The even Orlicz Minkowski problem

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Abstract

The classical Minkowski problem leads to the L_p Minkowski problem and now to the Orlicz Minkowski problem. Existence is demonstrated for the even Orlicz Minkowski problem. A byproduct is a new approach to the solution of the classical Minkowski problem.

1. Introduction

The Minkowski problem concerns the existence, uniqueness, and stability of convex hypersurfaces whose Gauss curvature (possibly in a generalized sense) is prescribed as a function of the outer unit normals. The Minkowski problem is one of the centerpieces of the classical Brunn-Minkowski theory. The complete solution to the Minkowski problem (for arbitrary "data" — with no smoothness assumptions) goes back exactly three-quarters of a century to the work of Aleksandrov and Fenchel and Jessen (see Schneider [59] for references). Of course, when the Minkowski "data" is discrete then the solution goes back to Minkowski's work at the turn of the 19th into the 20th Century.

Almost a century after Minkowski's original work, an L_p version of the Brunn-Minkowski theory began to emerge, beginning largely with [33, 34], and expanding rapidly thereafter (see e.g. [1–5, 7–10, 12, 13, 15–22, 25–39, 41–44, 47, 49, 50, 52–55, 58, 60–63, 65–68]). The L_p Minkowski problem is also of central importance in this new L_p-Brunn-Minkowski theory.

For a given continuous function $g : S^{n-1} \to \mathbb{R}$ (called the "data"), the *regular L_p Minkowski problem* seeks solutions $h : S^{n-1} \to (0, \infty)$ to the

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partial differential equation

$$h^{1-p}\,\det(h_{ij}+h\delta_{ij})=g \qquad \text{on } S^{n-1}, \tag{1}$$

where (h_{ij}) is the Hessian matrix of h with respect to an orthonormal frame on S^{n-1}. The case $p=1$ is classical with landmark contributions such as Cheng and Yau [6] and Pogorelov [56].

The *general L_p Minkowski problem* asks: Given a real p, what are the necessary and sufficient conditions on a Borel measure μ on the unit sphere, S^{n-1}, such that there exists a convex body K in $\mathbb{R}^n$ with support function h_K and surface area measure S_K (see §1 for definitions) so that

$$h_K^{1-p}\,dS_K=d\mu. \tag{2}$$

The solutions to the general L_p Minkowski problem (2) may be viewed as weak solutions to the regular L_p Minkowski problem (1). While regularity is a central focus in the fields of partial differential equations and differential geometry, the existence and the uniqueness of weak solutions of the Minkowski problem are of paramount interest to those working in convex geometric and functional analysis. This is because it is precisely these solutions that correspond to support functions of polytopes or to Banach norms (which often have limited smoothness).

The *even L_p Minkowski problem* seeks solutions under the assumption that the data measure μ is an even Borel measure on S^{n-1} (i.e. assumes the same values on antipodal Borel sets) or the data function g is an even function (i.e. assumes the same values on antipodal points of S^{n-1}). The L_p Minkowski problem with even data is not only natural but of practical interest because its solutions correspond to norms of Banach spaces.

In this paper the "L_p Minkowski problem" will always refer to the general L_p Minkowski problem (2). The case $p=1$ of the L_p Minkowski problem (2) is of course the classical Minkowski problem. For $p>1$, a solution to the even L_p Minkowski problem was given in [33] under the assumption that $p\neq n$. In [40], it was shown that, for $p\neq n$, the L_p Minkowski problem (2) has an equivalent volume-normalized formulation and a solution of the even volume-normalized L_p Minkowski problem was given for all $p>1$. The regular even L_p Minkowski problem was studied in [35].

In the plane ($n=2$), the L_p Minkowski problem was treated by Stancu [61, 62, 63], Umanskyi [65], Chen [5], and most recently by Jiang [24].

The L_p Minkowski problem (without the assumption that the data is even) was treated by Guan and Lin [15] and later by Chou and Wang [7].

Hug et al [23] gave an alternate approach to some of the results of Chou and Wang [7].

The solution of the even L_p Minkowski problem was a critical ingredient that allowed the authors of [39] to extend the affine Sobolev inequality [69] and obtain the L_p affine Sobolev inequality and later enabled Cianchi et al [8] to establish the affine Moser-Trudinger and the affine Morrey-Sobolev inequalities.

Recent work of Haberl and Schuster [19], [20], Haberl, Schuster, and Xiao [21], Ludwig and Reitzner [32], and Ludwig [31], shows the need to take the next step in the evolution of the Brunn-Minkowski theory towards an Orlicz Brunn-Minkowski theory. This is the third paper in a series ([45], [46]) that attempts to develop some of the elements of an Orlicz Brunn-Minkowski theory.

Suppose $\varphi : (0, \infty) \to (0, \infty)$ is a fixed continuous function. The aim of this paper is to study the even L_φ Minkowski problem: If μ is an even finite Borel measure on S^{n-1} which is not concentrated on a great subsphere of S^{n-1}, then does there exist an origin symmetric convex body K in $\mathbb{R}^n$ such that

$$c\varphi(h_K)\ dS_K = d\mu,$$

for some positive number c? We will show that under some mild assumptions on φ, the even L_φ Minkowski problem does have a solution. For example we shall establish:

Theorem. *Suppose $\varphi : (0, \infty) \to (0, \infty)$ is a continuous decreasing function. If μ is an even finite Borel measure on S^{n-1} which is not concentrated on a great subsphere of S^{n-1}, then there exists an origin symmetric convex body K in $\mathbb{R}^n$ such that*

$$c\varphi(h_K)\ dS_K = d\mu,$$

where c is a power of the volume of K — specifically, $c = V(K)^{\frac{1}{2n}-1}$.

When φ and the data μ are assumed to be sufficiently smooth, and under certain other restrictions, the existence of solutions to the L_φ Minkowski problem was already demonstrated by Chou and Wang [7]. For applications in analysis (such as in obtaining analytic inequalities of the type presented in [69] and [39]) and for applications in geometry (such as in obtaining affine isoperimetric inequalities), various Banach norms and convex bodies must be constructed. These constructions amount to solving an L_φ Minkowski problem — but usually with minimal restrictions on φ and the measure μ. (Note that a polytopal solution of the L_φ Minkowski problem corresponds to a measure μ whose support is a finite set.) In this paper we will establish the existence of solutions to the L_φ Minkowski problem with such minimal restrictions on φ and μ.

One interesting feature of our work is that it presents a new approach to the classical Minkowski problem as well as the even L_p Minkowski problem for $p > 0$.

2. Preliminaries

For quick later reference we develop some notation and basic facts about convex bodies. Good general references for the theory of convex bodies are provided by the books of Gardner [11], Gruber [14], Schneider [59], and Thompson [64].

Our setting will be Euclidean n-space $\mathbb{R}^n$ where $n \geq 2$. The standard inner product of the vectors $x, y \in \mathbb{R}^n$ is denoted by $x \cdot y$. We write $S^{n-1} = \{x \in \mathbb{R}^n : \ x \cdot x = 1\}$ for the boundary of the Euclidean unit ball B in $\mathbb{R}^n$.

The set of continuous functions on the sphere S^{n-1} will be denoted by $C(S^{n-1})$ and will always be viewed as equipped with the max-norm metric:

$$\|f - g\|_\infty = \max_{u \in S^{n-1}} |f(u) - g(u)|,$$

for $f, g \in C(S^{n-1})$. The subspace of positive continuous functions will be denoted by $C^+(S^{n-1})$ and $C^+_e(S^{n-1})$ will denote the subspace of $C^+(S^{n-1})$ consisting of only the even functions.

Write V for n-dimensional Lebesgue measure and $\mathcal{H}^{n-1}$ for $(n-1)$-dimensional Hausdorff measure. The letter μ will be used exclusively to denote a finite Borel measure on S^{n-1}. For such a measure μ, we denote by $|\mu|$ its total mass, i.e. $|\mu| = \mu(S^{n-1})$. The letter c (possibly with subscripts or other distinguishing features) will be used exclusively to denote a positive real number.

A *convex body* is a compact convex subset of $\mathbb{R}^n$ with non-empty interior. The set of convex bodies in $\mathbb{R}^n$ containing the origin in their interiors is denoted by $\mathcal{K}^n_o$. The set of convex bodies in $\mathbb{R}^n$ that are symmetric about the origin will be denoted by $\mathcal{K}^n_e$.

