements hold:*

(i) $$\inf\{\int_J \min\{\varrho_M(p, x)^a : p \in S\} d\omega_M(x) : S \subset M, \# S = n\} \sim \frac{\alpha(a)\omega_M(J)^b}{n^{\frac{a}{2}}}$$

as $n \to \infty$. This asymptotic formula continues to hold if we assume that $S \subset J$.

(ii) *If (S_n) is a sequence of sets in M such that $\# S_n = n$ and*

$$\int_J \min\{\varrho_M(p, x)^a : p \in S_n\} d\omega_M(x) \sim \frac{\alpha(a)\omega_M(J)^b}{n^{\frac{a}{2}}}$$

as $n \to \infty$, then S_n is asymptotically a regular hexagonal pattern in M where the edgelength of the hexagons is

$$\left(\frac{32\omega_M(J)}{3\sqrt{3}n}\right)^{\frac{1}{2}}.$$

The first step in the proof of this result is to establish a more precise stability version of Fejes Tóth's theorem in $\mathbb{E}^2$. The proof makes use of Dirichlet–Voronoi cells, the Euler polytope formula, and Jensen's inequality. This version is then extended to the above weak stability result for M using the local Euclidean nature of M.

It is interesting to note that sequences (S_n) satisfying the assumption of (ii) can be specified in a more or less explicit way.

A slight extension of Theorem 1, which may be useful for applications, is obtained by inserting into the integrals in (i) and (ii) a positive continuous weight factor w. The idea of the proof is to multiply the metric tensor of M by $w^{\frac{1}{b}}$ and to apply Theorem 1 to M endowed with the corresponding new Riemannian metric and area measure.

The following result shows that we may choose very special sequences of sets in M which still do quite well. Even more surprising is the fact that 'most' sequences are not too bad, at least infinitely often. Let $J \subset M$ be compact. Then a sequence of points in J may be considered as an element of the product space $J \times J \times \ldots = J^\infty$. If J^∞ is endowed with the product topology it is compact by Tychonov's theorem. Hence it is a *Baire space*, that is, the complement of each meager set is dense, where a set is *meager* if it is a countable union of nowhere dense sets. When speaking of *most* elements of a Baire space we mean all elements with a meager set of exceptions.

Theorem 2. *Let M be a 2-dimensional Riemannian manifold of class $\mathcal{C}^2$, let J be a compact Jordan measurable set in M with $\omega_M(J) > 0$ and let $0 < a \leq 2$. Then the following hold:*

(i) *There is a sequence $p_1, p_2, \ldots \in J$ such that*

$$\int_J \min\{\varrho_M(p_i, x)^a : i = 1, \ldots, n\} d\omega_M(x) = O(\frac{1}{n^{\frac{a}{2}}}) \text{ as } n \to \infty.$$

(ii) *For most sequences $q_1, q_2, \ldots \in J$,*

$$\int_J \min\{\varrho_M(q_i, x)^a : i = 1, \ldots, n\} d\omega_M(x) < \frac{\log n}{n^{\frac{a}{2}}}$$

for infinitely many n.

In the proof of (i), use is made of tools from the theory of uniform distribution, in particular of the notion of dispersion. The proof of (ii) relies on a result of the author concerning the irregularity of approximation, see [**?**].

Theorem 2 has implication both for the numerical integration and the approximation problem considered below. These applications are given elsewhere.

3 Nodes and weights in optimal numerical integration formulae

Let J be a Jordan measurable set on a d-dimensional Riemannian manifold M with measure ω_M. Given a class $\mathcal{F}$ of real functions on J, the problem arises to choose for $n = 1, 2, \ldots$, sets of nodes $N_n = \{p_1, \ldots, p_n\}$ in J and weights $W_n = \{w_1, \ldots, w_n\}$ in $I\!R$ such that the *error*

$$E(\mathcal{F}, N_n, W_n) = \sup_{f \in \mathcal{F}} \{ | \int_J f(x) d\omega_M(x) - \sum_{i=1}^{n} w_i f(p_i) | \}$$

is minimal for each n or at least asymptotically minimal as $n \to \infty$. (To simplify the notation we avoid writing p_{ni} and w_{ni}.) A related problem is to describe the optimal or asymptotically optimal choices of N_n and W_n. While the solution of these problems for arbitrary classes $\mathcal{F}$ and any sufficiently large n is hopeless, it is possible to settle the asymptotic problem for special classes $\mathcal{F}$ in case $d = 2$. We describe one such case. Let M be a 2-dimensional Riemannian manifold of class $\mathcal{C}^2$ with metric ϱ_M and area measure ω_M. Given a Jordan measurable set J in M with $\omega_M(J) > 0$ and $0 < a \leq 1$, consider the following class $\mathcal{H}^a$ of Hoelder continuous real functions on J:

$$\mathcal{H}^a = \mathcal{H}^a(J) = \{ f : J \to I\!R : |f(x) - f(y)| \leq \varrho_M(x, y)^a \text{ for } x, y \in J \}.$$

