AN ASYMMETRIC AFFINE PÓLYA–SZEGÖ PRINCIPLE

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Abstract. An affine rearrangement inequality is established which strengthens and implies the recently obtained affine Pólya–Szegö symmetrization principle for functions on $\mathbb{R}^n$. Several applications of this new inequality are derived. In particular, a sharp affine logarithmic Sobolev inequality is established which is stronger than its classical Euclidean counterpart.

1. Introduction

The classical Pólya–Szegö principle [61] states that the L^p norm of the gradient of a function on $\mathbb{R}^n$ does not increase under symmetric rearrangement. It plays a fundamental role in the solution to a number of variational problems in different areas such as isoperimetric inequalities, optimal forms of Sobolev inequalities, and sharp a priori estimates of solutions to second-order elliptic or parabolic boundary value problems; see, for example, [8, 40–42, 65, 66] and the references therein. In recent years, many important generalizations and variations have been obtained (see, e.g., [9, 16, 19–22, 28, 31]).

Based on the seminal work of Zhang [71], a full affine analogue of the classical Pólya–Szegö principle was recently established by Lutwak, Yang, and Zhang [55] (for $1 \leq p < n$) and by Cianchi, Lutwak, Yang, and Zhang [23] (for general $p \geq 1$). In this remarkable affine rearrangement inequality, an L^p affine energy replaces the standard L^p norm of the gradient leading to an inequality which is significantly stronger than its classical Euclidean counterpart. Moreover, Lutwak, Yang, and Zhang [55] and Cianchi et al. [23] obtained new sharp affine Sobolev, Moser–Trudinger and Morrey–Sobolev inequalities by applying their affine Pólya–Szegö principle, thereby demonstrating the power of this new affine symmetrization inequality.

In this article we establish a new affine Pólya–Szegö type inequality which strengthens and directly implies the affine Pólya–Szegö principle of Cianchi, Lutwak, Yang, and Zhang. We will show that an asymmetric L^p affine energy, which takes asymmetric parts of directional derivatives into account, leads to a stronger inequality. As an application of our affine rearrangement inequality we strengthen the previously known affine Moser–Trudinger and Morrey–Sobolev inequalities of Cianchi et al. and recover recent results by the first two authors [36] on asymmetric affine Sobolev inequalities. Among further applications is a new sharp affine logarithmic Sobolev inequality which is stronger than the classical Euclidean logarithmic Sobolev inequality.

For $p \geq 1$ and $n \geq 2$, let $W^{1,p}(\mathbb{R}^n)$ denote the space of real-valued L^p functions on $\mathbb{R}^n$ with weak L^p partial derivatives. We use $|\cdot|$ to denote the standard Euclidean norm on $\mathbb{R}^n$ and we write $\|f\|_p$ for the usual L^p norm of a function f on $\mathbb{R}^n$. For

$f \in W^{1,p}(\mathbb{R}^n)$, we set

$$\|\nabla f\|_p = \left(\int_{\mathbb{R}^n} |\nabla f|^p \, dx \right)^{1/p}.$$

Given $f \in W^{1,p}(\mathbb{R}^n)$, its *distribution function* $\mu_f \colon [0,\infty) \to [0,\infty]$ is defined by

$$\mu_f(t) = V(\{x \in \mathbb{R}^n : |f(x)| > t\}),$$

where V denotes Lebesgue measure on $\mathbb{R}^n$. The *decreasing rearrangement* $f^* : [0,\infty) \to [0,\infty]$ of f is defined by

$$f^*(s) = \inf\{t \geq 0 : \ \mu_f(t) \leq s\}.$$

The *symmetric rearrangement* of f is the function $f^\star : \mathbb{R}^n \to [0,\infty]$ defined by

$$f^\star(x) = f^*(\kappa_n |x|^n).$$

Here, $\kappa_n = \pi^{n/2}/\Gamma(1+\frac{n}{2})$ denotes the volume of the Euclidean unit ball in $\mathbb{R}^n$.
The classical Pólya–Szegö principle states that if $f \in W^{1,p}(\mathbb{R}^n)$ for some $p \geq 1$, then $f^\star \in W^{1,p}(\mathbb{R}^n)$ and

$$\|\nabla f^\star\|_p \leq \|\nabla f\|_p. \tag{1}$$

In the affine Pólya–Szegö inequality the L^p norm of the Euclidean length of the gradient is replaced by an affine invariant of functions, the (symmetric) L^p *affine energy*, defined, for $f \in W^{1,p}(\mathbb{R}^n)$, by

$$\mathcal{E}_p(f) = c_{n,p} \left(\int_{S^{n-1}} \|\mathrm{D}_u f\|_p^{-n} \, du \right)^{-1/n},$$

where $c_{n,p} = (n\kappa_n)^{1/n} (\frac{n\kappa_n \kappa_{p-1}}{2\kappa_{n+p-2}})^{1/p}$ and $\mathrm{D}_u f$ is the directional derivative of f in the direction u. Note that $c_{n,p}$ is chosen such that if $f \in W^{1,p}(\mathbb{R}^n)$, then

$$\mathcal{E}_p(f^\star) = \|\nabla f^\star\|_p. \tag{2}$$

We emphasize the remarkable and important fact that $\mathcal{E}_p(f)$ is invariant under volume preserving affine transformations on $\mathbb{R}^n$. In contrast, $\|\nabla f\|_p$ is invariant only under rigid motions.
The affine Pólya–Szegö principle established by Cianchi et al. [23] states that if $f \in W^{1,p}(\mathbb{R}^n)$, then

$$\mathcal{E}_p(f^\star) \leq \mathcal{E}_p(f). \tag{3}$$

It was shown in [55] that

$$\mathcal{E}_p(f) \leq \|\nabla f\|_p, \tag{4}$$

with equality if and only if $\|\mathrm{D}_u f\|_p$ is independent of $u \in S^{n-1}$. Thus, by (2), the affine inequality (3) is significantly stronger than its classical Euclidean counterpart (1).
Define the *asymmetric L^p affine energy* by

$$\mathcal{E}_p^+(f) = 2^{1/p} c_{n,p} \left(\int_{S^{n-1}} \|\mathrm{D}_u^+ f\|_p^{-n} \, du \right)^{-1/n},$$

where $\mathrm{D}_u^+ f(x) = \max\{\mathrm{D}_u f(x), 0\}$ denotes the positive part of the directional derivative of f in the direction u. Observe that only the even part of the directional derivatives of f contribute to $\mathcal{E}_p(f)$, while in $\mathcal{E}_p^+(f)$ also asymmetric parts are accounted for and that the asymmetric L^p affine energy $\mathcal{E}_p^+(f)$ is invariant under volume preserving affine transformations on $\mathbb{R}^n$.

The main result of this article is the following:

Theorem 1. *If $p \geq 1$ and $f \in W^{1,p}(\mathbb{R}^n)$, then $f^\star \in W^{1,p}(\mathbb{R}^n)$ and*

$$\mathcal{E}_p^+(f^\star) \leq \mathcal{E}_p^+(f). \tag{5}$$

In [36] it was shown that

$$\mathcal{E}_p^+(f) \leq \mathcal{E}_p(f), \tag{6}$$

with equality if and only if $\|\mathrm{D}_u^+ f\|_p$ is an even function on S^{n-1}. Thus, the new affine Pólya–Szegö inequality (5) is stronger than inequality (3) of Cianchi et al. In particular, inequality (5) is also stronger than the classical Pólya–Szegö inequality (1).