A compact, convex set $K \subset \mathbb{R}^n$ is uniquely determined by its *support function* $h_K : \mathbb{R}^n \to \mathbb{R}$, where $h_K(x) = \max\{x \cdot y : \ y \in K\}$, for each $x \in \mathbb{R}^n$. For example, the support function of the line segment $\bar{v}$ joining the points $\pm v \in \mathbb{R}^n$ is given by

$$h_{\bar{v}}(x) = |x \cdot v|, \qquad x \in \mathbb{R}^n.$$

We will need the trivial fact that for the support function of the dilate $cK = \{cx : x \in K\}$ of a compact, convex K we have

$$h_{cK} = ch_K, \qquad c > 0. \tag{3}$$

Note that support functions are positively homogeneous of degree 1 and subadditive. It follows immediately from the definition of support functions that for compact, convex $K, L \subset \mathbb{R}^n$,

$$K \subseteq L \quad \Longleftrightarrow \quad h_K \le h_L. \tag{4}$$

Consequently, the support function of a body $K \in \mathcal{K}_o^n$ is bounded from above and below by positive reals.

The set $\mathcal{K}_o^n$ will be viewed as equipped with the Hausdorff metric and thus for a sequence $\{K_i\}$ of bodies in $\mathcal{K}_o^n$ and a body $K \in \mathcal{K}_o^n$, we have $\lim_{i\to\infty} K_i = K$ provided that

$$\|h_{K_i} - h_K\|_\infty \to 0.$$

A boundary point $x \in \partial K$ is said to have $u \in S^{n-1}$ as an outer normal provided $x \cdot u = h_K(u)$. A boundary point is said to be *singular* if it has more than one unit normal vector. It is well known (see, e.g., [59]) that the set of singular boundary points of a convex body has $\mathcal{H}^{n-1}$-measure equal to 0.

For a convex body K and each Borel set $\omega \subset S^{n-1}$, the *inverse spherical image, $\tau(K,\omega)$, of K at ω* is the set of all boundary points of K which have an outer unit normal belonging to the set ω. Associated with each convex body $K \in \mathcal{K}_o^n$ is a Borel measure, S_K, on S^{n-1} called the Aleksandrov-Fenchel-Jessen *surface area measure* of K, defined by

$$S_K(\omega) = \mathcal{H}^{n-1}(\tau(K,\omega)), \tag{5}$$

for each Borel set $\omega \subseteq S^{n-1}$. Observe that for the surface area measure of the dilate cK of K we have

$$S_{cK} = c^{n-1} S_K, \qquad c > 0. \tag{6}$$

We will use the weak continuity of surface area measures; i.e., if $\{K_i\}$ is a sequence of bodies in $\mathcal{K}_o^n$ then

$$\lim_{i\to\infty} K_i = K \in \mathcal{K}_o^n \quad \Longrightarrow \quad \lim_{i\to\infty} S_{K_i} = S_K, \text{ weakly.} \tag{7}$$

The *mixed volume* $V_1(K,L)$ of the convex bodies $K, L \in \mathcal{K}_o^n$ may be defined by

$$V_1(K,L) = \frac{1}{n}\int_{S^{n-1}} h_L \, dS_K. \tag{8}$$

The fact that

$$V(K) = V_1(K,K),$$

or equivalently,

$$V(K) = \frac{1}{n}\int_{S^{n-1}} h_K \, dS_K, \tag{9}$$

is of critical importance. The fundamental inequality for mixed volumes is *Minkowski's mixed volume inequality*: For $K, L \in \mathcal{K}_o^n$,

$$V_1(K,L)^n \geq V(K)^{n-1}V(L) \tag{10}$$

with equality if and only if K and L are homothetic.

3. Aleksandrov bodies

A function $h \in C^+(S^{n-1})$ defines a family $\{H_u\}_{u\in S^{n-1}}$ of hyperplanes

$$H_u = \{x \in \mathbb{R}^n : \; x \cdot u = h(u)\}.$$

This family gives rise to concepts such as envelopes in classical differential geometry, generalized envelopes in convex geometric analysis (see, e.g., [51]), and hedgehogs (see, e.g., [48]).

We shall be interested in the intersection of the halfspaces that are associated to h by the family $\{H_u\}_{u\in S^{n-1}}$. This gives rise to the convex body

$$K = \bigcap_{u\in S^{n-1}} \{x \in \mathbb{R}^n : \; x \cdot u \leq h(u)\}.$$

The body K is called the *Aleksandrov body associated with h*. Note that since h is both positive and continuous its Aleksandrov body, K, must be an element of $\mathcal{K}_o^n$. The Aleksandrov body associated with h can alternatively be defined as the unique maximal element, with respect to set inclusion, of the set

$$\{Q \in \mathcal{K}_o^n : \; h_Q \leq h\}.$$

For the Aleksandrov body K associated with h we now examine equality (in S^{n-1}) in the inequality

$$h_K \leq h$$

in some detail. Aleksandrov showed that each element of $\tau(K,\omega_h)$, the inverse spherical image of K of the set

$$\omega_h = \{u \in S^{n-1} : \ h_K(u) < h(u)\},$$

must be a singular boundary point of K. Since the set of singular boundary points of a convex body has $\mathcal{H}^{n-1}$-measure zero, and since by (5), we know that $S_K(\omega_h) = \mathcal{H}^{n-1}(\tau(K,\omega_h))$, it follows that $S_K(\omega_h) = 0$. Consequently, while $h_K \le h$, in fact

$$h_K = h, \qquad \text{a.e. with respect to } S_K. \tag{11}$$

We will make use of the following three basic properties of Aleksandrov bodies. First, if h is the support function of a convex body $K \in \mathcal{K}_o^n$, then K itself is the Aleksandrov body associated with h. Second, as is easily shown, if h is an even function, then the Aleksandrov body associated with h is origin-symmetric. Third and critical, is *Aleksandrov's convergence lemma* (see, e.g., [59, Lemma 6.5.2]): If the functions $h_i \in C^+(S^{n-1})$ have associated Aleksandrov bodies $K_i \in \mathcal{K}_o^n$, then

$$h_i \to h \in C^+(S^{n-1}) \qquad \Longrightarrow \qquad K_i \to K,$$

where K is the Aleksandrov body associated with h.

The volume $V(h)$ of a function $h \in C^+(S^{n-1})$ is defined as the volume of the Aleksandrov body associated with h. Since the Aleksandrov body associated with the support function h_K of a convex body $K \in \mathcal{K}_o^n$ is the body K itself, we have

$$V(h_K) = V(K). \tag{12}$$

From Aleksandrov's convergence lemma and the continuity of ordinary volume on $\mathcal{K}_o^n$ we see that

$$V : C^+(S^{n-1}) \to \mathbb{R} \quad \text{is continuous.}$$

Let $I \subset \mathbb{R}$ be an interval containing 0 and suppose that $h_t(u) = h(t,u) : I \times S^{n-1} \to (0,\infty)$ is continuous. For fixed $t \in I$, let

$$K_t = \bigcap_{u \in S^{n-1}} \{x \in \mathbb{R}^n : \ x \cdot u \le h(t,u)\}$$

be the Aleksandrov body associated with h_t. The family of bodies $\{K_t\}_{t\in I}$ will be called *the family of Aleksandrov bodies associated with* h. Obviously, from (11) we have

$$h_{K_t} \le h_t \qquad \text{and} \qquad h_{K_t} = h_t, \text{ a.e. with respect to } S_{K_t}. \tag{13}$$

for each $t \in I$.

