Minkowski valuations intertwining the special linear group

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Abstract

All continuous Minkowski valuations which are compatible with the special linear group are completely classified. One consequence of these classifications is a new characterization of the projection body operator.

1 Introduction

Ever since they played a critical role in Dehn's solution of Hilbert's Third Problem, valuations have been a central focus in convex geometric analysis (see, e.g., [34, 50, 51]). In addition to the ongoing research concerning scalar valued valuations on convex sets (see, e.g., [1–4, 8, 11, 15, 20, 33, 36, 40, 41]), the study of valuations headed mainly in two different directions during the last years. First, a theory of valuations which are defined on much more general objects than convex sets emerged (see, e.g., [5–7,9,10,12–14,31,32,63]). Second, valuations with values other than scalars have been characterized (see, e.g., [24–26, 29, 30, 35, 37–39, 56–58]). In particular, Ludwig [35,37–39] developed a theory of body valued valuations which are compatible with the whole general linear group (see also [24–26, 58]). She thereby obtained simple characterizations of basic geometric operators. Recently, her results led to strengthenings and generalizations of several affine isoperimetric and Sobolev inequalities [27,28,47,48]. All proofs of such characterizations heavily rely on the assumption of homogeneity.

A central question in the subject has long been: Are the homogeneity assumptions necessitated only by the techniques used in the proofs? This is the first paper to indicate that this indeed may be the case.

Let $\mathcal{K}^n$ denote the set of convex bodies in $\mathbb{R}^n$ (i.e., non-empty compact convex subsets of $\mathbb{R}^n$) and write $\mathcal{K}_o^n$ for convex bodies containing the origin. We will view $\mathcal{K}^n$ as equipped with the Hausdorff metric. The Minkowski sum $K + L$ of two convex bodies $K, L \in \mathcal{K}^n$ is the usual vector sum of K and L. A *Minkowski valuation* is a map $\mathrm{Z} : \mathcal{K}_o^n \to \langle \mathcal{K}_o^n, + \rangle$ such that

$$\mathrm{Z}(K \cup L) + \mathrm{Z}(K \cap L) = \mathrm{Z}\,K + \mathrm{Z}\,L$$

whenever the union of $K, L \in \mathcal{K}_o^n$ is again convex. A map $Z : \mathcal{K}_o^n \to \mathcal{K}_o^n$ is said to be $\mathrm{SL}(n)$ *contravariant* if $Z(\phi K) = \phi^{-t} Z K$ for every $\phi \in \mathrm{SL}(n)$ and $K \in \mathcal{K}_o^n$. Here, ϕ^{-t} denotes the inverse of the transpose of ϕ. One of the main results in this paper shows that for $n \geq 3$ there exists (up to scalar multiples) a unique continuous $\mathrm{SL}(n)$ contravariant Minkowski valuation.

Theorem 1. *A map* $Z : \mathcal{K}_o^n \to \langle \mathcal{K}_o^n, + \rangle$ *is a continuous* $\mathrm{SL}(n)$ *contravariant Minkowski valuation if and only if there exists a non-negative constant* c *such that*

$$Z K = c \, \Pi K$$

for every $K \in \mathcal{K}_o^n$.

The *projection body* ΠK of K is defined via its support function (see Section 2 for details) by
$$h_{\Pi K}(u) = \mathrm{vol}_{n-1}(K | u^\perp), \qquad u \in S^{n-1},$$

where vol_{n-1} denotes $(n-1)$-dimensional volume and $K | u^\perp$ denotes the image of the orthogonal projection of K onto the subspace orthogonal to u.

Under the additional assumption of homogeneity in Theorem 1, a characterization of the projection body operator was previously given by Ludwig [38]. Projection bodies were introduced by Minkowski at the turn of the previous century and have since become a central notion in convex geometry. They also arise naturally in a number of different areas such as Minkowski geometry, geometric tomography, symbolic dynamics, and Sobolev inequalities (see, e.g., [17, 21, 27, 45, 61, 62, 66]).

A map $Z : \mathcal{K}_o^n \to \mathcal{K}_o^n$ is said to be $\mathrm{SL}(n)$ *covariant* if $Z(\phi K) = \phi \, Z K$ for every $\phi \in \mathrm{SL}(n)$ and $K \in \mathcal{K}_o^n$. We also establish the following characterization for $n \geq 3$.

Theorem 2. *A map* $Z : \mathcal{K}_o^n \to \langle \mathcal{K}_o^n, + \rangle$ *is a continuous* $\mathrm{SL}(n)$ *covariant Minkowski valuation if and only if there exist non-negative constants* $c_1, \ldots, c_4$ *such that*

$$Z K = c_1 K + c_2(-K) + c_3 \Gamma_+ K + c_4 \Gamma_+(-K)$$

for every $K \in \mathcal{K}_o^n$.

The *asymmetric centroid body* $\Gamma_+ K$ of K is the convex body whose support function is given by
$$h_{\Gamma_+ K}(u) = \int_K (u \cdot x)_+ \, dx, \qquad u \in S^{n-1}.$$

Here, integration is with respect to Lebesgue measure and $(u \cdot x)_+$ denotes the positive part of the standard Euclidean product of u and x. Note that the definition of the centroid body given above differs from the classical one by a dilation since it is not volume-normalized.

The identity and the centroid body operator were characterized before as $\mathrm{SL}(n)$ covariant Minkowski valuations which are positively homogeneous by Ludwig [38]. For symmetric bodies, centroid bodies are a classical concept which goes back to Dupin. A decade ago, a study was begun of their symmetric analogs (see, e.g., [16, 42, 49]) within the L_p Brunn-Minkowski theory (see, e.g., [16, 17, 19, 23,

24, 26–28, 37, 38, 42–45, 49, 54, 59, 60, 64, 65]). They became objects of interest in asymptotic geometric analysis (see, e.g., [18, 19, 53]), information theory (see, e.g., [46]), and even the theory of stable distributions (see, e.g., [52]).

In fact, we will prove more general characterizations of Minkowski valuations as those of Theorems 1 and 2. These results (see Section 3) deal with Minkowski valuations which are either defined on polytopes and are not necessarily continuous on the whole domain or their images do not have to contain the origin.

2   Notation and preliminaries

For quick later reference we develop some notation and basic facts about convex bodies. General references for the theory of convex bodies are provided by the books of Gardner [21], Gruber [22], Schneider [55], and Thompson [62].

We write $\mathbb{R}_+$ for the set of positive real numbers. The positive and negative part of a real number a is defined by

$$(a)_+ = \max\{a, 0\} \qquad \text{and} \qquad (a)_- = \max\{-a, 0\},$$

respectively.

Critical for us will be the solution to *Cauchy's functional equation*

$$f(x + y) = f(x) + f(y). \tag{1}$$

Let $f : \mathbb{R} \to \mathbb{R}$ be a function which satisfies (1) for all $x, y \in \mathbb{R}$ but is not linear. It is well known that the graph of such a function is a dense subset of $\mathbb{R}^2$. Every function $f : \mathbb{R}_+ \to \mathbb{R}$ which satisfies (1) only for positive real numbers can be extended (as an odd function) to a function which satisfies (1) for all $x, y \in \mathbb{R}$. We therefore obtain the following. If $f : \mathbb{R}_+ \to \mathbb{R}$ is bounded from below on some non-empty open interval $I \subset \mathbb{R}_+$, then

$$f(x + y) = f(x) + f(y) \quad \forall\, x, y \in \mathbb{R}_+ \quad \Longrightarrow \quad f(x) = x f(1) \quad \forall\, x \in \mathbb{R}_+. \tag{2}$$

The letter n will always denote an integer greater than one. Our setting will be Euclidean n-space $\mathbb{R}^n$. We write V for Lebesgue measure on $\mathbb{R}^n$. The standard basis vectors of $\mathbb{R}^n$ are denoted by $e_1, \ldots, e_n$. We write $x_1, \ldots, x_n$ for the coordinates of a vector $x \in \mathbb{R}^n$ with respect to the standard basis. The standard Euclidean inner product of two vectors $x, y \in \mathbb{R}^n$ is denoted by $x \cdot y$. Write $\|\cdot\|$ for the norm induced by this inner product and let S^{n-1} denote the Euclidean unit sphere, i.e. the set $\{x \in \mathbb{R}^n : \|x\| = 1\}$. For a unit vector $u \in S^{n-1}$, we write $u^\perp$ for its orthogonal complement. The linear and convex hull are denoted by lin and conv, respectively. As usual, we write $\mathrm{GL}(n)$ for the general linear group and $\mathrm{SL}(n)$ for the special linear group.

A *convex body* is a non-empty compact convex subset of $\mathbb{R}^n$. We write $\mathcal{K}^n$ for the set of convex bodies in $\mathbb{R}^n$ and denote by $\mathcal{K}_o^n$ the subset of convex bodies containing the origin. The dimension of a convex body is denoted by dim. Convex polytopes in $\mathbb{R}^n$ are denoted by $\mathcal{P}^n$ and we write $\mathcal{P}_o^n$ for convex polytopes containing the origin. Two special polytopes will be used frequently in the sequel. First,

a *line segment* is the convex hull of two distinct points in $\mathbb{R}^n$. The line segment joining distinct points $x, y \in \mathbb{R}^n$ is denoted by

$$[x, y] = \mathrm{conv}\{x, y\}.$$

Second, the *n-dimensional standard simplex* $T^n \in \mathcal{P}_o^n$ is given by

$$T^n = \mathrm{conv}\{o, e_1, \ldots, e_n\}.$$

A convex body $K \subset \mathcal{K}^n$ is uniquely determined by its *support function* $h_K :$ $\mathbb{R}^n \to \mathbb{R}$, where $h_K(x) = \max\{x \cdot y : y \in K\}$, for each $x \in \mathbb{R}^n$. Note that support functions are *sublinear*, i.e. for all $x, y \in \mathbb{R}^n$ and $\lambda \geq 0$ we have

$$h_K(\lambda x) = \lambda h_K(x) \qquad \text{and} \qquad h_K(x + y) \leq h_K(x) + h_K(y). \tag{3}$$

In other words, support functions are positively homogeneous of degree one and subadditive. Conversely, every sublinear function is the support function of a unique convex body. The sublinearity of support functions implies that they are continuous and uniquely determined by their values on the unit sphere S^{n-1}.

Next, let us collect three basic properties of support functions. For $K, L \in \mathcal{K}^n$ and non-negative numbers a and b we have

$$h_{aK+bL} = ah_K + bh_L. \tag{4}$$

Moreover, if $K \in \mathcal{K}^n$ and $\phi \in \mathrm{GL}(n)$ then

$$h_{\phi K} = h_K \circ \phi^t. \tag{5}$$

Suppose that $K \in \mathcal{K}^n$ is contained in $\mathrm{lin}\{e_1, \ldots, e_k\}$. Then

$$h_K(x) = h_K(x_1 e_1 + \cdots + x_k e_k), \qquad x \in \mathbb{R}^n. \tag{6}$$

We need the precise form of support functions of line segments and the standard simplex. The support function of the line segment $[o, v]$ joining the origin and a non-zero point $v \in \mathbb{R}^n$ is given by

$$h_{[o,v]}(x) = (x \cdot v)_+, \qquad x \in \mathbb{R}^n. \tag{7}$$

For the support function of the n-dimensional simplex T^n we have

$$h_{T^n}(x) = \max\{(x_1)_+, \ldots, (x_n)_+\}, \qquad x \in \mathbb{R}^n. \tag{8}$$

The set $\mathcal{K}^n$ will be viewed as equipped with the *Hausdorff metric*. The latter can be defined for $K, L \in \mathcal{K}^n$ by

$$\delta(K, L) = \sup_{u \in S^{n-1}} |h_K(u) - h_L(u)|.$$

Note that the set $\mathcal{P}_o^n$ of polytopes containing the origin is a dense subset of $\mathcal{K}_o^n$.

Associated with each convex body $K \in \mathcal{K}^n$ is a Borel measure, S_K, on S^{n-1} called *surface area measure* of K. It is defined as the unique finite Borel measure on S^{n-1} such that for all $L \in \mathcal{K}^n$

$$\int_{S^{n-1}} h_L \, dS_K = \lim_{\varepsilon \to 0^+} \frac{V(K + \varepsilon L) - V(K)}{\varepsilon}. \tag{9}$$

The surface area measure of a polytope is a discrete measure which is concentrated on the outer unit normals of its facets. Moreover, if $u \in S^{n-1}$ is an outer unit normal of a facet of $P \in \mathcal{P}^n$, then $S_P(\{u\})$ equals the $(n-1)$-dimensional volume of this facet.

Surface area measures are weakly continuous with respect to the Hausdorff metric, i.e., if $(K_i)_{i \in \mathbb{N}}$ is a sequence of bodies in $\mathcal{K}^n$ then

$$\lim_{i \to \infty} K_i = K \quad \Longrightarrow \quad \lim_{i \to \infty} S_{K_i} = S_K, \text{ weakly.} \tag{10}$$

For convex bodies $K, L \in \mathcal{K}^n$ with convex union $K \cup L$ we have

$$S_{K \cup L} + S_{K \cap L} = S_K + S_L. \tag{11}$$

Moreover, if $\lambda \geq 0$ and $K \in \mathcal{K}^n$, then

$$S_{\lambda K} = \lambda^{n-1} S_K. \tag{12}$$

Support functions of projection bodies can be expressed by surface area measures. Indeed, for $K \in \mathcal{K}^n$ one has

$$h_{\Pi K}(u) = \int_{S^{n-1}} (u \cdot v)_+ dS_K(v), \qquad u \in S^{n-1}.$$

This representation together with (9), (10), (11), and (12) shows that $\Pi : \mathcal{K}_o^n \to \mathcal{K}_o^n$ is a continuous $\mathrm{SL}(n)$ contravariant Minkowski valuation which is positively homogeneous of degree $n - 1$.

Let $\mathcal{Q}^n$ be a subset of $\mathcal{K}^n$. A map $\mathrm{Z} : \mathcal{Q}^n \to \langle \mathcal{G}, + \rangle$ with values in an abelian semigroup is called *valuation* if

$$\mathrm{Z}(P \cup Q) + \mathrm{Z}(P \cap Q) = \mathrm{Z} P + \mathrm{Z} Q$$

whenever $P, Q, P \cup Q, P \cap Q \in \mathcal{Q}^n$. If the semigroup $\langle \mathcal{G}, + \rangle$ is a subsemigroup of $\langle \mathcal{K}^n, + \rangle$, then Z is called *Minkowski valuation*. A map $\mathrm{Z} : \mathcal{Q}^n \to \mathcal{K}^n$ is said to be $\mathrm{SL}(n)$ *co- or contravariant* if $\mathrm{Z}(\phi Q)$ equals $\phi \mathrm{Z} Q$ or $\phi^{-t} \mathrm{Z} Q$ for every $\phi \in \mathrm{SL}(n)$ and $Q \in \mathcal{Q}^n$.