In the proof of Theorem 1 critical use is made of a new affine isoperimetric inequality recently established by the first two authors [35]. We will apply this crucial geometric inequality to convex bodies (associated with the given function) which occur as solutions to a family of (normalized) L^p Minkowski problems. These techniques clearly demonstrate that there are deep connections between the affine geometry of convex bodies and sharp affine functional inequalities (see also [23, 55, 57, 71]). The background material on the geometric core of Theorem 1 will be discussed in detail in Sections 3 and 4.

The classical Pólya–Szegö principle has important applications to a large class of variational problems, for example, it reduces the proof of sharp Sobolev inequalities to a considerably more manageable one-dimensional problem. It was shown in [23] and [55] that the affine Pólya–Szegö inequality (3) provides a similar unified approach to affine functional inequalities. In particular, sharp affine versions of L^p Sobolev, Moser–Trudinger and Morrey–Sobolev inequalities were derived from inequality (3), all of which are stronger than their Euclidean counterparts.

In Section 6 we aim to extend the picture given in [23], [55] and [71] by deriving new sharp (asymmetric) affine versions of a number of fundamental functional inequalities such as L^p Sobolev inequalities, Nash's inequality, logarithmic Sobolev inequalities and Gagliardo–Nirenberg inequalities. As an example, we state here our affine version of the sharp L^p logarithmic Sobolev inequality (the Euclidean analogue is due to Del Pino and Dolbeault [27]).

Corollary 1. *If $f \in W^{1,p}(\mathbb{R}^n)$, with $1 \leq p < n$, such that $\|f\|_p = 1$, then*

$$\int_{\mathbb{R}^n} |f|^p \log |f| \, dx \leq \frac{n}{p} \log\left(b_{n,p}\, \mathcal{E}_p^+(f)\right). \tag{7}$$

For $p > 1$, the optimal constant $b_{n,p}$ is given by

$$b_{n,p} = \left(\frac{p}{n}\right)^{1/p} \left(\frac{p-1}{e}\right)^{1-1/p} \left(\frac{\Gamma(1+\frac{n}{2})}{\pi^{n/2}\Gamma(1+\frac{n(p-1)}{p})}\right)^{1/n} \tag{8}$$

and $b_{n,1} = \lim_{p\to 1} b_{n,p}$. If $p = 1$, equality holds in (7) for characteristic functions of ellipsoids and for $p > 1$ equality is attained when

$$f(x) = \frac{\pi^{n/2}\Gamma(1+\frac{n}{2})}{a^{n(p-1)/p}\Gamma(1+\frac{n(p-1)}{p})} \exp\left(-\frac{1}{a}|\phi(x-x_0)|^{p/(p-1)}\right), \tag{9}$$

with $a > 0$, $\phi \in \mathrm{SL}(n)$ and $x_0 \in \mathbb{R}^n$.

2. Background material

For quick later reference we recall in this section some background material from the L^p Brunn–Minkowski theory of convex bodies. This theory has its origins in the work of Firey from the 1960's and has expanded rapidly over the last decade (see, e.g., [10,15,35,38,46–54,56,57]). We will also list some basic, and for the most part well known, facts from real analysis needed in the proof of Theorem 1.

A *convex body* is a compact convex subset of $\mathbb{R}^n$ with nonempty interior. We write $\mathcal{K}^n$ for the set of convex bodies in $\mathbb{R}^n$ endowed with the Hausdorff metric and we denote by $\mathcal{K}^n_{\mathrm{o}}$ the set of convex bodies containing the origin in their interiors. Each nonempty compact convex set K is uniquely determined by its *support function* $h(K,\cdot)$, defined by $h(K,x) = \max\{x \cdot y : \ y \in K\}$, $x \in \mathbb{R}^n$, where $x \cdot y$ denotes the usual inner product of x and y in $\mathbb{R}^n$. Note that $h(K,\cdot)$ is positively homogeneous of degree one and subadditive. Conversely, every function with these properties is the support function of a unique compact convex set.

If $K \in \mathcal{K}^n_{\mathrm{o}}$, then the polar body K^* of K is defined by

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \le 1 \text{ for all } y \in K\}.$$

From the polar formula for volume it follows that the n-dimensional Lebesgue measure $V(K^*)$ of the polar body K^* can be computed by

$$V(K^*) = \frac{1}{n}\int_{S^{n-1}} h(K,u)^{-n}\,du, \tag{10}$$

where integration is with respect to spherical Lebesgue measure.

If M and N are compact sets in $\mathbb{R}^n$, then the Brunn–Minkowski inequality (see, e.g., [33]) states that

$$V(M+N)^{1/n} \ge V(M)^{1/n} + V(N)^{1/n}.$$

Here, $M + N = \{x + y : x \in M \text{ and } y \in N\}$. For a compact set M and a convex body K in $\mathbb{R}^n$, define the mixed volume $V_1(M,K)$ by

$$nV_1(M,K) = \liminf_{\varepsilon \to 0^+} \frac{V(M+\varepsilon K) - V(M)}{\varepsilon}.$$

The Brunn-Minkowski inequality immediately gives the Minkowski inequality

$$V_1(M,K)^n \ge V(M)^{n-1}V(K). \tag{11}$$

We denote by $C_0^\infty(\mathbb{R}^n)$ the space of infinitely differentiable functions on $\mathbb{R}^n$ with compact support. If $f \in C_0^\infty(\mathbb{R}^n)$, then for every $t > 0$ we have

$$V_1(\{|f| \ge t\},K) = \frac{1}{n}\int_{\{|f|=t\}} h(K,\nu(x))\,d\mathcal{H}^{n-1}(x), \tag{12}$$

where $\nu(x)$ is the outward unit normal vector of the smooth $(n-1)$-dimensional submanifold $\{|f| = t\}$ at x and $\mathcal{H}^{n-1}$ denotes $(n-1)$-dimensional Hausdorff measure (cf. [71, Lemma 3.2]).

For real $p \ge 1$ and $\alpha, \beta > 0$, the L^p *Minkowski–Firey combination* of $K, L \in \mathcal{K}^n_{\mathrm{o}}$ is the convex body $\alpha \cdot K +_p \beta \cdot L$ defined by

$$h(\alpha \cdot K +_p \beta \cdot L, \cdot)^p = \alpha h(K,\cdot)^p + \beta h(L,\cdot)^p.$$

The L^p *mixed volume* $V_p(K,L)$ of K, $L \in \mathcal{K}^n_{\mathrm{o}}$ was defined in [50] by

$$V_p(K,L) = \frac{p}{n}\lim_{\varepsilon \to 0^+} \frac{V(K +_p \varepsilon \cdot L) - V(K)}{\varepsilon}.$$

Clearly, the diagonal form of V_p reduces to ordinary volume, i.e., for $K \in \mathcal{K}_o^n$,

$$V_p(K,K) = V(K). \tag{13}$$

It was also shown in [50] that for all convex bodies K and L

$$V_p(K,L) = \frac{1}{n}\int_{S^{n-1}} h(L,u)^p h(K,u)^{1-p}\, dS(K,u), \tag{14}$$

where the measure $S(K,\cdot)$ on S^{n-1} is the classical surface area measure of K. Recall that for a Borel set $\omega \subseteq S^{n-1}$, $S(K,\omega)$ is the $(n-1)$-dimensional Hausdorff measure of the set of all boundary points of K for which there exists a normal vector of K belonging to ω.