Lemma 1. *Let $I \subset \mathbb{R}$ be an interval containing 0 and some positive number and let $h(t,u) : I \times S^{n-1} \to (0,\infty)$ be continuous and such that the convergence in*

$$h'_+(0,u) = \lim_{t\to 0^+} \frac{h(t,u) - h(0,u)}{t} \tag{14}$$

is uniform on S^{n-1}. If $\{K_t\}_{t\in I}$ is the family of Aleksandrov bodies associated with h, then

$$\lim_{t\to 0^+} \frac{V(K_t) - V(K_0)}{t} = \int_{S^{n-1}} h'_+(0,u)\, dS_{K_0}(u).$$

Proof. The uniform convergence of (14) implies that $h_t \to h_0$, uniformly on S^{n-1}. Therefore, the Aleksandrov convergence lemma (mentioned above) yields

$$\lim_{t\to 0^+} K_t = K_0. \tag{15}$$

By (7) we conclude that the S_{K_t} converge weakly to S_{K_0} as $t \to 0$. Since the measures S_{K_t} are finite, converge weakly to S_{K_0} and since the convergence in

$$\lim_{t\to 0^+} \frac{h(t,u) - h(0,u)}{t}$$

is uniform on S^{n-1}, we obtain

$$\lim_{t\to 0^+} \int_{S^{n-1}} \frac{h_t(u) - h_0(u)}{t}\, dS_{K_t}(u) = \int_{S^{n-1}} h'_+(u,0)\, dS_{K_0}(u). \tag{16}$$

Formulas (9) and (13) imply

$$V(K_t) = \frac{1}{n} \int_{S^{n-1}} h_{K_t}(u)\, dS_{K_t}(u) = \frac{1}{n} \int_{S^{n-1}} h_t(u)\, dS_{K_t}(u). \tag{17}$$

From (17), (8), and inequality (13) at $t = 0$, we have

$$\begin{aligned} \liminf_{t\to 0^+} \frac{V(K_t) - V_1(K_t,K_0)}{t} &= \frac{1}{n} \liminf_{t\to 0^+} \int_{S^{n-1}} \frac{h_t(u) - h_{K_0}(u)}{t}\, dS_{K_t}(u) \\ &\geq \frac{1}{n} \liminf_{t\to 0^+} \int_{S^{n-1}} \frac{h_t(u) - h_0(u)}{t}\, dS_{K_t}(u), \end{aligned}$$

which when combined with (16) gives

$$\liminf_{t\to 0^+} \frac{V(K_t) - V_1(K_t,K_0)}{t} \geq \frac{1}{n} \int_{S^{n-1}} h'_+(u,0)\, dS_{K_0}(u). \tag{18}$$

For the sake of brevity set

$$l = \frac{1}{n} \int_{S^{n-1}} h'_+(u,0)\, dS_{K_0}(u).$$

Inequality (18) and Minkowski's mixed volume inequality (10) show

$$l \le \liminf_{t\to 0^+} \frac{V(K_t) - V_1(K_t, K_0)}{t} \le \liminf_{t\to 0^+} \frac{V(K_t) - V(K_t)^{1-\frac{1}{n}} V(K_0)^{\frac{1}{n}}}{t}.$$

But (15) gives $\lim_{t\to 0^+} V(K_t) = V(K_0)$ and hence

$$l \le V(K_0)^{1-\frac{1}{n}} \liminf_{t\to 0^+} \frac{V(K_t)^{\frac{1}{n}} - V(K_0)^{\frac{1}{n}}}{t}. \tag{19}$$

Now (8), the inequality in (13), and the uniform convergence in (14) give

$$\begin{aligned}
\limsup_{t\to 0^+} \frac{V_1(K_0, K_t) - V(K_0)}{t} &= \frac{1}{n} \limsup_{t\to 0^+} \int_{S^{n-1}} \frac{h_{K_t}(u) - h_0(u)}{t}\, dS_{K_0}(u) \\
&\le \frac{1}{n} \limsup_{t\to 0^+} \int_{S^{n-1}} \frac{h_t(u) - h_0(u)}{t}\, dS_{K_0}(u) \\
&= \frac{1}{n} \int_{S^{n-1}} h'_+(u,0)\, dS_{K_0}(u) \\
&= l.
\end{aligned}$$

This, together with Minkowski's mixed volume inequality (10), yields

$$l \ge \limsup_{t\to 0^+} \frac{V_1(K_0, K_t) - V(K_0)}{t} \ge \limsup_{t\to 0^+} \frac{V(K_0)^{1-\frac{1}{n}} V(K_t)^{\frac{1}{n}} - V(K_0)}{t},$$

and hence

$$l \ge V(K_0)^{1-\frac{1}{n}} \limsup_{t\to 0^+} \frac{V(K_t)^{\frac{1}{n}} - V(K_0)^{\frac{1}{n}}}{t}. \tag{20}$$

Combining (19) and (20) gives

$$l = V(K_0)^{1-\frac{1}{n}} \lim_{t\to 0^+} \frac{V(K_t)^{\frac{1}{n}} - V(K_0)^{\frac{1}{n}}}{t}. \tag{21}$$

Define a function $g : I \to \mathbb{R}$ by $g(t) = V(K_t)^{\frac{1}{n}}$. Identity (21) shows that the right derivative of g exists at 0. But this implies that the right derivative of g^n exists at 0 and that

$$\lim_{t\to 0^+} \frac{g(t)^n - g(0)^n}{t} = n g(0)^{n-1} \lim_{t\to 0^+} \frac{g(t) - g(0)}{t}.$$

Thus the definition of g and (21) show that

$$\lim_{t\to 0^+} \frac{V(K_t) - V(K_0)}{t} = nl,$$

which completes the proof of the lemma. $\square$

We shall require the following corollary of Lemma 1.

Corollary 1. *Let $I \subset \mathbb{R}$ be an interval containing 0 in its interior and let $h(t,u) : I \times S^{n-1} \to (0,\infty)$ be continuous such that the convergence in*

$$h'(0,u) = \lim_{t\to 0} \frac{h(t,u) - h(0,u)}{t}$$

is uniform on S^{n-1}. If $\{K_t\}_{t\in I}$ is the family of Aleksandrov bodies associated with h, then

$$\lim_{t\to 0} \frac{V(K_t) - V(K_0)}{t} = \int_{S^{n-1}} h'(0,u)\, dS_{K_0}(u).$$

Proof. From Lemma 1 we see that we only need to show that

$$\lim_{t\to 0^-} \frac{V(K_t) - V(K_0)}{t} = \int_{S^{n-1}} h'(0,u)\, dS_{K_0}(u). \tag{22}$$

To that end, define $\tilde{h}(t,u) : -I \times S^{n-1} \to (0,\infty)$ by $\tilde{h}(t,u) = h(-t,u)$. For the corresponding family $\{\tilde{K}_{-t}\}_{t\in I}$ of Aleksandrov bodies associated with $\tilde{h}$ we have $\tilde{K}_{-t} = K_t$ and $\tilde{K}_0 = K_0$. Thus, by Lemma 1,

$$\lim_{t\to 0^-} \frac{V(K_t) - V(K_0)}{-t} = \lim_{t\to 0^+} \frac{V(\tilde{K}_t) - V(\tilde{K}_0)}{t} = \int_{S^{n-1}} \tilde{h}'(0,u)\, dS_{K_0}(u).$$

Obviously, $\tilde{h}'(0,u) = -h'(0,u)$, which immediately implies (22). $\square$

The following lemma is a slight variant of a standard result about differentiability under an integral sign. For the sake of completeness, we include a proof.

Lemma 2. *Let $\phi : (0,\infty) \to (0,\infty)$ be continuously differentiable, $I \subset \mathbb{R}$ be an open interval, and*

$$h : I \times S^{n-1} \to (0,\infty), \qquad (t,u) \mapsto h(t,u)$$

be a continuous function such that the partial derivative $\frac{\partial h}{\partial t}(t,u)$ exists for all $(t,u) \in I \times S^{n-1}$. If $\frac{\partial h}{\partial t}$ is bounded and h is bounded from above and from below by positive numbers, then the function $H : I \to (0,\infty)$ defined by

$$H(t) = \int_{S^{n-1}} (\phi \circ h)(t,u)\, d\mu(u),$$

for $t \in I$, is differentiable on I and

$$H'(t) = \int_{S^{n-1}} \frac{\partial(\phi \circ h)}{\partial t}(t,u)\, d\mu(u). \tag{23}$$

Moreover, if $\partial(\phi \circ h)/\partial t$ is continuous with respect to t, then H' is continuous.