In the remaining part of this section we will construct several Minkowski valuations which are different from the projection body and the asymmetric centroid body operator. Suppose that $P \in \mathcal{P}_o^n$. We define $\mathcal{N}_o(P)$ as the set of all outer unit normals of facets of P containing the origin. Equivalently, for a unit vector $u \in S^{n-1}$ we have $u \in \mathcal{N}_o(P)$ if and only if $u \cdot x \leq 0$ for all $x \in P$ and $P \cap u^\perp$ is $(n-1)$-dimensional. Let $\Pi_o : \mathcal{P}_o^n \to \mathcal{K}_o^n$ be defined by

$$h_{\Pi_o P}(u) = \int_{\mathcal{N}_o(P)} (u \cdot v)_+ dS_P(v), \qquad u \in S^{n-1}.$$

The operator Π_o was defined by Ludwig [38]. It follows directly from (12) that Π_o is positively homogeneous of degree $n-1$. For the reader's convenience we collect other basic properties in the following lemma.

Lemma 1. *The map $\Pi_o : \mathcal{P}_o^n \to \mathcal{K}_o^n$ is an $\mathrm{SL}(n)$ contravariant valuation.*

Proof. For polytopes $P \in \mathcal{P}_o^n$ define a finite Borel measure μ_P on the sphere by

$$d\mu_P = \chi_{\mathcal{N}_o(P)} dS_P,$$

where χ denotes the characteristic function. Note that μ_P is a discrete measure with finite support. In fact, it is concentrated on the outer unit normals of facets of P which contain the origin. Clearly we have

$$h_{\Pi_o P}(u) = \int_{S^{n-1}} (u \cdot v)_+ \, d\mu_P(v), \qquad u \in S^{n-1}. \tag{13}$$

Let $P, Q \in \mathcal{P}_o^n$ with $P \cup Q \in \mathcal{P}_o^n$. Note that $\mathcal{N}_o(P \cup Q) \cup \mathcal{N}_o(P \cap Q) = \mathcal{N}_o(P) \cup \mathcal{N}_o(Q)$. This together with the valuation property (11) implies for every $v \in S^{n-1}$ that

$$\mu_{P \cup Q}(\{v\}) + \mu_{P \cap Q}(\{v\}) = \mu_P(\{v\}) + \mu_Q(\{v\}).$$

Since all measures which are involved in the last equation are discrete, we actually have $\mu_{P \cup Q} + \mu_{P \cap Q} = \mu_P + \mu_Q$. This and the representation (13) prove that Π_o is a valuation.

In order to establish the contravariance of Π_o suppose that $P \in \mathcal{P}_o^n$ and $\phi \in \mathrm{SL}(n)$. Write $\langle x \rangle = x/\|x\|$ for $x \in \mathbb{R}^n \setminus \{o\}$ and denote by $\langle \phi \rangle : S^{n-1} \to S^{n-1}$ the map with $\langle \phi \rangle(v) = \langle \phi^{-t} v \rangle$ for each $v \in S^{n-1}$. Clearly, $\langle \phi \rangle$ is a continuous bijection. For every unit vector $v \in S^{n-1}$ we have

$$\|\phi^{-t} v\| S_P(\{v\}) = S_{\phi P}(\{\langle \phi \rangle(v)\}) \quad \text{and} \quad v \in \mathcal{N}_o(P) \Longleftrightarrow \langle \phi \rangle(v) \in \mathcal{N}_o(\phi P).$$

Write $\langle \phi \rangle \mu_P$ for the image measure of μ_P with respect to $\langle \phi \rangle$. Thus

$$\langle \phi \rangle \mu_P(\{v\}) = \|\phi^t v\| \mu_{\phi P}(\{v\})$$

for every $v \in S^{n-1}$. Since all measures involved in the last equation are discrete with finite support we actually have $d\mu_{\phi P} = \|\phi^t \cdot\|^{-1} d\langle \phi \rangle \mu_P$. Representation (13) therefore yields $h_{\Pi_o \phi P} = h_{\Pi_o P} \circ \phi^{-1}$. From (5) we deduce the $\mathrm{SL}(n)$ contravariance of Π_o. $\square$

We finish our discussion of the operators Π and Π_o by two simple formulas. Since the volume of the n-dimensional standard simplex equals $1/n!$, the above interpretation of surface area measures for polytopes yields

$$h_{\Pi T^n}(x_1 e_1 + x_2 e_2) = \frac{1}{(n-1)!}[(x_1 + x_2)_+ + (x_1)_- + (x_2)_-], \tag{14}$$

as well as

$$h_{\Pi_o T^n}(x_1 e_1 + x_2 e_2) = \frac{1}{(n-1)!}[(x_1)_- + (x_2)_-]. \tag{15}$$

Let $K \in \mathcal{K}^n$. The *symmetric centroid body* ΓK of K is defined by

$$h_{\Gamma K}(u) = \int_K |u \cdot x| \, dx, \qquad u \in S^{n-1}.$$

It is easliy seen that $\Gamma K = \Gamma_+ K + \Gamma_+(-K)$. The *moment vector* $\mathrm{m}(K)$ of K is given by

$$h_{\mathrm{m}(K)}(u) = \int_K u \cdot x \, dx, \qquad u \in S^{n-1}.$$

Note that the moment vector is indeed an element of $\mathbb{R}^n$. Up to normalization, $\mathrm{m}(K)$ is equal to the centroid of K. Clearly, both $\Gamma, \mathrm{m} : \mathcal{K}^n \to \langle \mathcal{K}^n, + \rangle$ are continuous $\mathrm{SL}(n)$ covariant Minkowski valuations.

Finally, let $P \in \mathcal{P}_o^2$ and denote by $\mathcal{E}_o(P)$ the set of all edges of P containing the origin. We define an operator $\mathrm{E} : \mathcal{P}_o^2 \to \mathcal{P}_o^2$ by

$$\mathrm{E}\, P = \begin{cases} \{o\} & \mathcal{E}_o(P) = \emptyset, \\ 2F & \mathcal{E}_o(P) = \{F\}, \ P = F, \\ F & \mathcal{E}_o(P) = \{F\}, \ P \neq F, \\ F_1 + F_2 & \mathcal{E}_o(P) = \{F_1, F_2\}. \end{cases}$$

Note that $\mathrm{E} : \mathcal{P}_o^2 \to \langle \mathcal{K}_o^2, + \rangle$ is an $\mathrm{SL}(2)$ covariant Minkowski valuation.

3 Classification results of Minkowski valuations

Throughout this section let $n \geq 3$. In what follows, we state several classifications which are similar to those given in the introduction but hold under weaker assumptions.

3.1 Contravariant Minkowski valuations

Let us begin with a result on $\mathrm{SL}(n)$ contravariant Minkowski valuations which are not necessarily continuous.

Theorem 3. *If* $\mathrm{Z} : \mathcal{P}_o^n \to \langle \mathcal{K}_o^n, + \rangle$ *is an* $\mathrm{SL}(n)$ *contravariant Minkowski valuation, then there exist constants* c_1, c_2, c_3 *with* $c_1 \geq 0$ *and* $c_1 + c_2 + c_3 \geq 0$ *such that*

$$h_{\mathrm{Z}\,P} = c_1 h_{\Pi P} + c_2 h_{\Pi_o P} + c_3 h_{\Pi_o(-P)} \tag{16}$$

for every $P \in \mathcal{P}_o^n$.

We remark that there exist constants c_1, c_2, and c_3 such that c_2 or c_3 is negative but $\mathrm{Z} : \mathcal{P}_o^n \to \langle \mathcal{K}_o^n, + \rangle$ defined by (16) is an $\mathrm{SL}(n)$ contravariant Minkowski valuation. Under the assumption of continuity, we have the following characterization of Minkowski valuations whose images do not have to contain the origin a priori.

Theorem 4. *A map* $\mathrm{Z} : \mathcal{K}_o^n \to \langle \mathcal{K}^n, + \rangle$ *is a continuous* $\mathrm{SL}(n)$ *contravariant Minkowski valuation if and only if there exists a non-negative constant* c *such that*

$$\mathrm{Z}\, K = c\, \Pi K$$

for every $K \in \mathcal{K}_o^n$.

3.2 Covariant Minkowski valuations

For Minkowski valuations defined on polytopes we will prove the following.

Theorem 5. *A map* $Z : \mathcal{P}_o^n \to \langle \mathcal{K}_o^n, + \rangle$ *is an* $\mathrm{SL}(n)$ *covariant Minkowski valuation which is continuous at the line segment* $[o, e_1]$ *if and only if there exist non-negative constants* $c_1, \ldots, c_4$ *such that*

$$Z P = c_1 P + c_2(-P) + c_3 \Gamma_+ P + c_4 \Gamma_+(-P)$$

for every $P \in \mathcal{P}_o^n$.

Finally, the next result characterizes all continuous $\mathrm{SL}(n)$ covariant Minkowski valuations whose range is $\mathcal{K}^n$.

Theorem 6. *A map* $Z : \mathcal{K}_o^n \to \langle \mathcal{K}^n, + \rangle$ *is a continuous* $\mathrm{SL}(n)$ *covariant Minkowski valuation if and only if there exist non-negative constants* $c_1, \ldots, c_3$ *and a constant* $c_4 \in \mathbb{R}$ *such that*

$$Z K = c_1 K + c_2(-K) + c_3 \Gamma K + c_4 \,\mathrm{m}(K)$$

for every $K \in \mathcal{K}_o^n$.

4 Reduction to Simplices

The aim of this section is to show that $\mathrm{SL}(n)$ co- or contravariant Minkowski valuations are actually determined by their values on dilates of the standard simplex.

Let $P \in \mathcal{P}^n$ be n-dimensional. A finite set $\mathcal{T}_P$ of n-dimensional simplices is called *triangulation* of P if the union of all simplices in $\mathcal{T}_P$ equals P and no pair of simplices intersects in a set of dimension n. Suppose that $x \in P$. A *starring of P at x* is a triangulation such that each simplex in $\mathcal{T}_P$ has a vertex at x.

If $P \in \mathcal{P}^n$ is an n-dimensional polytope and $x \in P$, then it is well-known that there exists a starring of P at x. Indeed, for $n = 1$ it is trivial. Suppose that the assertion is true for $(n-1)$-dimensional polytopes and denote by F_j, $j = 1, \ldots, k$ the facets of an n-dimensional polytope P. We choose starrings $\mathcal{T}_{F_j}$ of F_j for those facets which do not contain the given point x. Thus the convex hulls of x and the $(n-1)$-dimensional simplices in $\mathcal{T}_{F_j}$ define the desired starring.

A real valued valuation $z : \mathcal{P}_o^n \to \langle \mathbb{R}, + \rangle$ is called *simple* if convex polytopes of dimension less than n are mapped to zero. The following result is a special case of [24, Lemma 3.2]. For the sake of completeness we give its proof here.

Lemma 2. *Let* $z : \mathcal{P}_o^n \to \langle \mathbb{R}, + \rangle$ *be a simple valuation. If* $z(S) = 0$ *for every n-dimensional simplex S having one vertex at the origin, then* $z(P) = 0$ *for every* $P \in \mathcal{P}_o^n$.

Proof. By what we have seen above, every $P \in \mathcal{P}_o^n$ has a starring at o. Therefore, it suffices to prove that for an n-dimensional polytope $P \in \mathcal{P}_o^n$

$$P = P_1 \cup P_2 \cup \cdots \cup P_k, \quad P, P_1, \ldots, P_k \in \mathcal{P}_o^n, \quad \dim(P_i \cap P_j) < n \quad \text{for } i \neq j$$

implies

$$z(P) = \sum_{i=1}^{k} z(P_i). \tag{17}$$

We proceed by induction on k. For $k = 1$ this is trivial. Assume that $k \geq 2$ and suppose that it is true for at most $k - 1$ polytopes. Without loss of generality assume that $\dim P_1 = n$.

Suppose that P_1 has at least one facet F_1 containing the origin such that $P \cap \text{int}(\text{lin } F_1)^+ \neq \emptyset$. Here, $\text{int}(\text{lin } F_1)^+$ denotes the interior of the halfspace determined by $\text{lin } F_1$ which does not contain P_1. For if no such facet exists, then $P_1 = P$. But this would imply $\dim P_i < n$ for $i = 2, \ldots, k$, and (17) would obviously hold by the simplicity of z.

Let $H_1, \ldots, H_l$ denote the linear hulls of those facets $F_1, \ldots, F_l$ of P_1 which contain the origin and satisfy

$$P \cap \text{int } H_i^+ \neq \emptyset \tag{18}$$

for $i = 1, \ldots, l$, where $\text{int } H_i^+$ denotes the interior of the halfspace determined by H_i which does not contain P_1. For $m = 1, \ldots, l$ and $i = 1, \ldots, k$ set $P^0 = P$, $P_i^0 = P_i$, and define

$$P^m = P \cap H_1^- \cap \ldots \cap H_m^- \quad \text{and} \quad P_i^m = P_i \cap H_1^- \cap \ldots \cap H_m^-.$$

Note that for each $m = 1, \ldots, l$ there exists a point p^m such that

$$p^m \in P^{m-1} \cap \text{int } H_m^+. \tag{19}$$

Indeed, for $m = 1$ this directly follows from (18). For $m > 1$ choose a point x in the relative interior of the facet $F_m = P_1 \cap H_m$. Then $x \in \text{int } H_i^-$ for all $i = 1, \ldots, m-1$. By (18) we know that there exists a $p \in P \cap \text{int } H_m^+$. Clearly, the set $[x, p] \setminus \{x\}$ is contained in $P \cap \text{int } H_m^+$. Moreover, points of this set which are sufficiently close to x are contained in $\text{int } H_1^- \cap \ldots \cap \text{int } H_{m-1}^-$. This proves (19).

Next, we are going to prove that for all $m = 1, \ldots, l$ the equality

$$z(P) = z(P^m) + \sum_{i=1}^{k} [z(P_i) - z(P_i^m)] \tag{20}$$

holds. To this end fix m and set

$$P^{m,+} = P^{m-1} \cap H_m^+ \quad \text{and} \quad P_i^{m,+} = P_i^{m-1} \cap H_m^+.$$

First, we want to prove

$$P^{m,+} = P_2^{m,+} \cup \ldots \cup P_k^{m,+}. \tag{21}$$

Note that since $P^{m,+} = P_1^{m,+} \cup \ldots \cup P_k^{m,+}$ it is actually enough to show that $P_1^{m,+} \subset P_2^{m,+} \cup \ldots \cup P_k^{m,+}$. Let $x \in P_1^{m,+} = P_1^{m-1} \cap H_m$. From (19) we infer that the set $[x, p^m] \setminus \{x\}$ has to be contained in $P_2^{m,+} \cup \ldots \cup P_k^{m,+}$. Thus there exists

9

an index $i \in \{2, \ldots, k\}$ such that points of the set $[x, p^m]\backslash\{x\}$ sufficiently close to x are contained in $P_i^{m,+}$. The closedness of $P_i^{m,+}$ concludes the proof of (21).