We turn now to the analytical preparations. Let $f \in W^{1,p}(\mathbb{R}^n)$ for some $p \geq 1$. Note that f, f^*, and $f^\star$ are equimeasureable, i.e., $\mu_f = \mu_{f^*} = \mu_{f^\star}$. Therefore, we have

$$\|f\|_\infty = f^*(0) = \|f^\star\|_\infty \tag{15}$$

and

$$V(\operatorname{sprt} f) = V(\operatorname{sprt} f^\star), \tag{16}$$

where $\operatorname{sprt} f$ stands for $\{x \in \mathbb{R}^n : \ |f(x)| > 0\}$. Moreover, the equality

$$\int_{\mathbb{R}^n} \Phi(|f(x)|)\, dx = \int_0^\infty \Phi(f^*(s))\, ds = \int_{\mathbb{R}^n} \Phi(f^\star(x))\, dx \tag{17}$$

holds for every Borel measurable function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$.

In the proof of Theorem 1, we will repeatedly apply Federer's coarea formula (see, e.g., [29, p. 258]). We state here a version which is sufficient for our purposes: If $f : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz and $g : \mathbb{R}^n \to [0,\infty)$ is measurable, then, for any Borel set $A \subseteq \mathbb{R}$,

$$\int_{f^{-1}(A)\cap\{|\nabla f|>0\}} g(x)\, dx = \int_A \int_{f^{-1}\{y\}} \frac{g(x)}{|\nabla f(x)|}\, d\mathcal{H}^{n-1}(x)\, dy. \tag{18}$$

In the following we collect the critical ingredients for the proof of Theorem 1. In the Euclidean setting, the Pólya–Szegö principle has the classical isoperimetric inequality at its core. In the affine setting, the geometric tools are an L^p affine isoperimetric inequality established by the first two authors [35] and the solution to the discrete data case of the normalized L^p Minkowski problem [38].

The *asymmetric L^p projection body* $\Pi_p^+ K$ of $K \in \mathcal{K}_o^n$, first considered in [51], is the convex body defined by

$$h(\Pi_p^+ K, u)^p = \int_{S^{n-1}} (u\cdot v)_+^p h(K,v)^{1-p}\, dS(K,v), \qquad u \in S^{n-1}, \tag{19}$$

where $(u\cdot v)_+ = \max\{u\cdot v, 0\}$. The (symmetric) *$L^p$ projection body* $\Pi_p K$ of $K \in \mathcal{K}_o^n$, defined in [54], is

$$\Pi_p K = \tfrac{1}{2}\cdot \Pi_p^+ K +_p \tfrac{1}{2}\cdot \Pi_p^- K,$$

where $\Pi_p^- K = \Pi_p^+(-K)$. When $p = 1$, asymmetric L^p projection bodies (and symmetric L^p projection bodies) coincide with the classical projection bodies introduced by Minkowski.

Within the Brunn–Minkowski theory, projection bodies have become a central notion. They arise naturally in a number of different areas such as functional analysis, stochastic geometry and geometric tomography. The fundamental affine isoperimetric inequality which connects the volume of a convex body with that of its polar projection body is the *Petty projection inequality* [60]. This inequality turned out to be far stronger than the classical isoperimetric inequality and has led to Zhang's affine Sobolev inequality [71].

In the new L^p Brunn–Minkowski theory, establishing an L^p analog of Petty's projection inequality became a major goal. This was accomplished for the symmetric L^p projection bodies by Lutwak, Yang, and Zhang [54] (see also Campi and Gronchi [10] for an independent approach): If $K \in \mathcal{K}_{\mathrm{o}}^n$, then

$$V(K)^{n/p-1} V(\Pi_p^* K) \leq \left(\frac{\kappa_n \kappa_{p-1}}{\kappa_{n+p-2}} \right)^{n/p}, \tag{20}$$

with equality if and only if K is an ellipsoid centered at the origin. This inequality forms the geometric core of the affine Pólya–Szegö principle (3) of Cianchi et al. [23].

Recently the first two authors [35] established a stronger L^p Petty projection inequality for asymmetric L^p projection bodies:

Theorem 2. *If $p > 1$ and $K \in \mathcal{K}_{\mathrm{o}}^n$, then*

$$V(K)^{n/p-1} V(\Pi_p^{+,*} K) \leq \left(\frac{\kappa_n \kappa_{p-1}}{\kappa_{n+p-2}} \right)^{n/p}, \tag{21}$$

where equality is attained if K is an ellipsoid centered at the origin.

Although this inequality was formulated in [35] for dimensions $n \geq 3$, we remark that it also holds true in dimension $n = 2$. The proof is verbally the same as the one given in [35].

It was also shown in [35] that inequality (21), for $p > 1$, strengthens and directly implies inequality (20) of Lutwak, Yang, and Zhang: If $K \in \mathcal{K}_{\mathrm{o}}^n$, then

$$V(\Pi_p^* K) \leq V(\Pi_p^{+,*} K).$$

If p is not an odd integer, equality holds precisely for origin-symmetric K.

We turn now to the second tool from the geometry of convex bodies needed in the proof of Theorem 1. The L^p Minkowski problem, essentially an elliptic Monge–Ampère PDE, deals with the existence and the uniqueness of convex bodies with prescribed L^p curvature (see, e.g., [15,38,56]). We will apply our affine isoperimetric inequality (21) to the bodies occurring as solutions to this problem for $p \geq 1$. Since the geometric inequality assumes that the convex bodies contain the origin in their interiors, its application is intricate in the asymmetric situation. Here, the origin can lie on the boundary of the bodies occurring as the solution to the L^p Minkowski problem. For this reason we will have to deal with a normalized version of the discrete-data case of the L^p Minkowski problem (see [38, Theorem 1.1]).

Theorem 3. *If $\alpha_1, \ldots, \alpha_k > 0$ and $u_1, \ldots, u_k \in S^{n-1}$ are not contained in a closed hemisphere, then, for any $p > 1$, there exists a polytope $P \in \mathcal{K}_{\mathrm{o}}^n$ such that*

$$V(P) h(P, \cdot)^{p-1} \sum_{j=1}^{k} \alpha_j \delta_{u_j} = S(P, \cdot).$$

Here, δ_{u_j} denotes the probability measure with unit point mass at $u_j \in S^{n-1}$.
Two more auxiliary results [38, Lemma 2.2 & 2.3] regarding the convex bodies which occur as solutions to the volume normalized L^p Minkowski problem will also be needed: Let μ be a positive Borel measure on S^{n-1}, and let $K \in \mathcal{K}^n$ contain the origin. Suppose that

$$V(K)h(K,\cdot)^{p-1}\mu = S(K,\cdot),$$

and that for some constant $c > 0$,

$$\int_{S^{n-1}} (u \cdot v)_+^p \, d\mu(v) \geq \frac{n}{c^p} \quad \text{for every } u \in S^{n-1}.$$

Then

$$V(K) \geq \kappa_n \left(\frac{n}{\mu(S^{n-1})} \right)^{n/p} \qquad \text{and} \qquad K \subset cB_n, \tag{22}$$

where B_n denotes the Euclidean unit ball in $\mathbb{R}^n$.