Proof. Since ϕ', the derivative of ϕ, is assumed to be continuous and h is bounded from above and from below by positive numbers, there exists a $c_1 \in (0,\infty)$ such that

$$|(\phi' \circ h)(t,u)| \le c_1$$

for all $(t,u) \in I \times S^{n-1}$. This, the chain rule, and the assumption that $\frac{\partial h}{\partial t}$ is bounded shows that there exists a $c_2 \in (0,\infty)$ such that

$$\left|\frac{\partial(\phi \circ h)}{\partial t}(t,u)\right| = \left|\left((\phi' \circ h)\cdot \frac{\partial h}{\partial t}\right)(t,u)\right| \le c_2, \tag{24}$$

for all $(t,u) \in I \times S^{n-1}$.

Let $\{t_i\}$ be a sequence such that $\lim_{i\to\infty} t_i = t \in I$ with $t_i \ne t$ for all i. Set

$$f_i = \frac{(\phi \circ h)(t_i,\cdot) - (\phi \circ h)(t,\cdot)}{t_i - t}.$$

The mean value theorem shows that for each i and $u \in S^{n-1}$ there exists a $t_i' = t_i'(u) \in I$ such that

$$|f_i(u)| = \left|\frac{\partial(\phi \circ h)}{\partial t}(t_i',u)\right| \le c_2,$$

where the last inequality follows from (24). Since the measure μ is assumed to be finite, we have shown that all the $|f_i|$ are bounded from above by the same integrable function. The dominated convergence theorem applied to the sequence $\{f_i\}$ gives the differentiability of H and (23).

The last assertion of the lemma is again an immediate consequence of (24) and the dominated convergence theorem. □

4. Orlicz norms

Throughout this section, let $\phi : [0,\infty) \to [0,\infty)$ be strictly increasing, continuously differentiable on $(0,\infty)$ with positive derivative, and satisfy $\lim_{t\to\infty} \phi(t) = \infty$. Note that under these assumptions, ϕ has an inverse $\phi^{-1} : \phi([0,\infty)) \to [0,\infty)$ which is continuously differentiable on $\phi((0,\infty))$.

Let μ be a finite Borel measure on the sphere S^{n-1}. For a continuous function $f : S^{n-1} \to [0,\infty)$, the *Orlicz norm* $\|f\|_\phi$ is defined by

$$\|f\|_\phi = \inf\left\{\lambda > 0 :\ \frac{1}{|\mu|}\int_{S^{n-1}} \phi\left(\frac{f}{\lambda}\right)\, d\mu \leq \phi(1)\right\}. \tag{25}$$

Observe that the Orlicz norm of a function depends not only on ϕ but also on μ although this is not reflected in our notation. Moreover, since ϕ may not be convex, the Orlicz norm defined above may not satisfy the triangle inequality. The usual L_p norm is obtained by taking $\phi(t) = t^p$. The reader is referred to [57] for background on Orlicz norms.

Observe that for continuous $f : S^{n-1} \to [0,\infty)$,

$$\|cf\|_\phi = c\|f\|_\phi, \qquad c > 0. \tag{26}$$

In particular we have

$$\|c\|_\phi = c, \qquad c > 0. \tag{27}$$

Moreover, it follows immediately from the monotonicity of ϕ that for continuous $f, g : S^{n-1} \to [0,\infty)$,

$$f \leq g \quad \Longrightarrow \quad \|f\|_\phi \leq \|g\|_\phi. \tag{28}$$

The following simple fact will prove useful.

Lemma 3. *Suppose μ is a finite Borel measure on S^{n-1} and the function $f : S^{n-1} \to [0,\infty)$ is continuous and such that $\mu(\{f \neq 0\}) > 0$. Then the Orlicz norm $\|f\|_\phi$ of f is positive and*

$$\|f\|_\phi = \lambda_0 \quad \Longleftrightarrow \quad \frac{1}{|\mu|}\int_{S^{n-1}} \phi\left(\frac{f}{\lambda_0}\right) d\mu = \phi(1).$$

Proof. Define a function $\psi : (0,\infty) \to [0,\infty)$, for $\lambda > 0$, by

$$\psi(\lambda) = \frac{1}{|\mu|}\int_{S^{n-1}} \phi\left(\frac{f}{\lambda}\right) d\mu.$$

Since ϕ is strictly increasing and $\mu(\{f \neq 0\}) > 0$, the function ψ is strictly decreasing. It therefore has an inverse $\psi^{-1} : \psi((0,\infty)) \to (0,\infty)$. The dominated convergence theorem and the continuity of ϕ on $(0, \infty)$, show that ψ is continuous, as well.

The non-negativity of ϕ, Fatou's lemma, and the fact that $\lim_{t\to\infty} \phi(t) = \infty$ give

$$\begin{aligned}
\liminf_{\lambda\to 0} \psi(\lambda) &\geq \liminf_{\lambda\to 0} \frac{1}{|\mu|} \int_{\{f>0\}} \phi\left(\frac{f}{\lambda}\right) d\mu \\
&\geq \frac{1}{|\mu|} \int_{\{f>0\}} \liminf_{\lambda\to 0} \phi\left(\frac{f}{\lambda}\right) d\mu \\
&= \infty,
\end{aligned}$$

and hence

$$\lim_{\lambda\to 0} \psi(\lambda) = \infty. \tag{29}$$

Next, we show that

$$\phi(1) \in \psi((0, \infty)). \tag{30}$$

Since f is continuous on S^{n-1}, there exists a $c \in (0, \infty)$ with $f(u) < c$ for every $u \in S^{n-1}$. Thus, the monotonicity of ϕ, yields

$$\psi(\lambda) = \frac{1}{|\mu|} \int_{S^{n-1}} \phi\left(\frac{f}{\lambda}\right) d\mu < \frac{1}{|\mu|} \int_{S^{n-1}} \phi\left(\frac{c}{\lambda}\right) d\mu = \phi(c/\lambda),$$

for every positive λ. In particular, $\psi(2c) < \phi(1/2) < \phi(1)$. From (29) and the continuity of ψ we therefore deduce (30) by the intermediate value theorem.

Finally (30) and the strict monotonicity of ψ yield

$$\|f\|_\phi = \inf\{\lambda > 0 : \ \psi(\lambda) \leq \phi(1)\} = \psi^{-1}(\phi(1)).$$

This shows that $\|f\|_\phi > 0$ and establishes the desired equivalence. $\square$

The continuity of the Orlicz norm is contained in the following lemma.

Lemma 4. *Suppose $f \in C(S^{n-1})$ with $\mu(\{f \neq 0\}) > 0$ and $\{f_i\}$ is a sequence of nonnegative functions in $C(S^{n-1})$. If*

$$f_i \to f \ \textit{in} \ C(S^{n-1}),$$

then

$$\|f_i\|_\phi \to \|f\|_\phi.$$

Proof. From the uniform convergence, it follows that there exists a real $c > 0$ such that $f_i(u) \le c$ for all i and all $u \in S^{n-1}$. From (27) and (28) we thus obtain

$$0 \le \|f_i\|_\phi \le \|c\|_\phi = c$$

for every i. Thus the sequence $\{\|f_i\|_\phi\}$ is bounded. To show that the sequence converges to $\|f\|_\phi$, we prove that every convergent subsequence converges to $\|f\|_\phi$. Denote an arbitrary convergent subsequence of $\{\|f_i\|_\phi\}$ by $\{\|f_i\|_\phi\}$ as well.

To see that $\lim_{i\to\infty} \|f_i\|_\phi > 0$, suppose the contrary; i.e., $\|f_i\|_\phi \to 0$. Then Lemma 3, the non-negativity of ϕ, Fatou's lemma, and the fact that $\lim_{t\to\infty} \phi(t) = \infty$ would produce the desired contradiction:

$$\begin{aligned} \phi(1) &= \lim_{i\to\infty} \frac{1}{|\mu|} \int_{S^{n-1}} \phi\left(\frac{f_i}{\|f_i\|_\phi}\right) d\mu \\ &\ge \liminf_{i\to\infty} \frac{1}{|\mu|} \int_{\{f>0\}} \phi\left(\frac{f_i}{\|f_i\|_\phi}\right) d\mu \\ &\ge \frac{1}{|\mu|} \int_{\{f>0\}} \liminf_{i\to\infty} \phi\left(\frac{f_i}{\|f_i\|_\phi}\right) d\mu \\ &= \infty. \end{aligned}$$

Thus $\lim_{i\to\infty} \|f_i\|_\phi > 0$. Note that since $\mu(\{f \ne 0\}) > 0$ and $f_i \to f$ in $C(S^{n-1})$ we have $\mu(\{f_i \ne 0\}) > 0$ for sufficiently large i. From the continuity of ϕ and (33) we therefore deduce

$$\frac{1}{|\mu|} \int_{S^{n-1}} \phi\left(\frac{f}{\lim_{i\to\infty} \|f_i\|_\phi}\right) d\mu = \lim_{i\to\infty} \frac{1}{|\mu|} \int_{S^{n-1}} \phi\left(\frac{f_i}{\|f_i\|_\phi}\right) d\mu = \phi(1).$$

Lemma 3 again shows $\lim_{i\to\infty} \|f_i\|_\phi = \|f\|_\phi$. □

We will need the following basic fact.