Now (21), the induction assumption, and the simplicity of z prove

$$z(P^{m,+}) = \sum_{i=2}^k z(P_i^{m,+}) = \sum_{i=1}^k z(P_i^{m,+}) = \sum_{i=1}^k [z(P_i^{m-1}) - z(P_i^m)].$$

By the valuation property of z and its simplicity we therefore obtain

$$z(P^{m-1}) = z(P^m) + z(P^{m,+}) = z(P^m) + \sum_{i=1}^k [z(P_i^{m-1}) - z(P_i^m)]. \qquad (22)$$

For $m = 1$ this is precisely (20). Let m be an integer greater than one and less than or equal l. Assume that (20) holds for $m - 1$. Then inserting (22) in this equation immediately shows that (20) also holds for m. Inductively we therefore proved (20) for all $m = 1, \ldots, l$.

Since $P_1^l = P^l$ we have $\dim P_i^l < n$ for $i = 2, \ldots, k$, and hence

$$z(P^l) = \sum_{i=1}^k z(P_i^l).$$

Inserting this in (20) for $m = l$ proves (17). $\qquad\square$

Lemma 3. *For $i = 1, \ldots, k$, let $c_i \in \mathbb{R}$ and suppose that $Z_i : \mathcal{P}_o^n \to \langle \mathcal{K}^n, + \rangle$ are Minkowski valuations. If all Z_i's are either $\mathrm{SL}(n)$ co- or contravariant and*

$$\sum_{i=1}^k c_i h_{Z_i(sT^n)} = 0 \qquad \text{for every } s > 0, \qquad (23)$$

then

$$\sum_{i=1}^k c_i h_{Z_i P} = 0 \qquad \text{for every } P \in \mathcal{P}_o^n.$$

Proof. Let $x \in \mathbb{R}^n$ and define a map $z : \mathcal{P}_o^n \to \langle \mathbb{R}, + \rangle$ by

$$z(P) = \sum_{i=1}^k c_i h_{Z_i P}(x).$$

From (4) we deduce that z is a valuation. Let S be an n-dimensional simplex with one vertex at the origin. Note that there exists a $\phi \in \mathrm{SL}(n)$ and a positive number s such that $\phi(sT^n) = S$. Since all Z_i's are assumed to be either $\mathrm{SL}(n)$ co- or contravariant, we obtain from (5) and (23) that $z(S) = 0$.

Let $0 \le k < n$ and S' be a k-dimensional simplex with one vertex at the origin. There exists a $(k + 1)$-dimensional simplex S with one vertex at the origin and a hyperplane H such that $S \cap H^+$ and $S \cap H^-$ are both k-dimensional simplices with

one vertex at the origin and $S' = S \cap H$. As before, $H^{\pm}$ denote the two halfspaces determined by H. The valuation property of z implies

$$z(S) + z(S') = z(S \cap H^+) + z(S \cap H^-).$$

Since z vanishes on n-dimensional simplices with one vertex at the origin, an obvious induction argument shows that z vanishes on all simplices with one vertex at the origin.

By induction on the dimension k, we will now show that z vanishes on k-dimensional polytopes. For $k = 0$ this obviously follows from the results of the last paragraph. Assume that z vanishes on k-dimensional polytopes and let $P \in \mathcal{P}_o^n$ be $(k + 1)$-dimensional. Set $H = \lim P$ and write $\pi : \mathbb{R}^{k+1} \to H$ for an arbitrary but fixed linear bijection. Then $\hat{z} : \mathcal{P}_o^{k+1} \to \langle \mathbb{R}, + \rangle$ defined by $\hat{z}(Q) = z(\pi Q)$ is a simple valuation which vanishes on $(k + 1)$-dimensional simplices with one vertex at the origin. Lemma 2 shows that $\hat{z} = 0$. Hence also $z(P) = 0$. $\square$

5 Functional equations

Throughout this section let $\lambda \in (0, 1)$ and $p, q \in \mathbb{R}$. Suppose that functions $g, h : (0, 1) \to \mathbb{R}_+$ are given. We define two families of linear maps on $\mathbb{R}^n$ by

$$\phi_{\lambda,g} e_2 = (1-\lambda)e_1 + \lambda e_2, \quad \phi_{\lambda,g} e_1 = e_1, \quad \phi_{\lambda,g} e_k = g(\lambda)e_k \quad \text{for } 3 \le k \le n \text{ if } n \ge 3,$$

and

$$\psi_{\lambda,h} e_1 = (1-\lambda)e_1 + \lambda e_2, \quad \psi_{\lambda,h} e_2 = e_2, \quad \psi_{\lambda,h} e_k = h(\lambda)e_k \quad \text{for } 3 \le k \le n \text{ if } n \ge 3.$$

If g and h are constant to one we set $\phi_\lambda = \phi_{\lambda,g}$ and $\psi_\lambda = \psi_{\lambda,h}$.

Let $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ be given. The aim of this section is to deduce properties of such functions f which satisfy at least one of the following three functional equations. The first one is

$$f(s, x) = \lambda^p f\left(s\lambda^q, \phi_{\lambda,g}^t x\right) + (1 - \lambda)^p f\left(s(1 - \lambda)^q, \psi_{\lambda,h}^t x\right). \tag{24}$$

The second one is just the special case of (24) where g and h are constant to one. It can be written as

$$f(s, x) = \lambda^p f\left(s\lambda^q, \phi_\lambda^t x\right) + (1 - \lambda)^p f\left(s(1 - \lambda)^q, \psi_\lambda^t x\right). \tag{25}$$

The third one reads as

$$f(s, x) = \lambda^p f\left(s\lambda^q, \phi_\lambda^{-1} x\right) + (1 - \lambda)^p f\left(s(1 - \lambda)^q, \psi_\lambda^{-1} x\right). \tag{26}$$

The statement that a function $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ satisfies (24) on $A \subset \mathbb{R}^n$ has to be understood as follows: There exist numbers $p, q \in \mathbb{R}$ and functions $g, h : (0, 1) \to \mathbb{R}_+$ such that for all $\lambda \in (0, 1)$, $s \in \mathbb{R}_+$, and $x \in A$ the function f satisfies equality (24). Similarly, f is said to satisfy (25) and (26) on $A \subset \mathbb{R}^n$ if there exist numbers $p, q \in \mathbb{R}$ such that for all $\lambda \in (0, 1)$, $s \in \mathbb{R}_+$, and $x \in A$ the function f satisfies equality (25) and (26), respectively.

5.1 Homogeneity

Let us start by showing that a function which solves (25) at certain points is positively homogeneous in its first argument.

Lemma 4. *Let $n \geq 3$. If for some $q \neq 0$ the function $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ satisfies equation (25) at the points $\pm e_3$ and the functions $f(\cdot, e_3)$ and $f(\cdot, -e_3)$ are bounded from below on some non-empty open intervals $I_+ \subset \mathbb{R}_+$ and $I_- \subset \mathbb{R}_+$, respectively, then*

$$f(s, e_3) = s^{(1-p)/q} f(1, e_3) \qquad and \qquad f(s, -e_3) = s^{(1-p)/q} f(1, -e_3)$$

for every $s > 0$.

Proof. Note that e_3 is a fixpoint of both ϕ_λ^t and ψ_λ^t. From (25) we see that

$$f(s, e_3) = \lambda^p f(s\lambda^q, e_3) + (1 - \lambda)^p f(s(1 - \lambda)^q, e_3) \qquad (27)$$

for every $s > 0$ and $\lambda \in (0, 1)$. Define a function $g : \mathbb{R}_+ \to \mathbb{R}$, for $s > 0$, by

$$g(s) = f(s^q, e_3).$$

Then, for every $s > 0$ and $\lambda \in (0, 1)$, equation (27) reads as

$$g(s^{1/q}) = \lambda^p g(s^{1/q}\lambda) + (1 - \lambda)^p g(s^{1/q}(1 - \lambda)). \qquad (28)$$

Let x and y be arbitrary positive real numbers. Set

$$s = (x + y)^q \qquad and \qquad \lambda = x(x + y)^{-1}.$$

If we insert these particular values of s and λ in (28), then we have for all $x, y > 0$

$$(x + y)^p g(x + y) = x^p g(x) + y^p g(y).$$

Thus the function $t \to t^p g(t)$ solves Cauchy's functional equation (1) on $\mathbb{R}_+$ and, by assumption, there exists a non-empty open interval $I \subset \mathbb{R}_+$ where it is bounded from below. We infer from (2) that $t^p g(t) = tg(1)$ and hence

$$g(t) = t^{1-p} g(1).$$

Finally, the definition of g immediately yields

$$f(s, e_3) = g(s^{1/q}) = s^{(1-p)/q} g(1) = s^{(1-p)/q} f(1, e_3).$$

Replacing e_3 by $-e_3$ in the above derivation concludes the proof of the lemma. $\square$

Corollary 1. *Let $n \geq 3$. Assume that for some $q \neq 0$ the function $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ satisfies equation (25) at the points $\pm e_3$, the functions $f(\cdot, e_3)$ and $f(\cdot, -e_3)$ are bounded from below on some non-empty open intervals $I_+ \subset \mathbb{R}_+$ and $I_- \subset \mathbb{R}_+$, respectively, and*

$$f(s, \pi e_3) = f(s, e_3) \qquad and \qquad f(s, -\pi e_3) = f(s, -e_3) \qquad (29)$$

for all $s > 0$ and $\pi \in \mathrm{SL}(n)$ induced by a permutation matrix. Then for all $i = 1, \ldots, n$ and every $s > 0$

$$f(s, e_i) = s^{(1-p)/q} f(1, e_i) \qquad and \qquad f(s, -e_i) = s^{(1-p)/q} f(1, -e_i).$$

Proof. Let $i \in \{1, \ldots, n\}$. Since the dimension n is assumed to be greater or equal than three, there exists a map $\pi \in \mathrm{SL}(n)$ which is induced by a permutation matrix and satisfies $e_i = \pi e_3$. From our assumption (29) and Lemma 4 we deduce

$$f(s, e_i) = f(s, \pi e_3) = f(s, e_3) = s^{(1-p)/q} f(1, e_3) = s^{(1-p)/q} f(1, e_i).$$

The argument for $-e_i$ is similar. $\qquad\square$

5.2 Uniqueness

In this subsection we deduce that, under certain circumstances, solutions to the equations (25) and (26) are uniquely determined by their values on a 2-dimensional subspace.

Lemma 5. *Let $n \geq 3$ and $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ be a function which is continuous in its second argument and satisfies (25) on $\mathbb{R}_+ \times \mathbb{R}^n$. Assume moreover that*

$$f(s, \pi x) = f(s, x) \tag{30}$$

for all $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^n$ and each $\pi \in \mathrm{SL}(n)$ which is induced by a permutation matrix. If

$$f(s, x) = 0 \qquad \text{for every} \qquad (s, x) \in \mathbb{R}_+ \times \mathrm{lin}\{e_1, e_2\},$$

then

$$f(s, x) = 0 \qquad \text{for every} \qquad (s, x) \in \mathbb{R}_+ \times \mathbb{R}^n.$$

Proof. Let $2 \leq j < n$. Using an inductional argument, it suffices to show that

$$f(s, x) = 0 \qquad \text{for every} \qquad (s, x) \in \mathbb{R}_+ \times \mathrm{lin}\{e_1, \ldots, e_j\} \tag{31}$$

implies

$$f(s, x) = 0 \qquad \text{for every} \qquad (s, x) \in \mathbb{R}_+ \times \mathrm{lin}\{e_1, \ldots, e_{j+1}\}. \tag{32}$$

Assume that (31) holds. The invariance property (30) therefore implies

$$f(s, x) = 0 \qquad \text{for every} \qquad (s, x) \in \mathbb{R}_+ \times \mathrm{lin}\{e_2, \ldots, e_{j+1}\} \tag{33}$$

and

$$f(s, x) = 0 \qquad \text{for every} \qquad (s, x) \in \mathbb{R}_+ \times \mathrm{lin}\{e_1, e_3, \ldots, e_{j+1}\}. \tag{34}$$

Let $x \in \mathrm{lin}\{e_1, \ldots, e_{j+1}\}$. Suppose that $0 < x_1/x_2 < 1$ and set $\lambda = x_1/x_2$. Then

$$(\psi_\lambda^{-t} x)_1 = (\phi_\lambda^t \psi_\lambda^{-t} x)_1 = \frac{x_1}{1-\lambda} - \frac{\lambda}{1-\lambda} x_2 = 0,$$
$$(\psi_\lambda^{-t} x)_i = (\phi_\lambda^t \psi_\lambda^{-t} x)_i = 0, \quad i = j+2, \ldots, n.$$

Note that (25) gives

$$f(s, \psi_\lambda^{-t} x) = \lambda^p f(s\lambda^q, \phi_\lambda^t \psi_\lambda^{-t} x) + (1-\lambda)^p f(s(1-\lambda)^q, x)$$

13

By (33) we conclude from this equality that $f(s(1-\lambda)^q, x) = 0$ for all $s > 0$. Hence (32) holds for $0 < x_1 < x_2$ and $x_2 < x_1 < 0$.

If $0 < (x_1 - x_2)/x_1 < 1$, then set $\lambda = (x_1 - x_2)/x_1$. Thus

$$(\phi_\lambda^{-t} x)_2 = (\psi_\lambda^t \phi_\lambda^{-t} x)_2 = -\frac{1-\lambda}{\lambda} x_1 + \frac{1}{\lambda} x_2 = 0,$$

$$(\phi_\lambda^{-t} x)_i = (\psi_\lambda^t \phi_\lambda^{-t} x)_i = 0, \quad i = j+2, \ldots, n.$$

Since we have by (25) that

$$f(s, \phi_\lambda^{-t} x) = \lambda^p f(s\lambda^q, x) + (1-\lambda)^p f(s(1-\lambda)^q, \psi_\lambda^t \phi_\lambda^{-t} x),$$

relation (34) proves (32) for $0 < x_2 < x_1$ and $x_1 < x_2 < 0$.