In addition to the error $E(\mathcal{H}^a, K_n, W_n)$ define

$$E(\mathcal{H}^a, N_n) = \inf_{W_n} E(\mathcal{H}^a, N_n, W_n),$$
$$E(\mathcal{H}^a, n) = \inf_{N_n} E(\mathcal{H}^a, N_n).$$

A sequence (W_n) of weights for a numerical integration formula for J with $\#W_n = n$, is *asymptotically uniform*, if there are Landau symbols $o(n)$ and (1) such that for all indices $i = 1, \ldots, n$, with at most $o(n)$ exceptions,

$$w_i = \frac{(1 \pm o(1))\omega_M(J)}{n}.$$

$\omega_M(J)/n$ then is called the *value* of W_n.

Theorem 3. *Let M be a 2-dimensional Riemannian manifold of class $\mathcal{C}^2$, let J be a Jordan measurable set in M with $\omega_M(J) > 0$, and let $0 < a \leq 1$ and $b = 1 + \frac{a}{2}$. Then the following claims hold:*

(i) $E(\mathcal{H}^a, n) \sim \dfrac{\alpha(a)\omega_M(J)^b}{n^{\frac{a}{2}}}$ *as $n \to \infty$.*

(ii) *If (N_n) and (W_n) are sequences of nodes in J and of weights, respectively, with $\#N_n = \#W_n = n$ and such that*

$$E(\mathcal{H}^a, N_n, W_n) \sim \frac{\alpha(a)\omega_M(J)^b}{n^{\frac{a}{2}}} \text{ as } n \to \infty,$$

then N_n is asymptotically a regular hexagonal pattern in J where the edge-length of the hexagons is

$$\left(\frac{32\omega_M(J)}{3\sqrt{3}n}\right)^{\frac{1}{2}}$$

and W_n is asymptotically uniform with value $\omega_M(J)/n$.

The proof of part (i) is simple and follows the lines of Babenko [**?**] who proved it for $M = \mathbb{E}^2$ and S^2; it makes use of part (i) of Theorem 1. In the proof of part (ii) the idea is to construct functions $f \in \mathcal{H}^a$ for which

$$|\int_J f(x)d\omega_M(x) - \sum_{i=1}^{n} w_i f(p_i)|$$

is large. Then part (ii) of Theorem 1 easily yields the statement about N_n, while the statement about W_n requires additional arguments.

As was the case for Theorem 1, one can, in principle, construct sequences (N_n) and (W_n) satisfying the assumptions of (ii). Again, it is possible to generalize Theorem 2 in such a way that the applications to numerical integration formulae include integrals of the form

$$\int_J f(x)w(x)d\omega_M(x)$$

where w is a positive continuous weight function on the closure of J. The proof is slightly different from the proof of the corresponding extension of Theorem 1.

4 The form of best approximating convex polytopes of a smooth convex body

Let C be a *convex body* in $\mathbb{E}^d$, that is a compact convex subset of $\mathbb{E}^d$ with non-empty interior, and let δ be a metric or some other notion of distance on the space of all convex bodies. Consider for $n = d+1, d+2, \ldots$, a class $\mathcal{P}_n$ of convex polytopes in $\mathbb{E}^d$, such as the classes of all convex polytopes with n vertices or n facets, respectively, or their subclasses of convex polytopes which are inscribed or circumscribed to C. Then the problems arise to determine or estimate

$$\delta(C, \mathcal{P}_n) = \inf\{\delta(C, P) : P \in \mathcal{P}_n\}$$

and to describe those polytopes $P_n \in \mathcal{P}_n$ for which the infimum is attained, the *best approximating polytopes* of C in $\mathcal{P}_n$ with respect to δ. For the numerous aspects of these problems see, for example, the surveys [**?**, **?**] of Gruber.

Disregarding trivial special cases, the problem to give precise descriptions of best approximating polytopes is out of reach, but it is possible to shed some light on the geometric form of such. First results in this direction which make use of ideas and notions of the theory of uniform distribution are due to Glasauer and Schneider [?] for the Hausdorff metric and to Glasauer and Gruber [?] for the symmetric difference metric and the mean width deviation. Much more precise results for $d = 3$ for the Hausdorff metric, the Banach–Mazur distance and a notion of distance due to Schneider, are due to Gruber [?]. The basic tool for the latter results is a stability theorem for thinnest coverings with solid circles.