Lemma 5. *If μ is a finite Borel measure on the sphere S^{n-1} which is not concentrated on a great subsphere of S^{n-1}, then there exists a real $c > 0$ such that $\|h_{\bar{v}}\|_\phi > c$ for every $v \in S^{n-1}$.*

Proof. Note that since μ is not concentrated on a great subsphere of S^{n-1}, we have, for every $v \in S^{n-1}$,

$$\mu(\{h_{\bar{v}} > 0\}) = \mu(S^{n-1} \backslash v^\perp) > 0,$$

where $v^\perp$ denotes the codimension 1 subspace orthogonal to v. Hence Lemma 3 shows that $\|h_{\bar{v}}\|_\phi > 0$ for every unit vector v. In order to establish the assertion of the lemma, it suffices to show that the function $v \mapsto \|h_{\bar{v}}\|_\phi$ is continuous.

Suppose $v_i \in S^{n-1}$ and $v_i \to v$. That $h_{\bar{v}_i} \to h_{\bar{v}}$, uniformly on S^{n-1} is easily seen. Thus $\|h_{\bar{v}_i}\|_\phi$ converges to $\|h_{\bar{v}}\|_\phi$ by Lemma 4. This establishes the desired continuity of $v \mapsto \|h_{\bar{v}}\|_\phi$. □

For a continuous $f : S^{n-1} \to [0,\infty)$ with $\|f\|_\phi > 0$, define

$$\langle f \rangle = \frac{f}{\|f\|_\phi}. \tag{31}$$

From (26) we see that we always have

$$\|\langle f \rangle\|_\phi = 1.$$

We will use the fact that $\langle h_K \rangle$ is homogeneous of degree 0 with respect to dilation of K. Indeed, from the definition of $\langle \cdot \rangle$, together with (3) and (26), we see that for every $c > 0$,

$$\langle h_K \rangle = \frac{h_K}{\|h_K\|_\phi} = \frac{h_{cK}}{\|h_{cK}\|_\phi} = \langle h_{cK} \rangle. \tag{32}$$

Lemma 3 shows that if μ is such that $\mu(\{f > 0\}) > 0$, then

$$\frac{1}{|\mu|} \int_{S^{n-1}} \phi \circ \langle f \rangle \, d\mu = \phi(1). \tag{33}$$

5. The even Orlicz Minkowski problem

The first of the two main results shows existence for the even Orlicz Minkowski problem under some mild assumptions regarding φ.

Theorem 1. *Suppose $\varphi : (0,\infty) \to (0,\infty)$ is a continuous decreasing function. If μ is an even finite Borel measure on S^{n-1} which is not concentrated on a great subsphere of S^{n-1}, then for each $\alpha \in (0,1)$ there exists an origin symmetric convex body K in $\mathbb{R}^n$ such that*

$$c\varphi(h_K)\, dS_K = d\mu,$$

with $c = V(K)^{\frac{\alpha}{n}-1}$.

Note that for $\varphi \equiv 1$, Theorem 1 provides a solution to the even Minkowski problem — by appealing to (6).

Proof. Define the function $\phi : [0, \infty) \to [0, \infty)$ by

$$\phi(t) = \int_0^t \frac{1}{\varphi(s)}\, ds.$$

Observe that since φ is decreasing and $0 < \alpha < 1$, it follows that for $c > 0$

$$\lim_{t\to\infty} \phi(ct)/t^\alpha = \infty, \tag{34}$$

and

$$\lim_{t\to 0^+} \phi(t)/t^\alpha = 0. \tag{35}$$

Since φ is decreasing, the derivative of ϕ is increasing and therefore the function ϕ is convex.

Furthermore, for $c > 0$ let $\psi_c : (0, \infty) \to \mathbb{R}$ be defined by

$$\psi_c(t) = \left(\tfrac{n}{\alpha} V(B)^{\frac{\alpha}{n}} - \frac{|\mu|\phi(ct)}{t^\alpha} \right) t^\alpha.$$

From (34) we conclude that $\lim_{t\to\infty} \psi_c(t) = -\infty$. In particular, for each c there exists a real $r_c > 0$ such that

$$t > r_c \Longrightarrow \psi_c(t) < 0. \tag{36}$$

Motivated by the work of Chou and Wang [7], we define the functional $\Phi : C_e^+(S^{n-1}) \to \mathbb{R}$ by

$$\Phi(f) = \tfrac{n}{\alpha} V(f)^{\frac{\alpha}{n}} - \int_{S^{n-1}} \phi \circ f \, d\mu,$$

for $f \in C_e^+(S^{n-1})$. Since, as seen in §3, the volume $V : C^+(S^{n-1}) \to \mathbb{R}$ is continuous, with respect to the max-norm metric, Φ is continuous as well.

Note that by (12), for every $r > 0$,

$$\Phi(h_{rB}) = \tfrac{n}{\alpha} r^\alpha V(B)^{\frac{\alpha}{n}} - \phi(r)|\mu| = \left(\tfrac{n}{\alpha} V(B)^{\frac{\alpha}{n}} - \frac{|\mu|\phi(r)}{r^\alpha} \right) r^\alpha.$$

From this and (35) we see that $\Phi(h_{rB})$ is positive for small positive r. Hence

$$\text{there exists } K \in \mathcal{K}_e^n \text{ such that } \Phi(h_K) > 0. \tag{37}$$

We are searching for a function at which Φ attains a maximum. The search can be restricted to support functions of origin symmetric convex bodies. To see this, recall from §3 that the Aleksandrov body K associated with a given function $h \in C_e^+(S^{n-1})$ is origin symmetric and has a support function h_K which satisfies $0 < h_K \leq h$. Since ϕ is increasing and $V(h) = V(h_K)$ by (12) we deduce $\Phi(h) \leq \Phi(h_K)$.

Next, we will show that the search for a function at which Φ attains a maximum can be further restricted to support functions of origin symmetric convex bodies contained in some ball of fixed radius. To this end, first note that the continuous function on S^{n-1},

$$v \longmapsto \int_{S^{n-1}} h_{\bar{v}} \, d\mu,$$

is positive since μ is not concentrated on a great subsphere. Thus, there exists a $c \in (0, \infty)$ such that

$$\frac{1}{|\mu|} \int_{S^{n-1}} h_{\bar{v}} \, d\mu \ \geq \ c, \tag{38}$$

for every $v \in S^{n-1}$. Let $K \in \mathcal{K}_e^n$ and choose $v_K \in S^{n-1}$ such that for a suitable real $r_K > 0$ the point $r_K v_K$ is an element of K with maximal distance from the origin. Since K is origin symmetric, the line segment with endpoints $\pm r_K v_K$ is contained in K. From (3) and (4) we deduce $r_K h_{\bar{v}_K} \leq h_K$. The monotonicity of ϕ, Jensen's inequality, and (38) therefore yield

$$\begin{aligned} \int_{S^{n-1}} \phi(h_K) \, d\mu \ &\geq \ \int_{S^{n-1}} \phi(r_K h_{\bar{v}_K}) \, d\mu \\ &\geq \ |\mu| \phi\left(\frac{1}{|\mu|} \int_{S^{n-1}} r_K h_{\bar{v}_K} \, d\mu \right) \\ &\geq \ |\mu| \phi(c r_K). \end{aligned}$$