3. Level sets and asymmetric L^p projection bodies

In order to apply the crucial L^p affine isoperimetric inequality (21) in the proof of our main result, it will be necessary to rewrite L^p gradient integrals over level sets in terms of L^p mixed volumes. This is done by constructing a family of convex bodies containing the origin in their interiors by solving a family of L^p Minkowski problems. In [23], this was done by using the normalized *even* L^p Minkowski problem. In the asymmetric situation, we have to deal with solutions to the general L^p Minkowski problem. Here, the bodies can contain the origin in their boundaries. Therefore, we will associate a family of convex polytopes to a given function which are obtained from the solution to the discrete-data case of the volume normalized L^p Minkowski problem. This ensures that the polytopes contain the origin in their interiors and allows us to apply Theorem 2.
We denote by $C_0^\infty(\mathbb{R}^n)$ the space of infinitely differentiable functions on $\mathbb{R}^n$ with compact support. If $f \in C_0^\infty(\mathbb{R}^n)$, we denote level sets by

$$[f]_t := \{x \in \mathbb{R}^n : |f(x)| > t\}.$$

By Sard's theorem, for almost every $t \in (0, \|f\|_\infty)$, the boundary

$$\partial[f]_t = \{x \in \mathbb{R}^n : |f(x)| = t\}$$

of $[f]_t$ is a smooth $(n-1)$-dimensional submanifold of $\mathbb{R}^n$ with everywhere nonzero normal vector $\nabla f(x)$.

Lemma 1. *Suppose that $p > 1$ and $f \in C_0^\infty(\mathbb{R}^n)$. Then, for almost every $t \in (0, \|f\|_\infty)$, the body $L_f^t \in \mathcal{K}_o^n$ given by*

$$h(L_f^t, u)^p = \int_{\partial[f]_t} (u \cdot \nabla f(x))_+^p |\nabla f(x)|^{-1} \, d\mathcal{H}^{n-1}(x), \qquad u \in S^{n-1}. \tag{23}$$

has the following properties. There exists a sequence of convex polytopes $P_k^t \in \mathcal{K}_o^n$, $k \in \mathbb{N}$, such that

$$\lim_{k \to \infty} P_k^t = K_f^t \in \mathcal{K}^n$$

and

$$\frac{1}{n} \int_{\partial[f]_t} h(K_f^t, \nabla f(x))^p |\nabla f(x)|^{-1} \, d\mathcal{H}^{n-1}(x) = 1, \tag{24}$$

as well as

$$\lim_{k\to\infty} V(P_k^t)^{-1/p}\Pi_p^+ P_k^t = L_f^t.$$

Proof. Let t be chosen such that $\partial[f]_t$ is a smooth manifold with everywhere nonzero normal vector $\nabla f(x)$ and denote by $\nu(x) = \nabla f(x)/|\nabla f(x)|$ the unit normal of $\partial[f]_t$ at x.

Let μ^t be the finite positive Borel measure on S^{n-1} satisfying

$$\int_{S^{n-1}} g(v)\, d\mu^t(v) = \int_{\partial[f]_t} g(\nu(x))|\nabla f(x)|^{p-1}\, d\mathcal{H}^{n-1}(x) \tag{25}$$

for every $g \in C(S^{n-1})$. From

$$\{\nu(x):\ x \in \partial[f]_t\} = S^{n-1},$$

it follows that for any $u \in S^{n-1}$,

$$\int_{S^{n-1}} (u\cdot v)_+\, d\mu^t(v) = \int_{\partial[f]_t} (u\cdot \nu(x))_+|\nabla f(x)|^{p-1}\, d\mathcal{H}^{n-1}(x) > 0.$$

Consequently, the measure μ^t is not concentrated in a closed hemisphere.

We can find a sequence μ_k^t, $k \in \mathbb{N}$, of discrete measures on S^{n-1} whose supports are not contained in a closed hemisphere and such that μ_k^t converges weakly to μ^t as $k \to \infty$ (see, e.g., [63, pp. 392-3]). By Theorem 3, for each $k \in \mathbb{N}$, there exists a polytope $P_k^t \in \mathcal{K}_{\mathrm{o}}^n$ such that

$$V(P_k^t)h(P_k^t,\cdot)^{p-1}\mu_k^t = S(P_k^t,\cdot). \tag{26}$$

From definition (19), relation (26) and the weak convergence of the measures μ_k^t it follows that for every $u \in S^{n-1}$,

$$h\left(V(P_k^t)^{-1/p}\Pi_p^+ P_k^t, u\right)^p = \int_{S^{n-1}} (u\cdot v)_+^p\, d\mu_k^t(v) \longrightarrow \int_{S^{n-1}} (u\cdot v)_+^p\, d\mu^t(v) > 0. \tag{27}$$

Since pointwise convergence of support functions implies uniform convergence on S^{n-1} (see, e.g., [63, Theorem 1.8.12]), there exists a $c > 0$ such that for all $k \in \mathbb{N}$,

$$\int_{S^{n-1}} (u\cdot v)_+^p\, d\mu_k^t(v) > c, \quad \text{for every } u \in S^{n-1}. \tag{28}$$

From (26), (28) and (22), it follows that the sequence P_k^t, $k \in \mathbb{N}$, is bounded and that the volumes $V(P_k^t)$ are also bounded from below by a positive constant. By the Blaschke selection theorem (see, e.g., [63, Theorem 1.8.6]), we can therefore select a subsequence of the P_k^t converging to a convex body K_f^t. After relabeling (if necessary) we may assume that $\lim_{k\to\infty} P_k^t = K_f^t$. By definition (25), we have

$$\frac{1}{n}\int_{\partial[f]_t} h(K_f^t,\nabla f(x))^p|\nabla f(x)|^{-1}\, d\mathcal{H}^{n-1}(x) = \frac{1}{n}\int_{S^{n-1}} h(K_f^t, v)^p\, d\mu^t(v).$$

Thus, from the uniform convergence of the support functions $h(P_k^t,\cdot)$ and the weak convergence of the measures μ_k^t to the finite measure μ^t, we obtain

$$\lim_{k\to\infty}\frac{1}{n}\int_{S^{n-1}} h(P_k^t, v)^p\, d\mu_k^t(v) = \frac{1}{n}\int_{\partial[f]_t} h(K_f^t,\nabla f(x))^p|\nabla f(x)|^{-1}\, d\mathcal{H}^{n-1}(x).$$

By (13), (14), and relation (26), we have for each $k \in \mathbb{N}$,

$$\frac{1}{n}\int_{S^{n-1}} h(P_k^t, v)^p\, d\mu_k^t(v) = 1,$$

which proves (24). Finally, we define $h(L_f^t,\cdot)$ by

$$h(L_f^t,u)^p=\int_{\partial[f]_t}(u\cdot\nabla f(x))_+^p|\nabla f(x)|^{-1}\,d\mathcal{H}^{n-1}(x),\qquad u\in S^{n-1}.$$

From Minkowski's integral inequality, it follows that $h(L_f^t,\cdot)$ is the support function of a compact convex set. From definition (25) and (27), we deduce that $L_f^t\in\mathcal{K}_o^n$ and that $\lim_{k\to\infty}V(P_k^t)^{-\frac{1}{p}}\Pi_p^+P_k^t=L_f^t$. □

The following lemma is a special case of Lemma 1 for functions with rotational symmetry arising from symmetric rearrangement.