We now describe a corresponding result for best approximating circumscribed polytopes with respect to the symmetric difference metric. Let C be a convex body in $\mathbb{E}^3$ (the boundary $\operatorname{bd} C$ of which is a surface) of class $\mathcal{C}^2$ with positive Gauss curvature. Let $\operatorname{bd} C$ be endowed with a Riemannian metric. For each p on $\operatorname{bd} C$ the Riemannian metric induces a Euclidean metric $\|\cdot\|_p$ on the tangent plane of $\operatorname{bd} C$ at p. Given a sequence (P_n) of convex polytopes circumscribed to C such that P_n has n facets, we say that P_n has *asymptotically regular hexagonal facets* with respect to the Riemannian metric if the following holds: there are Landau symbols $o(n)$ and $o(1)$ and a positive sequence (σ_n) such that each facet of P_n, with a set of at most $o(n)$ exceptions, has six vertices $v_1, \dots, v_6$, say, and for these vertices hold

$$\|v_i - p\|_p, \|v_i - v_{i+1}\|_p = (1 \pm o(1))\sigma_n \text{ for } i = 1, \dots, 6, \text{ where } v_7 = v_1.$$

Here p is the point where the facet touches $\operatorname{bd} C$. σ_n is called the *edgelength* of F.

Theorem 4. *Let C be a convex body in $\mathbb{E}^3$ of class $\mathcal{C}^2$ with positive Gauss curvature. For $n = 4, 5, \dots$, let P_n be a convex polytope with n facets circumscribed to C and of minimum volume, i.e. best approximating with respect to the symmetric difference metric. Then P_n has asymptotically regular hexagonal facets of the same edgelength with respect to the Riemannian metric of equiaffine differential geometry.*

For the definition of the latter see Blaschke [?], p. 104, or Li, Simon and Zhao [?], p. 40.

In the proof we make use of a relation between volume approximation of C by circumscribed convex polytopes and sums of moments for the Riemannian manifold $\operatorname{bd} C$. This then permits the application of Theorem 1.

We claim that one cannot expect a result similar to Theorem 4 for convex bodies of class $\mathcal{C}^1$. Our basis for this claim is the existence of a family of convex bodies of class $\mathcal{C}^1$, which is dense in the space of all convex bodies, such that each member C of the family satisfies the following: for $n = 4, 5, \dots$, let P_n be a convex polytope with n facets circumscribed to C and of minimum volume, then, for infinitely many n, all facets of P_n are approximately of triangular form. In fact, we presume that this holds not only for a dense family of convex bodies of class $\mathcal{C}^1$, but for most convex bodies in the sense of Baire categories.

It is very plausible that there is a result corresponding to Theorem 4, but for inscribed convex polytopes with n vertices. Using similar arguments to those in the proof of Theorem 4, it is possible to show a corresponding result where the symmetric difference metric is replaced by the mean width deviation.

5 The form of convex polytopes with minimum isoperimetric quotient

One among the large number of problems of isoperimetric type is the following: describe among all convex polytopes in $\mathbb{E}^d$ with n facets and given volume those of minimum surface area. Lindelöf [**?**] proved that such polytopes are necessarily circumscribed to a Euclidean ball; see Minkowski [**?**] for an alternative proof. Since there is a simple relation between the volume and the surface area of a convex polytope circumscribed to a ball, an application of Theorem 4 readily implies the following result, where the metric is the ordinary Euclidean metric.

Corollary of Theorem 4. *For $n = 4, 5, \ldots$, let P_n be a convex polytope in $\mathbb{E}^3$ of minimum surface area amongst those with n facets and given volume. Then P_n has asymptotically regular hexagonal facets of the same edgelength.*

For a discussion of related problems for particular values of n see L. Fejes Tóth [**?**] and Florian [**?**].

6 Acknowledgement

I am obliged to Professors G. Fejes Tóth, Goodey and Schnitzer for their advice and references to the literature. The work on the results of this article was begun when the author was a guest of the Mathematical Sciences Research Institute at Berkeley in April 1996. For this invitation thanks are due to Professors Milman and Thurston.

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P. M. Gruber
Abteilung für Analysis
Technische Universität Wien
Wiedner Hauptstraße 8–10/1142
A–1040 Vienna, Austria
pmgruber@pop.tuwien.ac.at