Now (12), the fact that $K \subset r_K B$, and the last inequality show that

$$\begin{aligned} \Phi(h_K) \ &= \ \tfrac{n}{\alpha} V(K)^{\frac{\alpha}{n}} - \int_{S^{n-1}} \phi(h_K) \, d\mu \\ &\leq \ \tfrac{n}{\alpha} r_K^\alpha V(B)^{\frac{\alpha}{n}} - |\mu| \phi(c r_K) \\ &= \ \left(\tfrac{n}{\alpha} V(B)^{\frac{\alpha}{n}} - \frac{|\mu| \phi(c r_K)}{r_K^\alpha} \right) r_K^\alpha \\ &= \ \psi_c(r_K). \end{aligned}$$

From (36) we therefore conclude that there exists a real $r = r_c > 0$ such that

$$r_K > r \Longrightarrow \Phi(h_K) < 0. \tag{39}$$

It follows from (37) and (39) that in order to find a maximum of the functional Φ on $C_e^+(S^{n-1})$, it is sufficient to search among support functions of members of the set

$$\mathcal{F} = \{K \in \mathcal{K}_e^n : K \subset rB\}.$$

Let $\{K_i\}$ be a maximizing sequence in $\mathcal{F}$ for Φ, i.e.

$$\lim_{i\to\infty} \Phi(h_{K_i}) = \sup\{\Phi(h_K) : K \in \mathcal{F}\}.$$

Obviously, the sequence $\{K_i\}$ is bounded. Blaschke's selection theorem (see, e.g., [59]) guarantees the existence of a convergent subsequence, which we also denote by $\{K_i\}$, with $\lim_{i\to\infty} K_i = K_0$. Since the $K_i \in \mathcal{K}_e^n$, the body K_0 is origin symmetric. Moreover, the continuity of volume, (12), and the positivity of $\lim_{i\to\infty} \Phi(h_{K_i})$ yield

$$\tfrac{n}{\alpha} V(K_0)^{\frac{\alpha}{n}} = \lim_{i\to\infty} \tfrac{n}{\alpha} V(K_i)^{\frac{\alpha}{n}} = \lim_{i\to\infty} \tfrac{n}{\alpha} V(h_{K_i})^{\frac{\alpha}{n}} \geq \lim_{i\to\infty} \Phi(h_{K_i}) > 0.$$

Consequently, the body K_0 has non-empty interior and thus $K_0 \in \mathcal{K}_e^n$. The continuity of Φ now shows that

$$\Phi(f) \leq \Phi(h_{K_0}),$$

for every $f \in C_e^+(S^{n-1})$.

Suppose $f \in C_e^+(S^{n-1})$ is arbitrary but fixed. For sufficiently small $\delta > 0$ we can define a function, $h : (-\delta, \delta) \times S^{n-1} \to (0, \infty)$, which is bounded from above and below by positive reals, by

$$h_t(u) := h(t, u) = h_{K_0}(u) + t f(u).$$

By Lemma 2 and Corollary 1, the function $t \mapsto \Phi \circ h_t$ is differentiable at 0. Since K_0 is a maximizer of the functional Φ and $h_0 = h_{K_0}$ we have

$$\frac{d}{dt}(\Phi \circ h_t)\bigg|_{t=0} = 0. \tag{40}$$

Note that by Corollary 1,

$$\frac{d}{dt} \tfrac{n}{\alpha} V(h_t)^{\frac{\alpha}{n}}\bigg|_{t=0} = V(K_0)^{\frac{\alpha}{n}-1} \int_{S^{n-1}} f \, dS_{K_0},$$

and by Lemma 2,

$$\frac{d}{dt}\int_{S^{n-1}}\phi\circ h_t\,d\mu\bigg|_{t=0}=\int_{S^{n-1}}\frac{1}{\varphi\circ h_{K_0}}f\,d\mu.$$

The definition of Φ together with (40) therefore give

$$V(K_0)^{\frac{\alpha}{n}-1}\int_{S^{n-1}}f\,dS_{K_0}-\int_{S^{n-1}}\frac{1}{\varphi\circ h_{K_0}}f\,d\mu=0.$$

Since this holds for all positive even continuous functions f on S^{n-1}, it holds for all even continuous f on S^{n-1} and hence

$$V(K_0)^{\frac{\alpha}{n}-1}\varphi(h_{K_0})\;dS_{K_0}=d\mu.$$

□

We now establish the following variant of our first theorem.

Theorem 2. *Suppose $\varphi:(0,\infty)\to(0,\infty)$ is a continuous function such that $\phi(t)=\int_0^t 1/\varphi(s)\,ds$ exists for every positive t and is unbounded as $t\to\infty$. If μ is an even finite Borel measure on S^{n-1} which is not concentrated on a great subsphere of S^{n-1}, then there exists an origin symmetric convex body K in $\mathbb{R}^n$ and a $c>0$ such that*

$$c\varphi(h_K)\;dS_K=d\mu, \tag{41}$$

and

$$\|h_K\|_\phi=1. \tag{42}$$

In order to establish this theorem we first establish four lemmas.

First extend the definition of ϕ to $[0,\infty)$ by

$$\phi(t)=\int_0^t\frac{1}{\varphi(s)}\,ds,\quad\text{for } t>0,\qquad\text{and}\qquad \phi(0)=\lim_{t\to0^+}\phi(t).$$

The function ϕ is strictly increasing and continuously differentiable on $(0,\infty)$, and $\phi'>0$. Thus ϕ has an inverse $\phi^{-1}:\phi([0,\infty))\to[0,\infty)$ which is continuously differentiable on $\phi((0,\infty))$. Observe that $\lim_{t\to\infty}\phi(t)=\infty$.

Suppose that $f\in C^+(S^{n-1})$ and $K\in\mathcal{K}_o^n$. Recall that the function ϕ is strictly increasing. There exists $\delta>0$ so that

$$\phi(1)-tf(u)\in\phi((\tfrac{1}{2},2))$$

for all $(t,u) \in (-\delta,\delta)\times S^{n-1}$. Define the function $\tilde{h} : (-\delta,\delta)\times S^{n-1} \to (0,\infty)$, at (t,u), by

$$\tilde{h}(t,u) = \frac{h_K(u)}{\phi^{-1}(\phi(1) - tf(u))}. \tag{43}$$

Clearly, $\tilde{h}$ is bounded from above and below by positive reals (dependent on f and K). From (27) and (28) we therefore have that also $\|\tilde{h}_t\|_\phi$ is bounded from above and below by positive reals.

The following lemma follows directly from the definitions.

Lemma 6. *The function $\tilde{h}$ is continuous and the partial derivative $\frac{\partial \tilde{h}}{\partial t}(t,u)$ exists on $(-\delta,\delta)\times S^{n-1}$ and is given by*

$$\frac{\partial \tilde{h}}{\partial t}(t,u) = \frac{h_K(u)f(u)(\phi^{-1})'(\phi(1) - tf(u))}{[\phi^{-1}(\phi(1) - tf(u))]^2}. \tag{44}$$

Moreover, it is continuous and bounded from above and below by positive reals.

We now establish:

Lemma 7. *The function $t \mapsto V(\tilde{h}_t)$ is differentiable at 0.*

Proof. By Lemma 6, the function $\tilde{h}$ is continuous. So in order to apply Corollary 1 it remains to show that the convergence in

$$\lim_{t\to 0} \frac{\tilde{h}_t(u) - \tilde{h}_0(u)}{t} \tag{45}$$

is uniform on S^{n-1}. Let $\delta' \in (0,\delta)$. By the mean value theorem, for every $t \in [-\delta',\delta']$ we can find a $t' \in [-t,t]$ such that

$$\left| \frac{\tilde{h}_t(u) - \tilde{h}_0(u)}{t} - \tilde{h}_0'(u) \right| = \left| \frac{\partial \tilde{h}}{\partial t}(t',u) - \frac{\partial \tilde{h}}{\partial t}(0,u) \right|.$$

Since $[-\delta',\delta']\times S^{n-1}$ is compact, Lemma 6 shows that the partial derivative $\frac{\partial \tilde{h}}{\partial t}(t,u)$ is uniformly continuous on $[-\delta',\delta']\times S^{n-1}$. Hence, the convergence in (45) is uniform on S^{n-1}. $\square$

Also needed will be:

Lemma 8. *The function $t \mapsto \|\tilde{h}_t\|_\phi$ is differentiable on $(-\delta,\delta)$ with bounded derivative.*