Lemma 2. *Suppose that $p>1$ and $f\in C_0^\infty(\mathbb{R}^n)$. Then, for almost every $t\in(0,\|f\|_\infty)$, there exists a real number $c_f^t>0$ such that*

$$h\left(V(c_f^tB_n)^{-1/p}\Pi_p^+(c_f^tB_n),u\right)^p=\int_{\partial[f^\star]_t}\frac{(u\cdot\nabla f^\star(x))_+^p}{|\nabla f^\star(x)|}\,d\mathcal{H}^{n-1}(x),\tag{29}$$

for every $u\in S^{n-1}$, and

$$\frac{1}{n}\int_{\partial[f^\star]_t}h\left(c_f^tB_n,\nabla f^\star(x)\right)^p|\nabla f^\star(x)|^{-1}\,d\mathcal{H}^{n-1}(x)=1.\tag{30}$$

Proof. Note that $f^\star$ is Lipschitz. For almost every $t\in(0,\|f\|_\infty)$, the set $\partial[f^\star]_t$ is the boundary of a ball of radius r_t with nonvanishing normal $\nabla f^\star$. Note that in this case, $|\nabla f^\star|$ is in fact constant on $\partial[f^\star]_t$. Define the real number c_f^t by

$$c_f^t=\left(\kappa_n^{-1}|\nabla f^\star|^{1-p}r_t^{1-n}\right)^{1/p}.$$

We write $\nu_*(x)=\nabla f^\star(x)/|\nabla f^\star(x)|$ for the unit normal vector of $\partial[f^\star]_t$. Since for every $g\in C(S^{n-1})$,

$$\int_{S^{n-1}}g(v)\,d\mathcal{H}^{n-1}(v)=r_t^{1-n}\int_{\partial[f^\star]_t}g(\nu_*(x))\,d\mathcal{H}^{n-1}(x),$$

the definition of asymmetric L^p projection bodies (19) yields, for $u\in S^{n-1}$,

$$h\left(V(c_f^tB_n)^{-1/p}\Pi_p^+(c_f^tB_n),u\right)^p=\frac{r_t^{1-n}}{(c_f^t)^p\kappa_n}\int_{\partial[f^\star]_t}(u\cdot\nu_*(x))_+^p\,d\mathcal{H}^{n-1}(x).$$

Thus, we obtain (29) from the definitions of c_f^t and $\nu_*(x)$. Finally, we have

$$\frac{1}{n}\int_{\partial[f^\star]_t}\frac{h(c_f^tB^n,\nabla f^\star(x))^p}{|\nabla f^\star(x)|}\,d\mathcal{H}^{n-1}(x)=\frac{(c_f^t)^p}{n}\int_{\partial[f^\star]_t}|\nabla f^\star(x)|^{p-1}\,d\mathcal{H}^{n-1}(x),$$

which yields (30) by the definition of c_f^t. □

4. Proof of the main result

We are now in a position to prove our main result. The approach we use to establish Theorem 1 is based on techniques developed in [23] and [55].
Before we begin, we want to point out that the asymmetric affine L^p energy $\mathcal{E}_p^+(f)$ is well defined. This follows from the fact that $\|\mathrm{D}_u^+f\|_p$ is positive for each $u\in S^{n-1}$ and every nontrivial $f\in W^{1,p}(\mathbb{R}^n)$ (see [36, Lemma 2]).

Proof of Theorem 1 In order to prove inequality (5), let us first assume that $p>1$ and that $f\in C_0^\infty(\mathbb{R}^n)$. Clearly, we may also assume that f is not identically zero.

Since f is Lipschitz, the reverse triangle inequality shows that $|f|$ is also Lipschitz and hence, by Rademacher's theorem, differentiable almost everywhere. Moreover, the equality $|\nabla f| = |\nabla |f||$ holds almost everywhere on $\mathbb{R}^n$ and on each level set $\{|f| = t\}$ where $t > 0$. Now, an application of the coarea formula (18) shows that

$$\begin{aligned} \|\mathrm{D}_u^+ f\|_p^p &= \int_{\{|\nabla f|>0\}} (u \cdot \nabla f(x))_+^p \, dx \\ &= \int_{\{|\nabla |f||>0\}} (u \cdot \nabla f(x))_+^p \, dx \\ &= \int_0^{\|f\|_\infty} \int_{\partial [f]_t} \frac{(u \cdot \nabla f(x))_+^p}{|\nabla f(x)|} \, d\mathcal{H}^{n-1}(x) \, dt. \end{aligned} \tag{31}$$

By Lemma 1 and (23), there exist convex bodies $L_f^t \in \mathcal{K}_\mathrm{o}^n$ such that

$$\mathcal{E}_p^+(f)^p = 2c_{n,p}^p \left(\int_{S^{n-1}} \left(\int_0^{\|f\|_\infty} h(L_f^t, u)^p \, dt \right)^{-n/p} du \right)^{-p/n}.$$

Since $h(L_f^t, \cdot)$ is positive for almost every t, we can apply Minkowski's inequality for integrals (see, e.g., [37, p. 148]), and obtain

$$\mathcal{E}_p^+(f)^p \geq 2c_{n,p}^p \int_0^{\|f\|_\infty} \left(\int_{S^{n-1}} h(L_f^t, u)^{-n} \, du \right)^{-p/n} dt.$$

Hence, the volume formula (10) yields

$$\mathcal{E}_p^+(f)^p \geq 2c_{n,p}^p \int_0^{\|f\|_\infty} \left(nV(L_f^{t,*}) \right)^{-p/n} dt. \tag{32}$$

By Lemma 1, there exists a sequence of polytopes $P_k^t \in \mathcal{K}_\mathrm{o}^n$ such that

$$\lim_{k\to\infty} P_k^t = K_f^t \in \mathcal{K}^n \qquad \text{and} \qquad \lim_{k\to\infty} V(P_k^t)^{-1/p} \Pi_p^+ P_k^t = L_f^t.$$

Thus, an application of Theorem 2 shows that

$$(nV(L_f^{t,*}))^{-p/n} = \lim_{k\to\infty} (nV(P_k^t)^{n/p} V(\Pi_p^{+,*} P_k^t))^{-p/n} \geq e_{n,p} V(K_f^t)^{-p/n}, \tag{33}$$

where

$$e_{n,p} = \frac{\kappa_{n+p-2}}{n^{p/n} \kappa_n \kappa_{p-1}}.$$

From (32) and (33), we deduce

$$\mathcal{E}_p^+(f)^p \geq n\kappa_n^{p/n} \int_0^{\|f\|_\infty} V(K_f^t)^{-p/n} \, dt. \tag{34}$$

By (24) and Hölder's integral inequality, we have

$$\left(\int_{\partial [f]_t} |\nabla f(x)|^{-1} \, d\mathcal{H}^{n-1}(x) \right)^{(p-1)/p} \geq n^{1-1/p} \, V_1(\{|f| \geq t\}, K_f^t),$$

where we have used representation (12) for the mixed volume $V_1(\{|f| \geq t\}, K_f^t)$. From the Minkowski inequality (11) and the fact that $V(\{|f| \geq t\}) = V(\{|f| > t\})$

for almost every t, we deduce further that

$$\left(\int_{\partial[f]_t} |\nabla f(x)|^{-1}\, d\mathcal{H}^{n-1}(x)\right)^{(p-1)/p} \geq n^{1-1/p}\, \mu_f(t)^{(n-1)/n} V(K_f^t)^{1/n} \tag{35}$$

for almost every t. By the coarea formula (18), we have for almost every t,

$$\mu_f(t) = V([f]_t \cap \{\nabla f = o\}) + \int_t^{\|f\|_\infty} \int_{\partial[f]_s} |\nabla f(x)|^{-1}\, d\mathcal{H}^{n-1}(x)\, ds. \tag{36}$$