For $x_1, x_2 \neq 0$ and $\mathrm{sgn}(x_1) \neq \mathrm{sgn}(x_2)$ set $\lambda = x_1/(x_1 - x_2)$. Then $\lambda \in (0,1)$ and

$$(\phi_\lambda^t x)_2 = (\phi_\lambda^t x)_i = (\psi_\lambda^t x)_1 = (\psi_\lambda^t x)_i = 0, \quad i = j+2, \ldots, n.$$

As before we conclude that (32) holds for $x_1 < 0$, $x_2 > 0$ and $x_1 > 0$, $x_2 < 0$. The assumed continuity of f in the second argument concludes the proof. $\square$

Next, let us establish a similar result for functions satisfying (26).

Lemma 6. *Let $n \geq 3$ and $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ be a function which is continuous in its second argument and satisfies (26) on $\mathbb{R}_+ \times \mathbb{R}^n$. Assume moreover that*

$$f(s, \pi x) = f(s, x) \tag{35}$$

for all $(s,x) \in \mathbb{R}_+ \times \mathbb{R}^n$ and each $\pi \in \mathrm{SL}(n)$ which is induced by a permutation matrix. If

$$f(s,x) = 0 \qquad for\ every \qquad (s,x) \in \mathbb{R}_+ \times \mathrm{lin}\{e_1, e_2\},$$

then

$$f(s,x) = 0 \qquad for\ every \qquad (s,x) \in \mathbb{R}_+ \times \mathbb{R}^n.$$

Proof. We proceed as in the proof of the previous lemma. Let $2 \leq j < n$. Using an inductional argument, it suffices to show that

$$f(s,x) = 0 \qquad \text{for every} \qquad (s,x) \in \mathbb{R}_+ \times \mathrm{lin}\{e_1, \ldots, e_j\} \tag{36}$$

implies

$$f(s,x) = 0 \qquad \text{for every} \qquad (s,x) \in \mathbb{R}_+ \times \mathrm{lin}\{e_1, \ldots, e_{j+1}\}. \tag{37}$$

Assume that (36) holds. The invariance property (35) therefore implies

$$f(s,x) = 0 \qquad \text{for every} \qquad (s,x) \in \mathbb{R}_+ \times \mathrm{lin}\{e_2, \ldots, e_{j+1}\} \tag{38}$$

and

$$f(s,x) = 0 \qquad \text{for every} \qquad (s,x) \in \mathbb{R}_+ \times \mathrm{lin}\{e_1, e_3, \ldots, e_{j+1}\}. \tag{39}$$

14

Assume that $x \in \lim\{e_1, \ldots, e_{j+1}\}$. If x_1 and x_2 are not both zero and $\operatorname{sgn}(x_1) = \operatorname{sgn}(x_2)$, then $0 < x_2/(x_1 + x_2) < 1$. Set $\lambda = x_2/(x_1 + x_2)$. Thus

$$
\begin{aligned}
(\phi_\lambda^{-1} x)_1 &= x_1 - \frac{1-\lambda}{\lambda} x_2 = 0, \\
(\psi_\lambda^{-1} x)_2 &= -\frac{\lambda}{1-\lambda} x_1 + x_2 = 0, \\
(\phi_\lambda^{-1} x)_i &= (\psi_\lambda^{-1} x)_i = 0, \quad i = j+2, \ldots, n.
\end{aligned}
$$

Note that (26) gives

$$
f(s, x) = \lambda^p f(s\lambda^q, \phi_\lambda^{-1} x) + (1-\lambda)^p f(s(1-\lambda)^q, \psi_\lambda^{-1} x).
$$

By (38) and (39) we therefore conclude that $f(s, x) = 0$ for all $s > 0$. Hence (37) holds for $x_1 > 0$, $x_2 > 0$ and $x_1 < 0$, $x_2 < 0$.

Suppose that $0 < -x_2/x_1 < 1$. Let $\lambda = -x_2/x_1$. Then

$$
\begin{aligned}
(\psi_\lambda x)_2 &= (\phi_\lambda^{-1} \psi_\lambda x)_2 = 0, \\
(\psi_\lambda x)_i &= (\phi_\lambda^{-1} \psi_\lambda x)_i = 0, \quad i = j+2, \ldots, n.
\end{aligned}
$$

Note that (26) gives

$$
f(s, \psi_\lambda x) = \lambda^p f(s\lambda^q, \phi_\lambda^{-1} \psi_\lambda x) + (1-\lambda)^p f(s(1-\lambda)^q, x).
$$

By (39) we therefore conclude from this equality that $f(s(1-\lambda)^q, x) = 0$ for all $s > 0$. Hence (37) holds for $0 < -x_2 < x_1$ and $x_1 < -x_2 < 0$.

If $0 < (x_1 + x_2)/x_2 < 1$, then set $\lambda = (x_1 + x_2)/x_2$. Thus

$$
\begin{aligned}
(\phi_\lambda x)_1 &= (\psi_\lambda^{-1} \phi_\lambda x)_1 = 0, \\
(\phi_\lambda x)_i &= (\psi_\lambda^{-1} \phi_\lambda x)_i = 0, \quad i = j+2, \ldots, n.
\end{aligned}
$$

From (26) and (38) we deduce as before that (37) holds for $0 < x_1 < -x_2$ and $-x_2 < x_1 < 0$. The assumed continuity of f in the second argument concludes the proof. $\qquad \square$

5.3   Representations

The following result solves the functional equation (24) on a subspace.

Lemma 7. *Suppose that $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ is positively homogeneous of degree one in the second argument and satisfies (24) on $\mathbb{R}_+ \times \mathbb{R}^n$. If there exists an $r \in \mathbb{R}$ such that for all $s > 0$ and $x \in \{\pm e_1, \pm e_2\}$*

$$
f(s, x) = s^r f(1, x), \tag{40}
$$

then for $x_1 > x_2 > 0$

$$
\begin{aligned}
f(s, x_1 e_1 + x_2 e_2) &= (x_1 - x_2)^{-p-rq} \left(x_1^{1+p+rq} - x_2^{1+p+rq} \right) f(s, e_1), \\
f(s, -x_1 e_1 - x_2 e_2) &= (x_1 - x_2)^{-p-rq} \left(x_1^{1+p+rq} - x_2^{1+p+rq} \right) f(s, -e_1),
\end{aligned}
$$

for $x_2 > x_1 > 0$

$$f(s, x_1 e_1 + x_2 e_2) = (x_2 - x_1)^{-p-rq} \left(x_2^{1+p+rq} - x_1^{1+p+rq} \right) f(s, e_2),$$

$$f(s, -x_1 e_1 - x_2 e_2) = (x_2 - x_1)^{-p-rq} \left(x_2^{1+p+rq} - x_1^{1+p+rq} \right) f(s, -e_2),$$

and for $x_1, x_2 > 0$

$$f(s, -x_1 e_1 + x_2 e_2) = (x_1 + x_2)^{-p-rq} \left(x_2^{1+p+rq} f(s, e_2) + x_1^{1+p+rq} f(s, -e_1) \right),$$

$$f(s, x_1 e_1 - x_2 e_2) = (x_1 + x_2)^{-p-rq} \left(x_1^{1+p+rq} f(s, e_1) + x_2^{1+p+rq} f(s, -e_2) \right).$$

Proof. Let $\lambda \in (0, 1)$. The functional equation (24) evaluated at e_1 together with the assumed homogeneity of degree one gives

$$f(s, e_1) = \lambda^p f(s\lambda^q, e_1 + (1 - \lambda)e_2) + (1 - \lambda)^{1+p} f(s(1 - \lambda)^q, e_1) \qquad (41)$$

for all $s > 0$. Let t be an arbitrary positive real number. Choose $s = t\lambda^{-q}$ in equation (41) in order to arrive at

$$f(t, e_1 + (1 - \lambda)e_2) = \lambda^{-p} \left[f(t\lambda^{-q}, e_1) - (1 - \lambda)^{1+p} f(t\lambda^{-q}(1 - \lambda)^q, e_1) \right] \qquad (42)$$

for every $t > 0$. Thus the homogeneity assumption (40) and equation (42) evaluated at $t = 1$ yield

$$
\begin{aligned}
f(t, e_1 + (1 - \lambda)e_2) &= t^r \lambda^{-p} \left[f(\lambda^{-q}, e_1) - (1 - \lambda)^{1+p} f(\lambda^{-q}(1 - \lambda)^q, e_1) \right] \\
&= t^r f(1, e_1 + (1 - \lambda)e_2).
\end{aligned}
$$

This and (40) show that we can rewrite (41) as

$$f(s, e_1 + (1 - \lambda)e_2) = \lambda^{-p-rq} \left(1 - (1 - \lambda)^{1+p+rq} \right) f(s, e_1). \qquad (43)$$

Let $x_1 > x_2 > 0$ and set $\lambda = 1 - x_2/x_1$. Since f is homogeneous of degree one in the second component we obtain

$$
\begin{aligned}
f(s, x_1 e_1 + x_2 e_2) &= x_1 f(s, e_1 + (1 - \lambda)e_2) \\
&= x_1 \left(1 - \frac{x_2}{x_1} \right)^{-p-rq} \left(1 - \left(\frac{x_2}{x_1} \right)^{1+p+rq} \right) f(s, e_1) \\
&= (x_1 - x_2)^{-p-rq} \left(x_1^{1+p+rq} - x_2^{1+p+rq} \right) f(s, e_1).
\end{aligned}
$$

Replacing e_1 and e_2 by $-e_1$ and $-e_2$, respectively, in the above derivation shows

$$f(s, -x_1 e_1 - x_2 e_2) = (x_1 - x_2)^{-p-rq} \left(x_1^{1+p+rq} - x_2^{1+p+rq} \right) f(s, -e_1)$$

for $x_1 > x_2 > 0$.

Let $0 < \lambda < 1$. The functional equation (24) evaluated at e_2 together with the assumed homogeneity of degree one gives

$$f(s, e_2) = \lambda^{1+p} f(s\lambda^q, e_2) + (1 - \lambda)^p f(s(1 - \lambda)^q, \lambda e_1 + e_2) \qquad (44)$$

for all $s > 0$. Let t be an arbitrary positive real number. Choose $s = t(1 - \lambda)^{-q}$ in the above equation (44) to get

$$f(t, \lambda e_1 + e_2) = (1 - \lambda)^{-p} \left[f(t(1 - \lambda)^{-q}, e_2) - \lambda^{1+p} f(t\lambda^q (1 - \lambda)^{-q}, e_2) \right] \qquad (45)$$

for every $t > 0$. Thus the homogeneity assumption (40) and equation (45) evaluated at $t = 1$ yield

$$
\begin{aligned}
f(t, \lambda e_1 + e_2) &= t^r (1 - \lambda)^{-p} \left[f((1 - \lambda)^{-q}, e_2) - \lambda^{1+p} f(\lambda^q (1 - \lambda)^{-q}, e_2) \right] \\
&= t^r f(1, \lambda e_1 + e_2).
\end{aligned}
$$

This and (40) show that we can rewrite (44) as

$$f(s, \lambda e_1 + e_2) = (1 - \lambda)^{-p-rq} \left(1 - \lambda^{1+p+rq} \right) f(s, e_2). \qquad (46)$$

Let $x_2 > x_1 > 0$ and set $\lambda = x_1/x_2$. Since f is homogeneous of degree one in the second component we obtain

$$
\begin{aligned}
f(s, x_1 e_1 + x_2 e_2) &= x_2 f(s, \lambda e_1 + e_2) \\
&= x_2 \left(1 - \frac{x_1}{x_2} \right)^{-p-rq} \left(1 - \left(\frac{x_1}{x_2} \right)^{1+p+rq} \right) f(s, e_2) \\
&= (x_2 - x_1)^{-p-rq} \left(x_2^{1+p+rq} - x_1^{1+p+rq} \right) f(s, e_2).
\end{aligned}
$$

Replacing e_1 and e_2 by $-e_1$ and $-e_2$, respectively, in the above derivation shows

$$f(s, -x_1 e_1 - x_2 e_2) = (x_2 - x_1)^{-p-rq} \left(x_2^{1+p+rq} - x_1^{1+p+rq} \right) f(s, -e_2)$$

for $x_2 > x_1 > 0$.

Let $\lambda \in (0, 1)$. The functional equation (24) evaluated at $-\lambda e_1 + (1 - \lambda)e_2$ together with the assumed homogeneity of degree one gives

$$f(s, -\lambda e_1 + (1 - \lambda)e_2) = \lambda^{1+p+rq} f(s, -e_1) + (1 - \lambda)^{1+p+rq} f(s, e_2). \qquad (47)$$

Suppose that both x_1 and x_2 are positive and set $\lambda = x_1/(x_1 + x_2)$. Since f is homogeneous of degree one in the second component we obtain

$$
\begin{aligned}
f(s, -x_1 e_1 + x_2 e_2) &= (x_1 + x_2) f(s, -\lambda e_1 + (1 - \lambda)e_2) \\
&= (x_1 + x_2)^{-p-rq} \left(x_2^{1+p+rq} f(s, e_2) + x_1^{1+p+rq} f(s, -e_1) \right).
\end{aligned}
$$

Replacing e_1 and e_2 by $-e_1$ and $-e_2$, respectively, in the above derivation shows

$$f(s, x_1 e_1 - x_2 e_2) = (x_1 + x_2)^{-p-rq} \left(x_1^{1+p+rq} f(s, e_1) + x_2^{1+p+rq} f(s, -e_2) \right).$$

$\square$

6 The contravariant case

6.1 Preliminaries

Let us first show that $SL(n)$ contravariant operators map $(n-1)$-dimensional simplices to line segments.