Proof. As previously mentioned (after the definition of $\tilde{h}$) there exist $c_1, c_2 \in (0, \infty)$ such that

$$c_1 < \tilde{h}_t(u) < c_2, \qquad \text{for all} \qquad (t, u) \in (-\delta, \delta) \times S^{n-1}.$$

Define the function $G : (-\delta, \delta) \times (c_1, c_2) \to \mathbb{R}$ at (t, λ) by

$$G(t, \lambda) = \frac{1}{|\mu|} \int_{S^{n-1}} \phi \left(\frac{\tilde{h}_t}{\lambda} \right) d\mu - \phi(1).$$

Furthermore, for fixed $\lambda \in (c_1, c_2)$, define $h_1 : (-\delta, \delta) \times S^{n-1} \to (0, \infty)$ by

$$h_1(t, u) = \tilde{h}(t, u)/\lambda,$$

and, for fixed $t \in (-\delta, \delta)$, define $h_2 : (c_1, c_2) \times S^{n-1} \to (0, \infty)$ by

$$h_2(\lambda, u) = \tilde{h}(t, u)/\lambda.$$

Clearly, the functions h_1 and h_2 are bounded from above and below by positive reals. Moreover, by Lemma 6, the derivatives $\frac{\partial h_1}{\partial t}$ and $\frac{\partial h_2}{\partial \lambda}$ exist, are continuous, and bounded on their domains. Thus, by applying Lemma 2 to h_1 and h_2 respectively, the partial derivatives $\partial G/\partial t$, $\partial G/\partial \lambda$ exist. Since $\partial(\phi \circ h_1)/\partial t$ and $\partial(\phi \circ h_2)/\partial \lambda$ are continuous by Lemma 6, another application of Lemma 2 shows that $\partial G/\partial t$ and $\partial G/\partial \lambda$ are in fact continuous. Since by Lemma 2 interchanging differentiation and integration are permitted, an elementary calculation shows that $\partial G/\partial t$ and $-\partial G/\partial \lambda$ are bounded from above and below by positive reals. In particular, $\partial G/\partial \lambda$ is always nonzero.

Let $t \in (-\delta, \delta)$. For sufficiently small ε, Lemma 3 and the mean value theorem give

$$\begin{aligned} 0 &= G(t + \varepsilon, \|\tilde{h}_{t+\varepsilon}\|_\phi) - G(t, \|\tilde{h}_t\|_\phi) \\ &= \varepsilon \frac{\partial G}{\partial t}(x_\varepsilon) + (\|\tilde{h}_{t+\varepsilon}\|_\phi - \|\tilde{h}_t\|_\phi) \frac{\partial G}{\partial \lambda}(x_\varepsilon), \end{aligned} \tag{46}$$

where x_ε is a point on the line segment joining the points $(t, \|\tilde{h}_t\|_\phi)$ and $(t + \varepsilon, \|\tilde{h}_{t+\varepsilon}\|_\phi)$.

Note that from the continuity of $\tilde{h}$, and compactness, follows the uniform continuity of $\tilde{h} : [-\delta', \delta'] \times S^{n-1}$ for arbitrary $0 < \delta' < \delta$. Thus the convergence in $\lim_{\varepsilon \to 0} \tilde{h}_{t+\varepsilon} = \tilde{h}_t$ is uniform and hence, by Lemma 4, $\lim_{\varepsilon \to 0} \|\tilde{h}_{t+\varepsilon}\|_\phi = \|\tilde{h}_t\|_\phi$. Hence $x_\varepsilon \to (t, \|\tilde{h}_t\|_\phi)$ as $\varepsilon \to 0$.

Since, as shown above, the partial derivatives of G are continuous and $\partial G/\partial\lambda$ is always nonzero, we conclude from (46) that $\|\tilde h_t\|_\phi$ is differentiable with

$$\frac{d\|\tilde h_t\|_\phi}{dt} = -\left[\frac{\partial G}{\partial t}(t, \|\tilde h_t\|_\phi)\right] \Big/ \left[\frac{\partial G}{\partial \lambda}(t, \|\tilde h_t\|_\phi)\right].$$

The fact that $\partial G/\partial t$ and $-\partial G/\partial\lambda$ are bounded from above and below by positive reals, finally shows that $d\|\tilde h_t\|_\phi/dt$ is bounded. $\square$

In the final lemma needed for the proof of Theorem 2, we establish some differentiability properties of functions involving $\tilde h$. Recall that $\langle \tilde h_t\rangle = \tilde h_t/\|\tilde h_t\|_\phi$.

Lemma 9. *The functions*

$$t \longmapsto \|\tilde h_t\|_\phi / V(\tilde h_t)^{1/n} \qquad \text{and} \qquad t \longmapsto \int_{S^{n-1}} \phi\circ\langle\tilde h_t\rangle \, d\mu$$

are differentiable at 0 with derivatives

$$\frac{1}{V(\tilde h_0)^{\frac{1}{n}}}\left(\frac{d}{dt}\|\tilde h_t\|_\phi\Big|_{t=0} - \frac{\|\tilde h_0\|_\phi}{nV(\tilde h_0)}\int_{S^{n-1}}\frac{\tilde h_0 f}{\phi'(1)}\, dS_K\right) \tag{47}$$

and

$$\frac{1}{\|\tilde h_0\|_\phi}\int_{S^{n-1}}\frac{1}{\varphi(\langle\tilde h_0\rangle)}\left(\frac{\tilde h_0 f}{\phi'(1)} - \langle\tilde h_0\rangle\frac{d}{dt}\|\tilde h_t\|_\phi\Big|_{t=0}\right) d\mu, \tag{48}$$

respectively.

Proof. First observe that $\tilde h_t$ is bounded from below by a positive real so its Alexandrov body contains the origin in its interior. The differentiability, at 0, of the function

$$t \longmapsto \|\tilde h_t\|_\phi/V(\tilde h_t)^{1/n}$$

is an immediate consequence of Lemmas 7 and 8. Formula (47) for the respective derivative follows directly from (44), Corollary 1, and

$$(\phi^{-1})'(\phi(1)) = \frac{1}{\phi'(1)}. \tag{49}$$

Since $\tilde h$ and $\|\tilde h_t\|_\phi$ are bounded from below and above by positive reals, so is $\langle\tilde h_t\rangle$. From Lemmas 6 and 8 we infer that $\langle\tilde h_t\rangle$ is differentiable with respect to t. An elementary calculation shows that

$$\frac{\partial\langle\tilde h_t\rangle}{\partial t} = \frac{1}{\|\tilde h_t\|_\phi}\left(\frac{d\tilde h_t}{dt} - \langle\tilde h_t\rangle\frac{d\|\tilde h_t\|_\phi}{dt}\right). \tag{50}$$

Since $\|\tilde{h}_t\|_\phi$ is bounded from below and above by positive reals and $\langle\tilde{h}_t\rangle$, $d\tilde{h}_t/dt$, $d\|\tilde{h}_t\|_\phi/dt$ are bounded, we therefore conclude that $\partial\langle\tilde{h}_t\rangle/\partial t$ is bounded as well. Lemma 2 yields the differentiability, at 0, of

$$t \longmapsto \int_{S^{n-1}} \phi \circ \langle\tilde{h}_t\rangle \, d\mu,$$

and together with (44), (49), and (50), the desired formula (48). $\square$

We are now in a position to establish Theorem 2.

Proof. For $f \in C_e^+(S^{n-1})$, define the functional $\Phi : C_e^+(S^{n-1}) \to \mathbb{R}$ by

$$\Phi(f) = \|f\|_\phi / V(f)^{1/n}.$$

Lemma 4, together with the fact that $V : C_e^+(S^{n-1}) \to \mathbb{R}$ is continuous, shows that Φ is continuous as well.

We are searching for a function at which Φ attains a minimum. As before, the search can be restricted to support functions of origin symmetric convex bodies: Indeed, recall that the Aleksandrov body K associated with a given function $h \in C_e^+(S^{n-1})$ is origin symmetric and has a support function h_K which satisfies $0 < h_K \le h$. The fact that $V(h) = V(h_K)$ together with (28) shows that $\Phi(h_K) \le \Phi(h)$.