Since μ_f is the sum of two nonincreasing functions, we obtain

$$-\mu_f(t)' \geq \int_{\partial[f]_t} |\nabla f(x)|^{-1}\, d\mathcal{H}^{n-1}(x) \tag{37}$$

for almost every t.
Combining (35) and (37), yields the estimate

$$V(K_f^t)^{-p/n} \geq n^{p-1} \frac{\mu_f(t)^{p(n-1)/n}}{(-\mu_f'(t))^{p-1}}. \tag{38}$$

Thus, by (34) and (38), we obtain

$$\mathcal{E}_p^+(f)^p \geq n^p \kappa_n^{p/n} \int_0^{\|f\|_\infty} \frac{\mu_f(t)^{p(n-1)/n}}{(-\mu_f'(t))^{p-1}}\, dt.$$

It remains to show that

$$\mathcal{E}_p^+(f^\star)^p = n^p \kappa_n^{p/n} \int_0^{\|f\|_\infty} \frac{\mu_f(t)^{p(n-1)/n}}{(-\mu_f'(t))^{p-1}}\, dt. \tag{39}$$

By (15), (18), and (29) we have

$$\mathcal{E}_p^+(f^\star)^p = 2c_{n,p}^p \left(\int_{S^{n-1}} \left(\int_0^{\|f\|_\infty} V(B_f^t)^{-1} h(\Pi_p^+ B_f^t, u)^p dt\right)^{-n/p} du\right)^{-p/n},$$

where B_f^t denotes the ball $c_f^t B_n$ whose existence is guaranteed by Lemma 2. Since $\Pi_p^+ B_f^t$ is a ball, $h(\Pi_p^+ B_f^t, \cdot)$ is a constant function on the sphere. Thus, we obtain as in the first part of the proof,

$$\mathcal{E}_p^+(f^\star)^p = n\kappa_n^{p/n} \int_0^{\|f\|_\infty} V(B_f^t)^{-p/n}\, dt. \tag{40}$$

From (30), Minkowski's inequality (11), and the fact that $[f^\star]_t$ and B_f^t are dilates, we have for almost every t on one hand

$$\left(\int_{\partial[f^\star]_t} |\nabla f^\star(x)|^{-1}\, d\mathcal{H}^{n-1}(x)\right)^{(p-1)/p} = n^{1-1/p} \mu_{f^\star}(t)^{(n-1)/n} V(B_f^t)^{1/n}$$

and, by (36) and [20, Lemma 2.4 & 2.6], on the other hand

$$-\mu_{f^\star}(t)' = \int_{\partial[f^\star]_t} |\nabla f^\star(x)|^{-1}\, d\mathcal{H}^{n-1}(x).$$

Hence, the equimeasurability of f and $f^\star$ yields

$$V(B_f^t)^{-p/n} = n^{p-1} \frac{\mu_f(t)^{p(n-1)/n}}{(-\mu_f'(t))^{p-1}}.$$

Combining this with (40) proves (39). Consequently, we have

$$\mathcal{E}_p^+(f^\star) \le \mathcal{E}_p^+(f) \tag{41}$$

for every $p > 1$ and every $f \in C_0^\infty(\mathbb{R}^n)$. Clearly, the case $p = 1$ of inequality (41) can be obtained by using a limiting argument as $p \to 1$.

In order to establish inequality (41) for an arbitrary $f \in W^{1,p}(\mathbb{R}^n)$ whose support has positive measure, consider a sequence $f_k \in C_0^\infty(\mathbb{R}^n)$, $k \in \mathbb{N}$, converging to f in $W^{1,p}(\mathbb{R}^n)$. Then, for every $k \in \mathbb{N}$,

$$\mathcal{E}_p^+(f_k^\star) \le \mathcal{E}_p^+(f_k). \tag{42}$$

By Minkowski's integral inequality, the function $u \mapsto \|\mathrm{D}_u^+ f\|_p$, $u \in S^{n-1}$, is a support function of a convex body for every $f \in W^{1,p}(\mathbb{R}^n)$. Thus, the pointwise convergence $\|\mathrm{D}_u^+ f_k\|_p \to \|\mathrm{D}_u^+ f\|_p$ on S^{n-1}, implies in fact that $\|\mathrm{D}_u^+ f_k\|_p$ converges to $\|\mathrm{D}_u^+ f\|_p$ uniformly (see, e.g., [63, Theorem 1.8.12]). Moreover, since $\|\mathrm{D}_u^+ f\|_p$ is strictly positive on S^{n-1} (see [35, Lemma 2]), also $\|\mathrm{D}_u^+ f_k\|_p^{-n} \to \|\mathrm{D}_u^+ f\|_p^{-n}$ uniformly on S^{n-1}. Hence,

$$\lim_{k\to\infty} \mathcal{E}_p^+(f_k) = \mathcal{E}_p^+(f). \tag{43}$$

On the other hand, the nonexpansivity of symmetric rearrangements (see, e.g., [14, 44] for the terminology and the corresponding result) implies $f_k^\star \to f^\star$ in $L^p(\mathbb{R}^n)$. Thus, the sequence $f_k^\star$, $k \in \mathbb{N}$, converges weakly to $f^\star$ in $W^{1,p}(\mathbb{R}^n)$. Since

$$\mathcal{E}_p^+(f_k^\star) = \|\nabla f_k^\star\|_p \qquad \text{and} \qquad \mathcal{E}_p^+(f^\star) = \|\nabla f^\star\|_p$$

and since the L^p norm of the gradient is lower semicontinuous with respect to weak convergence in $W^{1,p}(\mathbb{R}^n)$, we obtain

$$\liminf_{k\to\infty} \mathcal{E}_p^+(f_k^\star) \ge \mathcal{E}_p^+(f^\star)$$

which, by (42) and (43), concludes the proof.

5. Applications of the asymmetric affine Pólya-Szegö principle

In this section we will illustrate how Theorem 1 provides a direct unified approach to a number of affine functional inequalities. We will derive sharp affine versions of certain Gagliardo–Nirenberg inequalities, all of which are stronger than their Euclidean counterparts.

Affine L^p logarithmic Sobolev inequalities

The classical sharp logarithmic Sobolev inequality states that if $f \in W^{1,2}(\mathbb{R}^n)$ such that $\|f\|_2 = 1$, then

$$\int_{\mathbb{R}^n} |f|^2 \log|f|\,dx \le \frac{n}{2}\log\left(\left(\frac{2}{ne\pi}\right)^{1/2}\|\nabla f\|_2\right). \tag{44}$$

In this form, the logarithmic Sobolev inequality first appeared in [69]. However, it is well known that inequality (44) is equivalent to the logarithmic Sobolev inequality with respect to Gauss measure due to Stam [64] and Gross [34]. Different proofs and extensions of these inequalities have been the focus of a number of articles (see, e.g., [1, 5, 6, 11], and the references therein).