Lemma 8. *Let $n \geq 3$, $Z : \mathcal{P}_o^n \to \mathcal{K}^n$ be $SL(n)$ contravariant, and set $T' = T^n \cap e_1^\perp$. Then there exists a non-negative constant a with*

$$Z(sT') = as^{n-1}[-e_1, e_1], \qquad \text{for every } s > 0. \tag{48}$$

Proof. For arbitrary $s > 0$ and $2 \leq k \leq n$, let $\phi \in SL(n)$ be the map defined by

$$\phi e_1 = e_1 + se_k \qquad \text{and} \qquad \phi e_i = e_i, \quad i = 2, \ldots n.$$

Note that for $x \in \mathbb{R}^n$ we have $(\phi^{-t}x)_1 = x_1 - sx_k$. Clearly, $\phi T' = T'$. If $x \in ZT'$, then the contravariance of Z therefore implies that also $\phi^{-t}x \in ZT'$. Since ZT' is bounded, we conclude that $x_2 = \cdots = x_n = 0$. Thus there exist constants $a, b \in \mathbb{R}$ such that $-a \leq b$ and $ZT' = [-ae_1, be_1]$. Define $\psi \in SL(n)$ by

$$\psi e_1 = -e_1, \quad \psi e_2 = e_3, \quad \psi e_3 = e_2, \quad \text{and} \quad \psi e_i = e_i, \quad i = 4, \ldots, n \quad \text{if } n > 3.$$

Since $\psi T' = T'$, the $SL(n)$ contravariance of Z implies

$$[-ae_1, be_1] = ZT' = \psi^{-t} ZT' = [-be_1, ae_1],$$

and hence $a = b$. Finally, for $s > 0$, define a transformation $\tau \in SL(n)$ by

$$\tau e_1 = s^{1-n}e_1, \quad \text{and} \quad \tau e_i = se_i, \quad i = 2, \ldots, n.$$

Note that $\tau T' = sT'$ and $\tau^{-t}(a[-e_1, e_1]) = as^{n-1}[-e_1, e_1]$. The $SL(n)$ contravariance of Z therefore concludes the proof of the lemma. $\qquad \square$

Assume that $Z : \mathcal{P}_o^n \to \langle \mathcal{K}^n, + \rangle$ is an $SL(n)$ contravariant Minkowski valuation and let $\lambda \in (0,1)$. Now, we are going to derive a functional equation for the support function of Z. This equation is closely related to those treated in Section 5 and will be much needed. Denote by H_λ the hyperplane through o with normal vector $\lambda e_1 - (1-\lambda)e_2$ and set $T' = T^n \cap e_1^\perp$. Note that

$$(sT^n) \cap H_\lambda^+ = s\phi_\lambda T^n, \quad (sT^n) \cap H_\lambda^- = s\psi_\lambda T^n \quad \text{and} \quad (sT^n) \cap H_\lambda = s\phi_\lambda T'.$$

So the valuation property of Z implies

$$Z(sT^n) + Z(s\phi_\lambda T') = Z(s\phi_\lambda T^n) + Z(s\psi_\lambda T^n).$$

The $SL(n)$ contravariance of Z therefore gives

$$Z(sT^n) + \lambda^{1/n}\phi_\lambda^{-t} Z(s\lambda^{1/n}T')$$
$$= \lambda^{1/n}\phi_\lambda^{-t} Z(s\lambda^{1/n}T^n) + (1-\lambda)^{1/n}\psi_\lambda^{-t} Z(s(1-\lambda)^{1/n}T^n). \tag{49}$$

Let $\rho \in \mathrm{SL}(n)$ be defined by $\rho e_1 = e_2$, $\rho e_2 = -e_1$, and $\rho e_k = e_k$ for $k \geq 3$. Hence ρ is the counterclockwise rotation by an angle of $\pi/2$ in the plane spanned by the first two canonical basis vectors. For $g(\lambda) = \lambda$ and $h(\lambda) = 1 - \lambda$ we obtain

$$\rho \phi_\lambda^{-t} \rho^{-1} = \lambda^{-1} \phi_{\lambda,g} \qquad \text{and} \qquad \rho \psi_\lambda^{-t} \rho^{-1} = (1-\lambda)^{-1} \psi_{\lambda,h}. \tag{50}$$

Let $\hat{Z} : \mathcal{P}_o^n \to \mathcal{K}^n$ be defined by $\hat{Z}K = \rho\, Z\, K$. By (49) we therefore deduce

$$\hat{Z}(sT^n) + \lambda^{1/n-1} \phi_{\lambda,g} \hat{Z}(s\lambda^{1/n}T') =$$
$$\lambda^{1/n-1} \phi_{\lambda,g} \hat{Z}(s\lambda^{1/n}T^n) + (1-\lambda)^{1/n-1} \psi_{\lambda,h} \hat{Z}(s(1-\lambda)^{1/n}T^n). \tag{51}$$

After these preparations we are now in a position to prove our first main result.

6.2 The crucial classification

In this subsection we establish a theorem which has all main results on contravariant valuations stated in Sections 1 and 3 as consequences.

Theorem 7. *Let $n \geq 3$. Suppose that $Z : \mathcal{P}_o^n \to \langle \mathcal{K}^n, + \rangle$ is an $\mathrm{SL}(n)$ contravariant Minkowski valuation such that the two functions*

$$s \longmapsto h_{Z(sT^n)}(e_3) \qquad \text{and} \qquad s \longmapsto h_{Z(sT^n)}(-e_3), \qquad s > 0,$$

are bounded from below on some non-empty open intervals $I_+ \subset \mathbb{R}_+$ and $I_- \subset \mathbb{R}_+$, respectively. Then there exist constants c_1, c_2, c_3 with $c_1 \geq 0$ and $c_1 + c_2 + c_3 \geq 0$ such that

$$h_{Z\,P} = c_1 h_{\Pi P} + c_2 h_{\Pi_o P} + c_3 h_{\Pi_o(-P)}$$

for every $P \in \mathcal{P}_o^n$.

Proof. Define a function $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ by

$$f(s,x) = h_{\hat{Z}(sT^n)}(x) - h_{\hat{Z}(sT')}(x),$$

where as before $\hat{Z}K = \rho\, Z\, K$ and $T' = T^n \cap e_1^\perp$. Obviously we have $\psi_\lambda T' = T'$. By (50) this implies that

$$(1-\lambda)^{1/n-1} \psi_{\lambda,h} \hat{Z}(s(1-\lambda)^{1/n}T') = \hat{Z}(sT').$$

Hence we infer from (51) that f satisfies (24) for $p = 1/n - 1$ and $q = 1/n$, i.e.

$$f(s,x) = \lambda^{1/n-1} f\left(s\lambda^{1/n}, \phi_{\lambda,g}^t x\right) + (1-\lambda)^{1/n-1} f\left(s(1-\lambda)^{1/n}, \psi_{\lambda,h}^t x\right).$$

But f is positively homogeneous of degree one in x, and hence

$$f(s,\pm e_3) = \lambda^{1/n} f\left(s\lambda^{1/n}, \pm\phi_\lambda^t e_3\right) + (1-\lambda)^{1/n} f\left(s(1-\lambda)^{1/n}, \pm\psi_\lambda^t e_3\right).$$

In other words, the function f satisfies (25) at the points $\pm e_3$. Let $s > 0$. By Lemma 8 there exists a non-negative constant a such that

$$h_{\hat{Z}(sT')}(x) = as^{n-1}|x_2|, \qquad s > 0. \tag{52}$$

19

This and the assumed boundedness of the functions $s \longmapsto h_{Z(sT^n)}(\pm e_3)$ show that $f(\cdot, e_3)$ and $f(\cdot, -e_3)$ are bounded from below on some non-empty open intervals $I_+ \subset \mathbb{R}_+$ and $I_- \subset \mathbb{R}_+$, respectively. So Lemma 4 implies

$$f(s, e_3) = s^{n-1} f(1, e_3) \qquad \text{and} \qquad f(s, -e_3) = s^{n-1} f(1, -e_3).$$

Since $h_{\hat{Z}(sT')}(e_3) = 0$ by (52), we obtain

$$h_{\hat{Z}(sT^n)}(e_3) = f(s, e_3) = s^{n-1} f(1, e_3) = s^{n-1} h_{\hat{Z}T^n}(e_3).$$

Similarly, we deduce that $h_{\hat{Z}(sT^n)}(-e_3) = s^{n-1} h_{\hat{Z}T^n}(-e_3)$. Since both e_3 and $-e_3$ are fixpoints of ρ, we infer from (5) that

$$h_{Z(sT^n)}(e_3) = s^{n-1} h_{Z\,T^n}(e_3) \qquad \text{and} \qquad h_{Z(sT^n)}(-e_3) = s^{n-1} h_{Z\,T^n}(-e_3).$$

The SL(n) contravariance of Z proves

$$h_{Z(sT^n)}(\pi x) = h_{Z(sT^n)}(x), \qquad x \in \mathbb{R}^n, \tag{53}$$

for every map $\pi \in \mathrm{SL}(n)$ which is induced by a permutation matrix. Hence

$$h_{Z(sT^n)}(x) = s^{n-1} h_{Z\,T^n}(x) \qquad \text{for every } x \in \{\pm e_1, \pm e_2\}.$$

Since $\rho^t \{\pm e_1, \pm e_2\} = \{\pm e_1, \pm e_2\}$, we infer from (5) that for every $x \in \{\pm e_1, \pm e_2\}$

$$h_{\hat{Z}(sT^n)}(x) = h_{Z(sT^n)}(\rho^t x) = s^{n-1} h_{Z\,T^n}(\rho^t x) = s^{n-1} h_{\hat{Z}T^n}(x). \tag{54}$$

The equality (52) therefore implies

$$f(s, e_2) = h_{\hat{Z}(sT^n)}(e_2) - as^{n-1} = s^{n-1}[h_{\hat{Z}T^n}(e_2) - a] = s^{n-1} f(1, e_2).$$

Replacing e_2 by $-e_2$ in the above derivation proves $f(s, -e_2) = s^{n-1} f(1, -e_2)$. Moreover, since $f(s, \pm e_1) = h_{\hat{Z}(sT^n)}(\pm e_1)$ by (52), we obtain from (54) that $f(s, e_1) = s^{n-1} f(1, e_1)$ and $f(s, -e_1) = s^{n-1} f(1, -e_1)$. An application of Lemma 7 for $p = 1/n - 1$, $q = 1/n$, and $r = n - 1$ shows that for $x_1 > x_2 > 0$

$$\begin{aligned} f(s, x_1 e_1 + x_2 e_2) &= (x_1 - x_2)\, f(s, e_1), \\ f(s, -x_1 e_1 - x_2 e_2) &= (x_1 - x_2)\, f(s, -e_1), \end{aligned}$$

for $x_2 > x_1 > 0$

$$\begin{aligned} f(s, x_1 e_1 + x_2 e_2) &= (x_2 - x_1)\, f(s, e_2), \\ f(s, -x_1 e_1 - x_2 e_2) &= (x_2 - x_1)\, f(s, -e_2), \end{aligned}$$

and for $x_1, x_2 > 0$

$$\begin{aligned} f(s, -x_1 e_1 + x_2 e_2) &= x_2 f(s, e_2) + x_1 f(s, -e_1), \\ f(s, x_1 e_1 - x_2 e_2) &= x_1 f(s, e_1) + x_2 f(s, -e_2). \end{aligned}$$

20

The positive homogeneity of the functions $f(\cdot, \pm e_i)$, $i = 1, 2$, and the fact that

$$h_{Z(sT^n)}(x_1 e_1 + x_2 e_2) = f(s, -x_2 e_1 + x_1 e_2) + as^{n-1}|x_1|$$

proves for $x_1 > x_2 > 0$

$$
\begin{aligned}
h_{Z(sT^n)}(x_1 e_1 - x_2 e_2) &= s^{n-1}[x_1(f(1, e_2) + a) - x_2 f(1, e_2)], \\
h_{Z(sT^n)}(-x_1 e_1 + x_2 e_2) &= s^{n-1}[x_1(f(1, -e_2) + a) - x_2 f(1, -e_2)],
\end{aligned}
$$

for $x_2 > x_1 > 0$

$$
\begin{aligned}
h_{Z(sT^n)}(x_1 e_1 - x_2 e_2) &= s^{n-1}[x_1(a - f(1, e_1)) + x_2 f(1, e_1)], \\
h_{Z(sT^n)}(-x_1 e_1 + x_2 e_2) &= s^{n-1}[x_1(a - f(1, -e_1)) + x_2 f(1, -e_1)],
\end{aligned}
$$

and for $x_1, x_2 > 0$

$$
\begin{aligned}
h_{Z(sT^n)}(x_1 e_1 + x_2 e_2) &= s^{n-1}[x_1(a + f(1, e_2)) + x_2 f(1, -e_1)], \\
h_{Z(sT^n)}(-x_1 e_1 - x_2 e_2) &= s^{n-1}[x_1(a + f(1, -e_2)) + x_2 f(1, e_1)].
\end{aligned}
$$

Define a map $\phi \in \mathrm{SL}(n)$ with

$$\phi e_1 = e_3, \quad \phi e_2 = e_1, \quad \phi e_3 = e_2, \quad \phi e_i = e_i, \quad i = 4, \dots, n \quad \text{if } n > 3.$$

Then $\phi T^n = T^n$ and the contravariance of Z proves $h_{Z\,T^n}(e_1) = h_{Z\,T^n}(e_2)$ as well as $h_{Z\,T^n}(-e_1) = h_{Z\,T^n}(-e_2)$. Thus by the definition of f, (52), the definition of $\hat{Z}$, and (5) we arrive at

$$
\begin{aligned}
f(1, e_1) &= h_{\hat{Z}T^n}(e_1) = h_{Z\,T^n}(\rho^t e_1) = h_{Z\,T^n}(-e_2) = h_{Z\,T^n}(-e_1) = h_{\hat{Z}T^n}(-e_2) \\
&= f(1, -e_2) + a.
\end{aligned}
$$

Similarly, by replacing e_1 and e_2 by $-e_1$ and $-e_2$, respectively, in the above derivation one sees that $f(1, -e_1) = f(1, e_2) + a$. Define constants

$$
\begin{aligned}
c_1 &= (n-1)![f(1, e_2) + f(1, -e_2) + a], \\
c_2 &= -(n-1)!f(1, e_2), \\
c_3 &= -(n-1)!f(1, -e_2).
\end{aligned}
$$

Using formulas (14) and (15) as well as the positive homogeneity of degree $(n-1)$ of the operators Π and Π_o we see that

$$h_{Z(sT^n)}(x) = c_1 h_{\Pi(sT^n)}(x) + c_2 h_{\Pi_o(sT^n)}(x) + c_3 h_{\Pi_o(-sT^n)}(x), \quad x \in \mathrm{lin}\{e_1, e_2\}. \quad (55)$$

For all $s > 0$ and $x \in \mathbb{R}^n$ we have

$$h_{Z(sT')}(x) = as^{n-1}|x_1| = c_1 h_{\Pi(sT')}(x) + c_2 h_{\Pi_o(sT')}(x) + c_3 h_{\Pi_o(-sT')}(x).$$

Hence, by (49), the function $g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$g(s, x) = h_{Z(sT^n)}(x) - c_1 h_{\Pi(sT^n)}(x) - c_2 h_{\Pi_o(sT^n)}(x) - c_3 h_{\Pi_o(-sT^n)}(x)$$

satisfies (26). From (53), (55), and the $\mathrm{SL}(n)$ contravariance of the operators Π and Π_o we see that g satisfies the conditions of Lemma 6. Thus the equality (55) actually holds for all $x \in \mathbb{R}^n$. Lemma 3 implies that for all $P \in \mathcal{P}_o^n$

$$h_{\mathrm{Z}\,P} = c_1 h_{\Pi P} + c_2 h_{\Pi_o P} + c_3 h_{\Pi_o(-P)}. \tag{56}$$

Note that by the definition of the constants c_1, c_2, and c_3 and the non-negativity of a we have $c_1 + c_2 + c_3 = a(n-1)! \geq 0$. Let $P \in \mathcal{P}_o^n$ be chosen such that the origin is an interior point of P. Thus $h_{\mathrm{Z}\,P} = c_1 h_{\Pi P}$ and we conclude that $c_1 h_{\Pi P}$ is a support function. From (3) we infer

$$0 \leq c_1 [h_{\Pi P}(x) + h_{\Pi P}(y) - h_{\Pi P}(x + y)]$$

for all $x, y \in \mathbb{R}^n$. In particular, if we take $P = [-1, 1]^n$, $x = e_1$, and $y = -e_1$, then we get $c_1 \geq 0$. $\qquad\square$

6.3 Proofs of the main theorems

Using the previous result, we are now in a position to establish all theorems on $\mathrm{SL}(n)$ contravariant Minkowski valuations stated in Sections 1 and 3.