Since Φ is positively homogeneous of degree 0, the search can be further restricted to support functions of convex bodies of unit volume. Let $c_1 = \Phi(h_{B'})$, where B' is the dilate of B chosen so that $V(B') = 1$. It follows that in order to find a minimum of the functional Φ on $C_e^+(S^{n-1})$, it is sufficient to search among the support functions of the members of the set

$$\mathcal{F} = \{K \in \mathcal{K}_e^n : \ \Phi(h_K) \le c_1 \ \text{ and } \ V(K) = 1\}.$$

Let $\{K_i\}$ be a minimizing sequence, of bodies in $\mathcal{F}$, for the functional Φ, i.e.,

$$\lim_{i\to\infty} \Phi(h_{K_i}) = \inf\{\Phi(h_K) : \ K \in \mathcal{F}\}.$$

We now show that the sequence $\{K_i\}$ is bounded. For each i, let $v_i \in S^{n-1}$ be chosen such that for suitable $r_i > 0$ the points $r_i v_i$ are elements of K_i with maximal distance from the origin. Since each K_i is origin symmetric, the segments with endpoints $\pm r_i v_i$ are contained in K_i. From (3) and (4) we deduce $r_i h_{\bar{v}_i} \le h_{K_i}$. Hence (26), (28), and the fact that the K_i belong to $\mathcal{F}$ imply

$$r_i \|h_{\bar{v}_i}\|_\phi = \|r_i h_{\bar{v}_i}\|_\phi \le \|h_{K_i}\|_\phi = \Phi(h_{K_i}) \le c_1. \tag{51}$$

By Lemma 5, there exists a $c_2 \in (0,\infty)$ such that

$$c_2 \le \|h_{\bar{v}_i}\|_\phi \tag{52}$$

for all i. Combining (51) and (52) we see that the r_i are bounded from above and hence the sequence $\{K_i\}$ is bounded.

Now Blaschke's selection theorem guarantees the existence of a convergent subsequence of $\{K_i\}$, which we also denote by $\{K_i\}$, with $\lim_{i\to\infty} K_i = K_0$. Clearly, the body K_0 is again an origin symmetric compact, convex set. Since obviously $V(K_0) = 1$, we see that $K_0 \in \mathcal{K}_e^n$. The continuity of Φ now implies that $K_0 \in \mathcal{F}$ and thus that

$$\Phi(h_{K_0}) \le \Phi(f)$$

for every $f \in C_e^+(S^{n-1})$.

Choose a fixed but arbitrary $f \in C_e^+(S^{n-1})$. As in (43), define the function $\tilde{h} : (-\delta,\delta) \times S^{n-1} \to (0,\infty)$ by

$$\tilde{h}(t,u) = \frac{h_{K_0}(u)}{\phi^{-1}(\phi(1) - tf(u))}.$$

Lemma 9 shows that the function $t \mapsto \Phi \circ \tilde{h}_t$ is differentiable at 0. Since h_{K_0} is a minimizer of the functional Φ and $\tilde{h}_0 = h_{K_0}$ we have

$$\frac{d}{dt}(\Phi \circ \tilde{h}_t)\bigg|_{t=0} = 0.$$

The expression for the above derivative given in (47) implies

$$\frac{d}{dt}\|\tilde{h}_t\|_\phi\bigg|_{t=0} = \frac{\|\tilde{h}_0\|_\phi}{nV(\tilde{h}_0)} \int_{S^{n-1}} \frac{\tilde{h}_0 f}{\phi'(1)}\, dS_{K_0}. \tag{53}$$

For each t such that $|t| < \delta$, we have from (33)

$$\frac{1}{|\mu|}\int_{S^{n-1}} \phi \circ \langle \tilde{h}_t \rangle \, d\mu = \phi(1).$$

Thus, the derivative (with respect to t) of the function on the left is 0. This fact, at $t = 0$, together with (48) now gives

$$\int_{S^{n-1}} \frac{1}{\varphi \circ \langle \tilde{h}_0 \rangle} \left(\frac{\tilde{h}_0 f}{\phi'(1)} - \langle \tilde{h}_0 \rangle \frac{d}{dt}\|\tilde{h}_t\|_\phi\bigg|_{t=0} \right) d\mu = 0.$$

In this substitute the value of the derivative given by (53), use (12), and the fact that $\tilde{h}_0 = h_{K_0}$, to get

$$\frac{1}{nV(K_0)} \int_{S^{n-1}} h_{K_0} f \, dS_{K_0} \int_{S^{n-1}} \frac{h_{K_0}}{\varphi \circ \langle h_{K_0} \rangle} \, d\mu = \int_{S^{n-1}} \frac{h_{K_0} f}{\varphi \circ \langle h_{K_0} \rangle} \, d\mu.$$

By (3), (6), (32), and the homogeneity of volume (of degree n), the last equation remains unchanged if we replace K_0 by a dilate of K_0. In particular, if we choose a dilate K_0' of K_0 such that

$$\|h_{K_0'}\|_\phi = 1,$$

then we obtain

$$c \int_{S^{n-1}} h_{K_0'} f \, dS_{K_0'} = \int_{S^{n-1}} \frac{h_{K_0'} f}{\varphi(h_{K_0'})} \, d\mu,$$

where

$$c = \frac{1}{nV(K_0')} \int_{S^{n-1}} \frac{h_{K_0'}}{\varphi(h_{K_0'})} d\mu.$$

Since this holds for all positive even continuous f on S^{n-1}, it holds for all even continuous f on S^{n-1}, and hence

$$c \, dS_{K_0'} = \frac{1}{\varphi(h_{K_0'})} d\mu,$$

or equivalently

$$c \, \varphi(h_{K_0'}) \, dS_{K_0'} = d\mu.$$

□

From Theorem 2, we obtain the solution to the even L_p Minkowski problem for all positive p, when $0 < p \neq n$, and the solution to the even volume-normalized L_p Minkowski problem for all positive p.

Corollary 2. *If μ is an even finite Borel measure on the sphere S^{n-1} which is not concentrated on a great subsphere of S^{n-1}, then*

(i) for $p > 0$, there exists an origin symmetric convex body K in $\mathbb{R}^n$ such that

$$c \, h_K^{1-p} \, dS_K = d\mu, \tag{54}$$

where $c = 1/V(K)$.

(ii) for $0 < p \neq n$, there exists an origin symmetric convex body K in $\mathbb{R}^n$ such that

$$h_K^{1-p} \, dS_K = d\mu.$$

Proof. Note that from (3) and (6) it follows that the maps

$$K \longmapsto h_K^{1-p}\, dS_K \qquad \text{and} \qquad K \longmapsto \frac{1}{V(K)} h_K^{1-p}\, dS_K$$

are homogeneous of degree $n-p$ and $-p$, respectively; i.e., for $K \in \mathcal{K}_e^n$ and $\lambda > 0$,

$$h_{\lambda K}^{1-p}\, dS_{\lambda K} = \lambda^{n-p}\, h_K^{1-p}\, dS_K, \tag{55}$$

and

$$\frac{1}{V(\lambda K)} h_{\lambda K}^{1-p}\, dS_{\lambda K} = \lambda^{-p} \frac{1}{V(K)} h_K^{1-p}\, dS_K. \tag{56}$$

We first prove (i). Take $\varphi(t) = t^{1-p}$. From (41) we have (54) and from (42) that h_K satisfies

$$\frac{1}{|\mu|} \int_{S^{n-1}} h_K^p d\mu = 1. \tag{57}$$

We now show that a dilation of K gives the desired constant. By using (9), integration in (41) gives

$$\int_{S^{n-1}} h_K^p d\mu = cnV(K). \tag{58}$$

From (57) and (58) it follows that $1/c = nV(K)/|\mu|$. To complete the proof, let $K' = \lambda K$, where $\lambda^p = n/|\mu|$ is chosen so that by (56)

$$\frac{|\mu|}{nV(K)} h_K^{1-p} dS_K = \frac{1}{V(K')} h_{K'}^{1-p} dS_{K'}.$$

To see that (ii) follows from (i), observe that by (55),

$$c\, h_K^{1-p}\, dS_K = h_{K'}^{1-p}\, dS_{K'},$$

where $K' = \lambda K$, where λ is chosen so that $\lambda^{n-p} = c$. $\square$

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