The natural problem to find a sharp L^p analogue of inequality (44) was solved by Ledoux [43] for $p = 1$ and recently by Del Pino and Dolbeault [27] for $1 < p < n$: If $f \in W^{1,p}(\mathbb{R}^n)$, with $1 \le p < n$, such that $\|f\|_p = 1$, then

$$\int_{\mathbb{R}^n} |f|^p \log |f| \, dx \le \frac{n}{p} \log \left(b_{n,p} \, \|\nabla f\|_p \right), \tag{45}$$

where the optimal constant $b_{n,p}$ is given by (8). Beckner [6] proved that, for $p = 1$, the only extremals in inequality (45) are the characteristic functions of balls. Carlen [11], for $p = 2$, and Del Pino and Dolbeault [27], for general $1 < p < n$, showed that equality holds in (45) if and only if for some $a > 0$ and $x_0 \in \mathbb{R}^n$,

$$f(x) = \frac{\pi^{n/2}\Gamma(1+\frac{n}{2})}{a^{n(p-1)/p}\Gamma(1+\frac{n(p-1)}{p})} \exp\left(-\frac{1}{a}|x - x_0|^{p/(p-1)} \right). \tag{46}$$

The first application of our new affine Pólya–Szegö principle is an asymmetric affine version of (45), which in light of (4) and (6), is stronger than (45) and is stated as Corollary 1 in the Introduction.

Proof of Corollary 1 By (17) and (45), we have

$$\int_{\mathbb{R}^n} |f|^p \log |f| \, dx = \int_{\mathbb{R}^n} |f^\star|^p \log |f^\star| \, dx \le \frac{n}{p} \log \left(b_{n,p} \, \|\nabla f^\star\|_p \right) \tag{47}$$

for every $f \in W^{1,p}(\mathbb{R}^n)$ such that $\|f\|_p = 1$. Since

$$\mathcal{E}_p^+(f^\star) = \left(\int_0^\infty \left(n\kappa_n^{1/n} s^{(n-1)/n} (-f^{*\prime}(s)) \right)^p ds \right)^{1/p} = \|\nabla f^\star\|_p,$$

we deduce from (47) and Theorem 1 that

$$\int_{\mathbb{R}^n} |f|^p \log |f| \, dx \le \frac{n}{p} \log \left(b_{n,p} \, \mathcal{E}_p^+(f^\star) \right) \le \frac{n}{p} \log \left(b_{n,p} \, \mathcal{E}_p^+(f) \right) \tag{48}$$

which proves inequality (7). Equality holds in (47) for any function having the form (46) with $x_0 = o$. Any such function is spherically symmetric, so that equality holds in Theorem 1 and, thus, also in inequality (48). Equality continues to hold in (7) for any function of the form (9), as (7) is invariant under volume preserving affine transformations. □

Affine L^p Sobolev inequalities

The classical sharp L^p Sobolev inequality states that if $f \in W^{1,p}(\mathbb{R}^n)$, with $1 \le p < n$, then

$$\|f\|_{p^*} \le a_{n,p} \, \|\nabla f\|_p, \tag{49}$$

where $p^* = np/(n-p)$. For $p > 1$, the optimal constants $a_{n,p}$ are given by

$$a_{n,p} = n^{-1/p} \left(\frac{p-1}{n-p} \right)^{1-1/p} \left(\frac{\Gamma(n)}{\kappa_n \Gamma(\frac{n}{p}) \Gamma(n+1-\frac{n}{p})} \right)^{1/n},$$

and $a_{n,1} = \lim_{p \to 1} a_{n,p}$. They go back to Federer and Fleming [30] and Maz'ya [58] for $p = 1$ and to Aubin [2] and Talenti [65] for $p > 1$. Extremal functions for inequality (49) are the characteristic functions of balls for $p = 1$ and for $p > 1$ equality is attained if for some $a, b > 0$ and $x_0 \in \mathbb{R}^n$,

$$f(x) = (a + b|(x - x_0)|^{p/(p-1)})^{1-n/p}.$$

The sharp L^p Sobolev inequality plays a central role in the theory of partial differential equations and functional analysis. Generalizations of (49) and related problems have been much studied (see, e.g., [3, 8, 20, 25, 26, 42, 57, 66, 70]), and the references therein).

An affine version of inequality (49), which is in light of (4) stronger than (49), was established by Zhang [71] for $p = 1$ and Lutwak, Yang, and Zhang [55] for $1 < p < n$. It states that if $f \in W^{1,p}$, with $1 \le p < n$, then

$$\|f\|_{p^*} \le a_{n,p}\, \mathcal{E}_p(f). \tag{50}$$

If $p = 1$, equality holds in (50) for characteristic functions of ellipsoids and for $p > 1$ equality is attained when

$$f(x) = (a + |\phi(x - x_0)|^{p/(p-1)})^{1-n/p}, \tag{51}$$

with $a > 0$, $\phi \in \mathrm{GL}(n)$ and $x_0 \in \mathbb{R}^n$.

Using (6), a strengthened asymmetric version of the affine Sobolev inequality (50) was recently established by the first two authors [36]. It is now an immediate consequence of Theorem 1 and (49):

Corollary 2. *If $f \in W^{1,p}(\mathbb{R}^n)$, with $1 \le p < n$, then*

$$\|f\|_{p^*} \le a_{n,p}\, \mathcal{E}_p^+(f). \tag{52}$$

If $p = 1$, equality holds in (52) for characteristic functions of ellipsoids and for $p > 1$ equality is attained for functions of the form (51).

We turn now to the limiting case $p = n$ of inequality (49). It is well known that functions $f \in W^{1,n}(\mathbb{R}^n)$, whose support has finite Lebesgue measure, are exponentially summable (cf., e.g., [68]). The sharp Moser–Trudinger inequality [59, 68] states that there exists a constant $m_n > 0$ such that

$$\frac{1}{V(\operatorname{sprt} f)} \int_{\operatorname{sprt} f} \exp\left(\frac{n\kappa_n^{1/n}|f(x)|}{\|\nabla f\|_n}\right)^{n/(n-1)} dx \le m_n \tag{53}$$

for every $f \in W^{1,n}(\mathbb{R}^n)$ with $0 < V(\operatorname{sprt} f) < \infty$. Inequality (53) and its variants have been the focus of investigations by specialists in different areas (see, e.g., [4, 17, 18, 24, 32, 45, 62]).

The constant $n\kappa_n^{1/n}$ is optimal, in that inequality (53) would fail for any real number m_n if $n\kappa_n^{1/n}$ were to be replaced by a larger number. The best constant m_n is characterized as follows

$$m_n = \sup_g \int_0^\infty \exp\left(g(t)^{n/(n-1)} - t\right) dt,$$

where the supremum ranges over all nondecreasing and locally absolutely continuous functions g on $[0, \infty)$ such that $g(0) = 0$ and $\int_0^\infty g'(t)^n\, dt \le 1$. In [13] Carleson and Chang showed that spherically symmetric extremals do exist for the Moser–Trudinger inequality (53).