Proof of Theorem 4. Assume that $\mathrm{Z} : \mathcal{K}_o^n \to \mathcal{K}^n$ is a continuous $\mathrm{SL}(n)$ contravariant Minkowski valuation. The continuity of Z implies that the two functions

$$s \longmapsto h_{\mathrm{Z}(sT^n)}(e_3) \qquad \text{and} \qquad s \longmapsto h_{\mathrm{Z}(sT^n)}(-e_3), \qquad s > 0,$$

are continuous. Hence they are bounded from below on some non-empty open intervals $I_+ \subset \mathbb{R}_+$ and $I_- \subset \mathbb{R}_+$, respectively. Theorem 7 shows that there exist a non-negative constant c_1 as well as constants $c_2, c_3 \in \mathbb{R}$ with

$$h_{\mathrm{Z}\,P} = c_1 h_{\Pi P} + c_2 h_{\Pi_o P} + c_3 h_{\Pi_o(-P)}.$$

for every $P \in \mathcal{P}_o^n$. Since Z and Π are continuous but Π_o is not continuous at polytopes containing the origin on their boundaries, we have $h_{\mathrm{Z}\,P} = c_1 h_{\Pi P}$ for every $P \in \mathcal{P}_o^n$. The continuity of Z and Π as well as the fact that $\mathcal{P}_o^n$ is a dense subset of $\mathcal{K}_o^n$ prove

$$h_{\mathrm{Z}\,K} = c_1 h_{\Pi K}$$

for every $K \in \mathcal{K}_o^n$. $\qquad\square$

Proof of Theorem 3. Let $s > 0$. Since the origin is contained in $\mathrm{Z}(sT^n)$, we have $h_{\mathrm{Z}(sT^n)} \geq 0$. Hence the functions $h_{\mathrm{Z}(sT^n)}(\pm e_3)$ are bounded from below on $\mathbb{R}_+$. By Theorem 7 there exist a non-negative constant c_1 as well as constants $c_2, c_3 \in \mathbb{R}$ with $c_1 + c_2 + c_3 \geq 0$ and

$$h_{\mathrm{Z}\,P} = c_1 h_{\Pi P} + c_2 h_{\Pi_o P} + c_3 h_{\Pi_o(-P)}.$$

for every $P \in \mathcal{P}_o^n$. $\qquad\square$

Proof of Theorem 1. Theorem 1 is an immediate consequence of Theorem 4. $\qquad\square$

7 The Covariant Case

7.1 Preliminaries

Let us collect some basic facts about covariant operators. The following statement is an immediate consequence of the definition of covariance and (5).

Lemma 9. *Let $n \geq 3$, $1 \leq k < n$. For $i = 1, \ldots, m$ let $c_i \in \mathbb{R}$ and suppose that $Z_i : \mathcal{P}_o^n \to \langle \mathcal{K}^n, + \rangle$ are $\mathrm{SL}(n)$ covariant maps. If $\sum_{i=1}^m c_i h_{Z_i P} = 0$ for every k-dimensional convex polytope $P \in \mathcal{P}_o^n$ which is contained in $\mathrm{lin}\{e_1, \ldots, e_k\}$, then $\sum_{i=1}^m c_i h_{Z_i P} = 0$ for every k-dimensional convex polytope $P \in \mathcal{P}_o^n$.*

The next Lemma is due to Ludwig [38]. For the sake of completeness we include a proof here.

Lemma 10. *Let $n \geq 3$. If $Z : \mathcal{P}_o^n \to \mathcal{K}^n$ is $\mathrm{SL}(n)$ covariant, then $Z P \subset \mathrm{lin}\, P$ for every $P \in \mathcal{P}_o^n$.*

Proof. The statement is trivial if P is n-dimensional. Next, assume that $\dim P = 0$. Hence $P = \{o\}$. Define a map $\phi \in \mathrm{SL}(n)$ by

$$\phi e_1 = 2e_1, \quad \phi e_2 = 2^{-1} e_2, \quad \phi e_k = e_k, \quad 3 \leq k \leq n.$$

Clearly, $\phi P = P$ and so the covariance of Z together with (5) and the fact that support functions are positively homogeneous of degree one yield

$$h_{Z P}(e_1) = h_{Z \phi P}(e_1) = h_{\phi Z P}(e_1) = h_{Z P}(\phi^t e_1) = 2 h_{Z P}(e_1).$$

Hence $h_{Z P}(e_1) = 0$. For each unit vector $u \in S^{n-1}$ it is possible to find a rotation $\vartheta \in \mathrm{SL}(n)$ such that $u = \vartheta e_1$. This, (5), the covariance of Z, and the obvious equality $\vartheta^t P = P$ imply

$$h_{Z P}(u) = h_{Z P}(\vartheta e_1) = h_{\vartheta^t Z P}(e_1) = h_{Z \vartheta^t P}(e_1) = h_{Z P}(e_1) = 0.$$

Since support functions are positively homogeneous of degree one we conclude that $h_{Z P} = 0$. This proves $Z P = \{o\}$ and therefore settles the 0-dimensional case of our lemma.

So let us finally assume that $0 < \dim P = k < n$ and, without loss of generality, that $P \subset \mathrm{lin}\{e_1, \ldots, e_k\}$. For arbitrary $s > 0$, let $\phi \in \mathrm{SL}(n)$ be defined by

$$\phi e_i = e_i, \quad i = 1, \ldots k, \qquad \text{and} \qquad \phi e_i = s e_{i-k} + e_i, \quad i = k+1, \ldots n.$$

Clearly, $\phi P = P$. If $x \in Z P$, then the covariance of Z therefore implies that also $\phi x \in Z P$. Since $Z P$ is bounded and s can be arbitrarily large we conclude that $x_{k+1} = \cdots = x_n = 0$. $\square$

Corollary 2. *Suppose that $n \geq 3$ and $Z : \mathcal{P}_o^n \to \mathcal{K}^n$ is $\mathrm{SL}(n)$ covariant. If $0 < k < n$ and P is contained in $\mathrm{lin}\{e_1, \ldots, e_k\}$, then*

$$Z(sP) = s\, Z P \tag{57}$$

for any positive real number s. If $1 < k < n$ and $P \subset \mathrm{lin}\{e_1, \ldots, e_k\}$, then

$$Z(\phi_\lambda P) = \phi_\lambda\, Z P \qquad \text{and} \qquad Z(\psi_\lambda P) = \psi_\lambda\, Z P. \tag{58}$$

Proof. For $s > 0$ define a map $\phi \in \mathrm{SL}(n)$ by

$$\phi e_i = s e_i \quad i = 1, \ldots, n-1, \qquad \text{and} \qquad \phi e_n = s^{1-n} e_n.$$

Clearly, $\phi P = sP$. Lemma 10 shows that also $\phi \mathrm{Z} P = s \mathrm{Z} P$. From the $\mathrm{SL}(n)$ covariance of Z we deduce (57).

The $\mathrm{SL}(n)$ covariance of Z together with (57) proves that $\mathrm{Z}(\phi P) = \phi \mathrm{Z} P$ for every $P \subset \mathrm{lin}\{e_1, \ldots, e_k\}$ and all invertible linear transformation ϕ with positive determinant which fix $\mathrm{lin}\{e_1, \ldots, e_k\}$. Hence (58) holds. $\qquad\square$

Assume that $\mathrm{Z} : \mathcal{P}_o^n \to \langle \mathcal{K}^n, + \rangle$ is an $\mathrm{SL}(n)$ covariant Minkowski valuation. Now, we are going to derive some functional equations for support functions of the covariant Minkowski valuation Z. These equations are closely related to those treated in Section 5 and will be much needed. Let s be a positive real number, $\lambda \in (0,1)$, and denote by H_λ the hyperplane passing through o with normal vector $\lambda e_1 - (1-\lambda)e_2$. Then we have

$$(sT^n) \cap H_\lambda^+ = s\phi_\lambda T^n, \quad (sT^n) \cap H_\lambda^- = s\psi_\lambda T^n \quad \text{and} \quad (sT^n) \cap H_\lambda = s\phi_\lambda T',$$

where $T' = T^n \cap e_1^\perp$. So the valuation property of Z implies

$$\mathrm{Z}(sT^n) + \mathrm{Z}(s\phi_\lambda T') = \mathrm{Z}(s\phi_\lambda T^n) + \mathrm{Z}(s\psi_\lambda T^n). \tag{59}$$

The $\mathrm{SL}(n)$ covariance of Z therefore gives

$$\begin{aligned} \mathrm{Z}(sT^n) &+ \lambda^{-1/n}\phi_\lambda \, \mathrm{Z}(s\lambda^{1/n}T') \\ &= \lambda^{-1/n}\phi_\lambda \, \mathrm{Z}(s\lambda^{1/n}T^n) + (1-\lambda)^{-1/n}\psi_\lambda \, \mathrm{Z}(s(1-\lambda)^{1/n}T^n). \end{aligned} \tag{60}$$

So from (4) and (5) we obtain

$$\begin{aligned} h_{\mathrm{Z}(sT^n)}(x) &+ \lambda^{-1/n}h_{\mathrm{Z}(s\lambda^{1/n}T')}(\phi_\lambda^t x) \\ &= \lambda^{-1/n}h_{\mathrm{Z}(s\lambda^{1/n}T^n)}(\phi_\lambda^t x) + (1-\lambda)^{-1/n}h_{\mathrm{Z}(s(1-\lambda)^{1/n}T^n)}(\psi_\lambda^t x). \end{aligned} \tag{61}$$

An $\mathrm{SL}(n)$ covariant Minkowski valuation $\mathrm{Z} : \mathcal{P}_o^n \to \langle \mathcal{K}^n, + \rangle$ gives rise to the following valuation $\hat{\mathrm{Z}}$. For $n \geq 3$ and $1 < k < n$ we can define, by Lemma 10, a map $\hat{\mathrm{Z}} : \mathcal{P}_o^k \to \langle \mathcal{K}^k, + \rangle$ by

$$\hat{\mathrm{Z}}P = \pi_k\left(\mathrm{Z}(\pi_k^{-1}P)\right),$$

where $\pi_k : \mathrm{lin}\{e_1, \ldots, e_k\} \to \mathbb{R}^k$ denotes the projection of a vector to its first k coordinates with respect to the standard basis vectors. Note that π_k is a linear bijection. It is easily seen that $\hat{\mathrm{Z}}$ is in fact an $\mathrm{SL}(k)$ covariant Minkowski valuation. We also remark that for $P \subset \mathrm{lin}\{e_1, \ldots, e_k\}$ we infer from (6) and Lemma 10 that

$$h_{\mathrm{Z}\,P}(x) = h_{\hat{\mathrm{Z}}(\pi_k P)}(\pi_k(x_1 e_1 + \cdots + x_k e_k)), \qquad x \in \mathbb{R}^n. \tag{62}$$

Moreover, Corollary 2 shows that for $P \in \mathcal{P}_o^k$

$$\hat{\mathrm{Z}}(\phi_\lambda P) = \phi_\lambda \hat{\mathrm{Z}} P \qquad \text{and} \qquad \hat{\mathrm{Z}}(\psi_\lambda P) = \psi_\lambda \hat{\mathrm{Z}} P, \tag{63}$$

as well as

$$\hat{\mathrm{Z}}(sP) = s\hat{\mathrm{Z}} P \tag{64}$$

for any positive real number s. In particular, we get from (4), (5), (59), and (63)

$$h_{\hat{\mathrm{Z}}(sT^k)}(x) + h_{\hat{\mathrm{Z}}(sT')}(\phi_\lambda^t x) = h_{\hat{\mathrm{Z}}(sT^k)}(\phi_\lambda^t x) + h_{\hat{\mathrm{Z}}(sT^k)}(\psi_\lambda^t x). \tag{65}$$

7.2 The crucial classification

In this subsection we establish a theorem which has all main results on covariant valuations stated in Sections 1 and 3 as consequences. We start by clarifying the behavior of $\mathrm{SL}(n)$ covariant Minkowski valuations on lower dimensional sets.

Lemma 11. *Suppose that $n \geq 3$ and $\mathrm{Z} : \mathcal{P}_o^n \to \langle \mathcal{K}^n, + \rangle$ is an $\mathrm{SL}(n)$ covariant Minkowski valuation which is continuous at the line segment $[o, e_1]$. Then there exist non-negative constants c_1 and c_2 such that for every convex polytope $P \in \mathcal{P}_o^n$ of dimension less than n one has*

$$\mathrm{Z}\, P = c_1 P + c_2(-P).$$

Proof. From Lemma 10 we know that there exists constants c_1 and c_2 such that $-c_2 \leq c_1$ and $\mathrm{Z}[o, e_1] = [-c_2 e_1, c_1 e_1]$. By (4) it suffices to prove that the constants c_1 and c_2 are non-negative and

$$h_{\mathrm{Z}\, P} - c_1 h_P - c_2 h_{-P} = 0 \qquad \text{for every } P \in \mathcal{P}_o^n \text{ with } \dim P < n. \qquad (66)$$

By induction on the dimension, we are now going to show that (66) holds. If $\dim P = 0$, then $\mathrm{Z}\, P = \{o\}$ by Lemma 10 and hence (66) obviously holds. It easily follows from (7) that

$$h_{[-c_2 e_1, c_1 e_1]} = c_1 h_{[o, e_1]} + c_2 h_{-[o, e_1]}.$$

Hence, by the definition of the constants c_1 and c_2, equation (66) also holds for the segment $[o, e_1]$. Next, let P be a line segment of the form $[o, x]$ for some non-zero $x \in \mathbb{R}^n$. Let $\vartheta \in \mathrm{SL}(n)$ be a rotation with $x = \|x\| \vartheta e_1$. Then the covariance of Z, (57), (5), and the already established equality (66) for $[o, e_1]$ give

$$h_{\mathrm{Z}[o,x]} = \|x\| h_{\mathrm{Z}[o,e_1]} \circ \vartheta^t = \|x\| (c_1 h_{[o,e_1]} + c_2 h_{-[o,e_1]}) \circ \vartheta^t = c_1 h_{[o,x]} + c_2 h_{-[o,x]}. \quad (67)$$

More generally, let $P \in \mathcal{P}_o^n$ be of the form $P = [-ax, bx]$ for some non-zero $x \in \mathbb{R}^n$ and positive constants a and b. The valuation property, the fact that $\mathrm{Z}\{o\} = \{o\}$ (which follows from Lemma 10), and (67) give

$$h_{\mathrm{Z}\, P} = h_{\mathrm{Z}[o,-ax]} + h_{\mathrm{Z}[o,bx]} = c_1 h_P + c_2 h_{-P}.$$

Thus (66) holds for 1-dimensional bodies.

Let $2 \leq k \leq n - 1$ and assume that (66) holds for $(k-1)$-dimensional bodies. As before, define $\hat{\mathrm{Z}} : \mathcal{P}_o^k \to \langle \mathcal{K}^k, + \rangle$ by $\hat{\mathrm{Z}} P = \pi_k \left(\mathrm{Z}(\pi_k^{-1} P) \right)$. Recall that $\hat{\mathrm{Z}}$ is an $\mathrm{SL}(k)$ covariant Minkowski valuation. Note that $\hat{\mathrm{Z}}$ is also continuous at the line segment $[o, e_1]$. Moreover, since we assume that (66) holds for $(k-1)$-dimensional polytopes, we have

$$h_{\hat{\mathrm{Z}} P} - c_1 h_P - c_2 h_{-P} = 0 \qquad \text{for every } P \in \mathcal{P}_o^k \text{ with } \dim P < k. \qquad (68)$$

Define a function $f : \mathbb{R}_+ \times \mathbb{R}^k \to \mathbb{R}$ by

$$f(s, x) = h_{\hat{\mathrm{Z}}(sT^k)}(x) - c_1 h_{sT^k}(x) - c_2 h_{-sT^k}(x).$$

Let ϕ be a linear map with

$$\phi e_1 = e_2, \quad \phi e_2 = e_1, \quad \phi e_j = e_j \quad \text{if } k \geq 3 \text{ and } 3 \leq j \leq k.$$

Clearly we have $\phi^t = \phi$ and $\phi T^k = T^k$. Recall that n is assumed to be greater or equal than three and k is less than n. The $SL(n)$ covariance of Z therefore implies $\hat{Z}(\phi T^k) = \phi \hat{Z} T^k$ and we obtain from (5) that

$$f(1, e_1) = f(1, e_2) \qquad \text{and} \qquad f(1, -e_1) = f(1, -e_2). \tag{69}$$

From (64) and relation (4) one immediately deduces

$$f(s, x) = s f(1, x) \qquad \text{for every} \qquad (s, x) \in \mathbb{R}_+ \times \mathbb{R}^k. \tag{70}$$

Note that equation (65) holds in dimension k for $\hat{Z}$, c_1 times the identity as well as c_2 times the reflection at the origin. Subtracting the respective equalities and using (68) therefore gives for all $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^k$

$$f(s, x) = f(s, \phi_\lambda^t x) + f(s, \psi_\lambda^t x). \tag{71}$$

Thus f satisfies the functional equation (25) for $p = q = 0$. By appealing to (70) we can apply Lemma 7 for $r = 1$ and conclude that for $x_1 > x_2 > 0$

$$\begin{aligned}
f(s, x_1 e_1 + x_2 e_2) &= s(x_1 - x_2) f(1, e_1), \\
f(s, -x_1 e_1 - x_2 e_2) &= s(x_1 - x_2) f(1, -e_1),
\end{aligned}$$

for $x_2 > x_1 > 0$

$$\begin{aligned}
f(s, x_1 e_1 + x_2 e_2) &= s(x_2 - x_1) f(1, e_2), \\
f(s, -x_1 e_1 - x_2 e_2) &= s(x_2 - x_1) f(1, -e_2),
\end{aligned}$$

and for $x_1, x_2 > 0$

$$\begin{aligned}
f(s, -x_1 e_1 + x_2 e_2) &= s\left[x_2 f(1, e_2) + x_1 f(1, -e_1)\right], \\
f(s, x_1 e_1 - x_2 e_2) &= s\left[x_1 f(1, e_1) + x_2 f(1, -e_2)\right].
\end{aligned}$$

Define constants
$$a = f(1, e_1) \qquad \text{and} \qquad b = f(1, -e_1).$$

Using these definitions, the symmetry relation (69), and the continuity of f, it is readily verified that for all $x_1, x_2 \in \mathbb{R}$

$$\begin{aligned}
f(1, x_1 e_1 + x_2 e_2) &= a[2\max\{(x_1)_+, (x_2)_+\} - (x_1)_+ - (x_2)_+] \\
&\quad + b[2\max\{(x_1)_-, (x_2)_-\} - (x_1)_- - (x_2)_-].
\end{aligned}$$

Relations (7), (8), and the definition of E imply for $x \in \lin\{e_1, e_2\}$

$$f(1, x) = 2ah_{T^2}(x) + 2bh_{-T^2}(x) - ah_{\mathrm{E}\,T^2}(x) - bh_{\mathrm{E}(-T^2)}(x).$$

26

Assume first that $k = 2$. By the definition of f, (70), and the homogeneity of E we have

$$h_{\hat{Z}(sT^2)} = (2a + c_1)h_{sT^2} + (2b + c_2)h_{-sT^2} - ah_{E(sT^2)} - bh_{E(-sT^2)}$$

From Lemma 3 we therefore know that for all $P \in \mathcal{P}_o^2$

$$h_{\hat{Z}P} = (2a + c_1)h_P + (2b + c_2)h_{-P} - ah_{E\,P} - bh_{E(-P)}. \tag{72}$$

For sufficiently small $\varepsilon > 0$ define $P_\varepsilon \in \mathcal{P}_o^2$ as $P_\varepsilon = [o, e_1] - \varepsilon e_1 + \varepsilon[-e_2, e_2]$. Note that $\lim_{\varepsilon \to 0} P_\varepsilon = [o, e_1]$. From (72) we infer by the continuity of $\hat{Z}$ at $[o, e_1]$ and the fact that for every ε the origin is an interior point of P_ε that

$$
\begin{aligned}
c_1 &= h_{\hat{Z}[o,e_1]}(e_1) = \lim_{\varepsilon \to 0^+} h_{\hat{Z}P_\varepsilon}(e_1) \\
&= \lim_{\varepsilon \to 0^+} ((2a + c_1)h_{P_\varepsilon}(e_1) + (2b + c_2)h_{-P_\varepsilon}(e_1)) \\
&= 2a + c_1.
\end{aligned}
$$

This shows that $a = 0$. By performing the same computation as before but for $-e_1$ instead of e_1, one obtains that $b = 0$. Thus $h_{\hat{Z}P} = c_1h_P + c_2h_{-P}$. From (62), we obtain that $h_{Z\,P} = c_1h_P + c_2h_{-P}$ for all $P \subset \operatorname{lin}\{e_1, e_2\}$. Now Lemma 9 proves that (66) holds for all 2-dimensional polytopes.

Assume now that $k \geq 3$. Since $\pm e_3$ are fixpoints of both ϕ_λ and ψ_λ, equation (71) immediately shows that $f(s, \pm e_3) = 0$ for all $s > 0$. Define $\phi \in \mathrm{SL}(k)$ by

$$\phi e_1 = e_2 \quad \phi e_2 = e_3, \quad \phi e_3 = e_1, \quad \phi e_l = e_l \text{ for } 3 < l \leq k \text{ if } k > 3.$$

Since $\phi^t T^k = T^k$ we obtain from (5) and the covariance of $\hat{Z}$ that

$$
\begin{aligned}
f(s, e_1) &= f(s, \phi e_3) \\
&= h_{\hat{Z}(sT^k)}(\phi e_3) - c_1h_{sT^k}(\phi e_3) - c_2h_{-sT^k}(\phi e_3) \\
&= h_{\hat{Z}(sT^k)}(e_3) - c_1h_{sT^k}(e_3) - c_2h_{-sT^k}(e_3) \\
&= f(s, e_3) \\
&= 0.
\end{aligned}
$$

If we replace in the last argument e_1 and e_3 by $-e_1$ and $-e_3$, respectively, then we see that $f(s, -e_1) = 0$. From (69) we conclude that also $f(s, \pm e_2) = 0$ for all $s > 0$. Thus $f(s, x) = 0$ for every $x \in \operatorname{lin}\{e_1, e_2\}$. Since $f(s, \pi x) = f(s, x)$ for every $\pi \in \mathrm{SL}(k)$ which is induced by a permutation matrix, Lemma 5 implies that $f(s, x) = 0$ for every $x \in \mathbb{R}^k$. Hence $h_{\hat{Z}(sT^k)} = c_1h_{sT^k} + c_2h_{-sT^k}$ for all positive s. From Lemma 3 we therefore know that $h_{\hat{Z}P} = c_1h_P + c_2h_{-P}$. As above, one deduces that (66) holds for all k-dimensional polytopes.

It remains to show that the constants c_1 and c_2 are non-negative. By (66), $c_1h_P + c_2h_{-P}$ has to be a support function for each $P \in \mathcal{P}_o^n$ with dimension less than n. We infer from (3) that

$$0 \leq c_1[h_P(x) + h_P(y) - h_P(x + y)] + c_2[h_{-P}(x) + h_{-P}(y) - h_{-P}(x + y)]$$

for all $x, y \in \mathbb{R}^n$. Evaluate this inequality at the 2-dimensional standard simplex $T^2 = \operatorname{conv}\{o, e_1, e_2\}$, $x = e_1$, and $y = e_2$. Then (6) and (8) immediately imply the non-negativity of c_1. Similarly, by looking at $P = -T^2$, $x = e_1$, and $y = e_2$, one sees that also c_2 has to be nonnegative. $\qquad \square$

We are now in a position to establish the main classification for $\mathrm{SL}(n)$ covariant Minkowski valuations.

Theorem 8. *Let $n \geq 3$. Suppose that $\mathrm{Z} : \mathcal{P}_o^n \to \langle \mathcal{K}^n, + \rangle$ is an $\mathrm{SL}(n)$ covariant Minkowski valuation which is continuous at the line segment $[o, e_1]$ and such that the two functions*

$$s \longmapsto h_{\mathrm{Z}(sT^n)}(e_3) \qquad \text{and} \qquad s \longmapsto h_{\mathrm{Z}(sT^n)}(-e_3), \qquad s > 0,$$

are bounded from below on some non-empty open intervals $I_+ \subset \mathbb{R}_+$ and $I_- \subset \mathbb{R}_+$, respectively. Then there exist non-negative constants c_1 and c_2 as well as constants $c_3, c_4 \in \mathbb{R}$ such that

$$h_{\mathrm{Z}\,P} = c_1 h_P + c_2 h_{-P} + c_3 h_{\Gamma_+ P} + c_4 h_{\Gamma_+(-P)}$$

for every $P \in \mathcal{P}_o^n$.

Proof. Let c_1 and c_2 be the constants from Lemma 11. For $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^n$ set

$$f(s, x) = h_{\mathrm{Z}(sT^n)}(x) - c_1 h_{sT^n}(x) - c_2 h_{-sT^n}(x).$$

By assumption, the functions $f(\cdot, e_3)$ and $f(\cdot, -e_3)$ are bounded from below on some non-empty open intervals $I_+ \subset \mathbb{R}_+$ and $I_- \subset \mathbb{R}_+$, respectively. Note that for every map $\pi \in \mathrm{SL}(n)$ which is induced by a permutation matrix we have

$$f(s, \pi x) = f(s, x). \tag{73}$$

Equation (61) is true for Z, c_1 times the identity as well as c_2 times the reflection at the origin. Subtracting the respective equalities and using Lemma 11 therefore gives

$$f(s, x) = \lambda^{-\frac{1}{n}} f(s\lambda^{\frac{1}{n}}, \phi_\lambda^t x) + (1 - \lambda)^{-\frac{1}{n}} f(s(1 - \lambda)^{\frac{1}{n}}, \psi_\lambda^t x).$$

From Corollary 1 we deduce that $f(s, \pm e_i) = s^{n+1} f(1, \pm e_i)$. An application of Lemma 7 for $p = -1/n$, $q = 1/n$, and $r = n + 1$ yields for $x_1 > x_2 > 0$

$$\begin{aligned}
f(s, x_1 e_1 + x_2 e_2) &= (x_1 + x_2) f(s, e_1), \\
f(s, -x_1 e_1 - x_2 e_2) &= (x_1 + x_2) f(s, -e_1),
\end{aligned}$$

for $x_2 > x_1 > 0$

$$\begin{aligned}
f(s, x_1 e_1 + x_2 e_2) &= (x_1 + x_2) f(s, e_2), \\
f(s, -x_1 e_1 - x_2 e_2) &= (x_1 + x_2) f(s, -e_2),
\end{aligned}$$

and for $x_1, x_2 > 0$

$$\begin{aligned}
f(s, -x_1 e_1 + x_2 e_2) &= (x_1 + x_2)^{-1} \left(x_2^2 f(s, e_2) + x_1^2 f(s, -e_1) \right), \\
f(s, x_1 e_1 - x_2 e_2) &= (x_1 + x_2)^{-1} \left(x_1^2 f(s, e_1) + x_2^2 f(s, -e_2) \right).
\end{aligned}$$

Being the sum of continuous functions, f is continuous in x for each fixed s. This continuity at $xe_1 + xe_2$ immediately shows

$$f(s, e_1) = f(s, e_2) \qquad \text{and} \qquad f(s, -e_1) = f(s, -e_2).$$

Define constants c_3 and c_4 by

$$c_3 = (n+1)! f(1, e_1) \qquad \text{and} \qquad c_4 = (n+1)! f(1, -e_1).$$

An elementary calculation proves

$$f(s, x_1 e_1 + x_2 e_2) = c_3 h_{\Gamma_+(sT^n)}(x_1 e_1 + x_2 e_2) + c_4 h_{\Gamma_+(-sT^n)}(x_1 e_1 + x_2 e_2).$$

By what we already proved, the function f satisfies the assumptions of Lemma 5 but also $c_3 h_{\Gamma_+(sT^n)}(x) + c_4 h_{\Gamma_+(-sT^n)}(x)$ does. Applying Lemma 5 to the difference of these two functions shows

$$f(s, x) = c_3 h_{\Gamma_+(sT^n)}(x) + c_4 h_{\Gamma_+(-sT^n)}(x)$$

for every $x \in \mathbb{R}^n$. The definition of f gives

$$h_{\mathrm{Z}(sT^n)} = c_1 h_{sT^n} + c_2 h_{-sT^n} + c_3 h_{\Gamma^+(sT^n)} + c_4 h_{\Gamma^+(-sT^n)}$$

for all positive s. From Lemma 3 we therefore know that

$$h_{\mathrm{Z}P} = c_1 h_P + c_2 h_{-P} + c_3 h_{\Gamma_+ P} + c_4 h_{\Gamma_+(-P)}.$$

$\square$

7.3 Proofs of the main theorems

Using the results of the previous subsection, we are now in a position to establish all theorems on $\mathrm{SL}(n)$ covariant Minkowski valuations stated in Sections 1 and 3.

Proof of Theorem 6. Clearly, the identity, the reflection at the origin, and the asymmetric centroid body operator are continuous $\mathrm{SL}(n)$ covariant Minkowski valuations. Assume that $\mathrm{Z} : \mathcal{K}_o^n \to \mathcal{K}^n$ is a continuous $\mathrm{SL}(n)$ covariant Minkowski valuation. The continuity of Z implies that the two functions

$$s \longmapsto h_{\mathrm{Z}(sT^n)}(e_3) \qquad \text{and} \qquad s \longmapsto h_{\mathrm{Z}(sT^n)}(-e_3), \qquad s > 0,$$

are continuous. Hence they are bounded from below on some non-empty open intervals $I_+ \subset \mathbb{R}_+$ and $I_- \subset \mathbb{R}_+$, respectively. Theorem 8 together with the continuity of Z and the fact that $\mathcal{P}_o^n$ is a dense subset of $\mathcal{K}_o^n$ show that there exist non-negative constants c_1 and c_2 as well as constants $\hat{c}_3, \hat{c}_4 \in \mathbb{R}$ with

$$h_{\mathrm{Z}K} = c_1 h_K + c_2 h_{-K} + \hat{c}_3 h_{\Gamma_+ K} + \hat{c}_4 h_{\Gamma_+(-K)}$$

for every $K \in \mathcal{K}_o^n$. Set $c_3 = (\hat{c}_3 + \hat{c}_4)/2$ and $c_4 = (\hat{c}_3 - \hat{c}_4)/2$. Then it is easy to see that

$$\hat{c}_3 h_{\Gamma_+ K} + \hat{c}_4 h_{\Gamma_+(-K)} = c_3 h_{\Gamma K} + c_4 h_{\mathrm{m}(K)}.$$

Consequently, for all $K \in \mathcal{K}_o^n$ we have

$$h_{Z\,K} = c_1 h_K + c_2 h_{-K} + c_3 h_{\Gamma K} + c_4 h_{\mathrm{m}(K)}.$$

It remains to show that c_3 is non-negative. Suppose that K is origin-symmetric. Then $\mathrm{m}(K) = o$ and the positive homogeneity of the symmetric centroid body operator implies for $s > 0$ that

$$s^{-n-1} h_{Z(sK)} = s^{-n}[c_1 h_K + c_2 h_{-K}] + c_3 h_{\Gamma K}.$$

Thus for every positive s, the function $s^{-n}[c_1 h_K + c_2 h_{-K}] + c_3 h_{\Gamma K}$ is a support function and hence sublinear. The pointwise limit of sublinear functions is sublinear. So if s tends to infinity we deduce that $c_3 h_{\Gamma K}$ is sublinear, i.e.

$$0 \leq c_3 [h_{\Gamma K}(x) + h_{\Gamma K}(y) - h_{\Gamma K}(x + y)]$$

for all $x, y \in \mathbb{R}^n$. In particular, for $K = [-1, 1]^n$, $x = e_1$, and $y = -e_1$, it immediately follows that $c_3 \geq 0$. $\qquad\square$

Proof of Theorem 5. Let $s > 0$. Since the origin is contained in $Z(sT^n)$, we have $h_{Z(sT^n)} \geq 0$. Hence the functions $h_{Z(sT^n)}(\pm e_3)$ are bounded from below on $\mathbb{R}_+$. By Theorem 8 there exist non-negative constants c_1 and c_2 as well as constants $c_3, c_4 \in \mathbb{R}$ with

$$h_{Z\,P} = c_1 h_P + c_2 h_{-P} + c_3 h_{\Gamma_+ P} + c_4 h_{\Gamma_+(-P)}$$

for every $P \in \mathcal{P}_o^n$. It remains to show that the constants c_3 and c_4 are non-negative. Let $s > 0$. Since the origin is contained in $Z(sT^n)$, evaluating the last equation at sT^n and e_1 gives

$$0 \leq h_{Z(sT^n)}(e_1) = s c_1 h_{T^n}(e_1) + s^{n+1} c_3 h_{\Gamma_+ T^n}(e_1).$$

Since $h_{\Gamma_+ T^n}(e_1) > 0$, taking the limit $s \to \infty$ in $0 \leq s^{-n} c_1 + c_3 h_{\Gamma_+ T^n}(e_1)$ proves $c_3 \geq 0$. Similarly, by looking at $-e_1$ instead of e_1, we get $c_4 \geq 0$. $\qquad\square$

Proof of Theorem 2. Theorem 2 is an immediate consequence of Theorem 5 and the denseness of the set P_o^n in $\mathcal{K}_o^n$. $\qquad\square$

References

[1] Alesker S., Continuous rotation invariant valuations on convex sets. *Ann. of Math. (2)* **149** (1999), 977–1005.

[2] Alesker S., Description of translation invariant valuations on convex sets with solution of P. McMullen's conjecture. *Geom. Funct. Anal.* **11** (2001), 244–272.

[3] Alesker S., Hard Lefschetz theorem for valuations, complex integral geometry, and unitarily invariant valuations. *J. Differential Geom.* **63** (2003), 63–95.

[4] Alesker S., The multiplicative structure on continuous polynomial valuations. *Geom. Funct. Anal.* **14** (2004), 1–26.

[5] Alesker S., Theory of valuations on manifolds. I. Linear spaces. *Israel J. Math.* **156** (2006), 311–339.

[6] Alesker S., Theory of valuations on manifolds. II. *Adv. Math.* **207** (2006), 420–454.

[7] Alesker S., Theory of valuations on manifolds. IV. New properties of the multiplicative structure. In *Geometric aspects of functional analysis*, volume 1910 of *Lecture Notes in Math.* Springer, Berlin, 2007, 1–44.

[8] Alesker S. and Bernstein J., Range characterization of the cosine transform on higher Grassmannians. *Adv. Math.* **184** (2004), 367–379.

[9] Alesker S. and Fu J. H. G., Theory of valuations on manifolds. III. Multiplicative structure in the general case. *Trans. Amer. Math. Soc.* **360** (2008), 1951–1981.

[10] Bernig A., Valuations with Crofton formula and Finsler geometry. *Adv. Math.* **210** (2007), 733–753.

[11] Bernig A., A Hadwiger-type theorem for the special unitary group. *Geom. Funct. Anal.* **19** (2009), 356–372.

[12] Bernig A., A product formula for valuations on manifolds with applications to the integral geometry of the quaternionic line. *Comment. Math. Helv.* **84** (2009), 1–19.

[13] Bernig A. and Bröcker L., Valuations on manifolds and Rumin cohomology. *J. Differential Geom.* **75** (2007), 433–457.

[14] Bernig A. and Fu J. H. G., Convolution of convex valuations. *Geom. Dedicata* **123** (2006), 153–169.

[15] Bernig A. and Fu J. H. G., Hermitian integral geometry. *Ann. of Math (2)*, in press.

[16] Campi S. and Gronchi P., The L^p-Busemann-Petty centroid inequality. *Adv. Math.* **167** (2002), 128–141.

[17] Cianchi A., Lutwak E., Yang D. and Zhang G., Affine Moser-Trudinger and Morrey-Sobolev inequalities. *Calc. Var. Partial Differential Equations* **36** (2009), 419–436.

[18] Dafnis N. and Paouris G., Small ball probability estimates, ψ_2-behavior and the hyperplane conjecture. *J. Funct. Anal.* **258** (2010), 1933–1964.

[19] Fleury B., Guedon O. and Paouris G., A stability result for mean width of L_p-centroid bodies. *Adv. Math.* **214** (2007), 865–877.

[20] Fu J. H. G., Structure of the unitary valuation algebra. *J. Differential Geom.* **72** (2006), 509–533.

[21] Gardner R. J., *Geometric tomography*, volume 58 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, Cambridge, second edition, 2006.

[22] Gruber P., *Convex and Discrete Geometry.* Springer, Berlin, 2007.

[23] Haberl C., L_p intersection bodies. *Adv. Math.* **217** (2008), 2599–2624.

[24] Haberl C., Star body valued valuations. *Indiana Univ. Math. J.* **58** (2009), 2253–2276.

[25] Haberl C., Blaschke valuations. *Amer. J. Math.* , **133** (2011), 717–751.

[26] Haberl C. and Ludwig M., A characterization of L_p intersection bodies. *Int. Math. Res. Not.* (2006), Article ID 10548, 29 pages.

[27] Haberl C. and Schuster F.E., Asymmetric affine L_p Sobolev inequalities. *J. Funct. Anal.* **257** (2009), 641–658.

[28] Haberl C. and Schuster F.E., General L_p affine isoperimetric inequalities. *J. Differential Geom.* **83** (2009), 1–26.

[29] Hug D., Schneider R. and Schuster R., The space of isometry covariant tensor valuations. *Algebra i Analiz* **19** (2007), 194–224.

[30] Kiderlen M. and Weil W., Measure-valued valuations and mixed curvature measures of convex bodies. *Geom. Dedicata* **76** (1999), 291–329.

[31] Klain D. A., Star valuations and dual mixed volumes. *Adv. Math.* **121** (1996), 80–101.

[32] Klain D. A., Invariant valuations on star-shaped sets. *Adv. Math.* **125** (1997), 95–113.

[33] Klain D. A., Even valuations on convex bodies. *Trans. Amer. Math. Soc.* **352** (2000), 71–93.

[34] Klain D. A., and Rota G.-C., *Introduction to geometric probability.* Cambridge University Press, Cambridge, 1997.

[35] Ludwig M., Projection bodies and valuations. *Adv. Math.* **172** (2002), 158–168.

[36] Ludwig M., Valuations of polytopes containing the origin in their interiors. *Adv. Math.* **170** (2002), 239–256.

[37] Ludwig M., Ellipsoids and matrix valued valuations. *Duke Math. J.* **119** (2003), 159–188.

[38] Ludwig M., Minkowski valuations. *Trans. Amer. Math. Soc.* **357** (2005), 4191–4213.

[39] Ludwig M., Intersection bodies and valuations. *Amer. J. Math.* **128** (2006), 1409–1428.

[40] Ludwig M., and Reitzner M., A characterization of affine surface area. *Adv. Math.* **147** (1999), 138–172.

[41] Ludwig M., and Reitzner M., A classification of SL(n) invariant valuations. *Ann. Math. (2)*, **172** (2010), 1223–1271.

[42] Lutwak E., Yang D. and Zhang G., L_p affine isoperimetric inequalities. *J. Differential Geom.* **56** (2000), 111–132.

[43] Lutwak E., Yang D. and Zhang G., A new ellipsoid associated with convex bodies. *Duke Math. J.* **104** (2000), 375–390.

[44] Lutwak E., Yang D. and Zhang G., The Cramer-Rao inequality for star bodies. *Duke Math. J.* **112** (2002), 59–81.

[45] Lutwak E., Yang D. and Zhang G., Sharp affine L_p Sobolev inequalities. *J. Differential Geom.* **62** (2002), 17–38.

[46] Lutwak E., Yang D. and Zhang G., Moment-entropy inequalities. *Ann. Probab.* **32** (2004), 757–774.

[47] Lutwak E., Yang D. and Zhang G., Orlicz projection bodies. *Adv. Math.* **223** (2010), 220–242.

[48] Lutwak E., Yang D. and Zhang G., Orlicz centroid bodies. *J. Differential Geom.*, **84** (2010), 365–387.

[49] Lutwak E. and Zhang G., Blaschke-Santaló inequalities. *J. Differential Geom.* **47** (1997), 1–16.

[50] McMullen P., Valuations and dissections. In *Handbook of Convex Geometry, Vol. B*, (edited by P. Gruber and J. Wills). North-Holland, Amsterdam, 1993, 933–990.

[51] McMullen P. and Schneider R., Valuations on convex bodies. In *Convexity and its applications*, (edited by P. Gruber and J. Wills). Birkhäuser, 1983, 170–247.

[52] Molchanov I., Convex and star-shaped sets associated with multivariate stable distributions. I. Moments and densities. *J. Multivariate Anal.* **100** (2009), 2195–2113.

[53] Paouris G., Concentration of mass on convex bodies. *Geom. Funct. Anal.* **16** (2006), 1021–1049.

[54] Ryabogin D. and Zvavitch A., The Fourier transform and Firey projections of convex bodies. *Indiana Univ. Math. J.* **53** (2004), 667–682.

[55] Schneider R., *Convex bodies: the Brunn–Minkowski theory*. Cambridge Univ. Press, Cambridge, 1993.

[56] Schneider R. and Schuster F. E., Rotation equivariant Minkowski valuations. *Int. Math. Res. Not.* (2006), Article ID 72894.

[57] Schuster F. E., Crofton measures and Minkowski valuations. *Duke Math. J.*, **154** (2010), 1-30.

[58] Schuster F. E. and Wannerer T., GL(n) contravariant Minkwoski valuations. *Trans. Amer. Math. Soc.*, in press.

[59] Stancu A., The discrete planar L_0-Minkowski problem. *Adv. Math.* **167** (2002), 160–174.

[60] Stancu A., On the number of solutions to the discrete two-dimensional L_0-Minkowski problem. *Adv. Math.* **180** (2003), 290–323.

[61] Steineder C., Subword complexity and projection bodies. *Adv. Math.* **217** (2008), 2377–2400.

[62] Thompson A. C., *Minkowski geometry*, volume 63 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1996.

[63] Tsang, A., Valuations on L^p-spaces. *Int. Math. Res. Not.*, 2010, no. 20, 3993-4023.

[64] Werner E. and Ye D., New L_p affine isoperimetric inequalities. *Adv. Math.* **218** (2008), 762–780.

[65] Yaskin V. and Yaskina M., Centroid bodies and comparison of volumes. *Indiana Univ. Math. J.* **55** (2006), 1175–1194.

[66] Zhang G., The affine Sobolev inequality. *J. Differential Geom.* **53** (1999), 183–202.