An affine version of the Moser–Trudinger inequality, stronger than (53), was recently established by Cianchi et al. [23]. It states that if $f \in W^{1,n}(\mathbb{R}^n)$ with $0 < V(\operatorname{sprt} f) < \infty$, then

$$\frac{1}{V(\operatorname{sprt} f)} \int_{\operatorname{sprt} f} \exp\left(\frac{n\kappa_n^{1/n}|f(x)|}{\mathcal{E}_n(f)}\right)^{n/(n-1)} dx \le m_n. \tag{54}$$

The constants $n\kappa_n^{1/n}$ and m_n are again best possible. Composing any extremal f for the Moser–Trudinger inequality with any element of $\mathrm{GL}(n)$ will also yield an extremal for inequality (54).
From Theorem 1 we can derive a strengthened asymmetric version of the affine Moser–Trudinger inequality (54):

Corollary 3. *If $f \in W^{1,n}(\mathbb{R}^n)$ with $0 < V(\operatorname{sprt} f) < \infty$, then*

$$\frac{1}{V(\operatorname{sprt} f)} \int_{\operatorname{sprt} f} \exp\left(\frac{n\kappa_n^{1/n}|f(x)|}{\mathcal{E}_n^+(f)}\right)^{n/(n-1)} dx \le m_n. \tag{55}$$

The constant $n\kappa_n^{1/n}$ is optimal, in that (55) would fail for any real number m_n if $n\kappa_n^{1/n}$ were to be replaced by a larger number. Composing any extremal f for inequality (53) with any element of $\mathrm{GL}(n)$ will also yield an extremal for inequality (55).

We will omit the proof of Corollary 3 since it is almost verbally the same as the one for inequality (54) given in [23] when [23, Theorem 2.1] is replaced by Theorem 1.

Finally, we come to the case $p > n$. The sharp Morrey–Sobolev inequality [67] states that if $f \in W^{1,p}(\mathbb{R}^n)$, $p > n$, such that $V(\operatorname{sprt} f) < \infty$, then

$$\|f\|_\infty \le \alpha_{n,p}\, V(\operatorname{sprt} f)^{(p-n)/np} \|\nabla f\|_p, \tag{56}$$

where the optimal constant $\alpha_{n,p}$ is given by

$$\alpha_{n,p} = n^{-1/p} \kappa_n^{-1/n} \left(\frac{p-1}{p-n}\right)^{(p-1)/p}.$$

Equality holds in inequality (56) if for some $a, b \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$,

$$f(x) = a\left(1 - |b(x - x_0)|^{(p-n)/(p-1)}\right)_+.$$

The affine counterpart of (56) established by Cianchi et al. [23] states that

$$\|f\|_\infty \le \alpha_{n,p}\, V(\operatorname{sprt} f)^{(p-n)/np} \mathcal{E}_p(f). \tag{57}$$

By (4), the affine inequality (57) is significantly stronger than (56). As an immediate consequence of Theorem 1, (15), (16) and (56) we obtain the following strengthened asymmetric affine Morrey–Sobolev inequality:

Corollary 4. *If $f \in W^{1,p}(\mathbb{R}^n)$, $p > n$, such that $V(\operatorname{sprt} f) < \infty$, then*

$$\|f\|_\infty \le \alpha_{n,p}\, V(\operatorname{sprt} f)^{(p-n)/np} \mathcal{E}_p^+(f). \tag{58}$$

Equality is attained in (58) if for some $a \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$, and $\phi \in \mathrm{GL}(n)$,

$$f(x) = a\left(1 - |\phi(x - x_0)|^{(p-n)/(p-1)}\right)_+.$$

Let $W^{1,\infty}(\mathbb{R}^n)$ denote the space of real-valued L^∞ functions on $\mathbb{R}^n$ with weak L^∞ derivatives. For $f \in W^{1,\infty}(\mathbb{R}^n)$ define the *asymmetric L^∞ affine energy* by

$$\mathcal{E}_\infty^+(f) = (n\kappa_n)^{1/n} \left(\int_{S^{n-1}} \|\mathrm{D}_u^+ f\|_\infty^{-n}\, du\right)^{-1/n}.$$

We are now in a position to prove the following Faber-Krahn type inequality.

Corollary 5. *If $f \in W^{1,\infty}(\mathbb{R}^n)$ such that $V(\operatorname{sprt} f) < \infty$, then*

$$\|f\|_\infty \leq \kappa_n^{-1/n} V(\operatorname{sprt} f)^{1/n} \mathcal{E}_\infty^+(f). \tag{59}$$

Equality is attained in (59) if for some $a \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$, and $\phi \in \mathrm{GL}(n)$,

$$f(x) = a\left(1 - |\phi(x - x_0)|\right)_+ .$$

Proof. We take a limit in (58) to get the desired estimate. Indeed, by Fatou's lemma we get

$$\begin{aligned}
\|f\|_\infty &\leq \kappa_n^{-1/n} V(\operatorname{sprt} f)^{1/n} (n\kappa_n)^{1/n} \limsup_{q\to\infty} \mathcal{E}_q^+(f) \\
&\leq \kappa_n^{-1/n} V(\operatorname{sprt} f)^{1/n} (n\kappa_n)^{1/n} \Big(\liminf_{q\to\infty} \int_{S^{n-1}} \|D_v^+ f\|_q^{-n}\, dv \Big)^{-\frac{1}{n}} \\
&\leq \kappa_n^{-1/n} V(\operatorname{sprt} f)^{1/n} (n\kappa_n)^{1/n} \Big(\int_{S^{n-1}} \liminf_{q\to\infty} \|D_v^+ f\|_q^{-n}\, dv \Big)^{-\frac{1}{n}} \\
&\leq \kappa_n^{-1/n} V(\operatorname{sprt} f)^{1/n} (n\kappa_n)^{1/n} \Big(\int_{S^{n-1}} \|D_v^+ f\|_\infty^{-n}\, dv \Big)^{-\frac{1}{n}}.
\end{aligned}$$

The corresponding equality case can be verified by a straightforward computation. □

Affine Nash inequality

Nash's inequality in its optimal form, established by Carlen and Loss [12], states that if $f \in L^1(\mathbb{R}^n) \cap W^{1,2}(\mathbb{R}^n)$, then

$$\|f\|_2^{1+2/n} \leq \beta_n \|\nabla f\|_2 \|f\|_1^{2/n}. \tag{60}$$

The best constant β_n is given by

$$\beta_n^2 = \frac{2\left(1 + \frac{n}{2}\right)^{1+n/2}}{n\lambda_n \kappa_n^{2/n}},$$

where λ_n denotes the first nonzero Neumann eigenvalue of the Laplacian $-\Delta$ on radial functions on B_n. There is equality in (60) if and only if up to normalization and scaling

$$f(x) = \begin{cases} u(|x - x_0|) - u(1), & \text{if } |x| \leq 1 \\ 0, & \text{if } |x| \geq 1, \end{cases}$$

for some $x_0 \in \mathbb{R}^n$. Here, u is the normalized eigenfunction of the Neumann Laplacian on B_n with eigenvalue λ_n. Note the striking feature that all of the extremals have compact support. Nash's inequality and its variants have proven to be very useful in a number of contexts (see, e.g., [3, 5, 7, 39] and the references therein).

From an application of Theorem 1 together with (60), we immediately obtain a new stronger asymmetric affine version of Nash's inequality.

Corollary 6. *If $f \in L^1(\mathbb{R}^n) \cap W^{1,2}(\mathbb{R}^n)$, then*

$$\|f\|_2^{1+2/n} \leq \beta_n\, \mathcal{E}_2^+(f) \|f\|_1^{2/n}. \tag{61}$$

Equality is attained in (61) if up to normalization and scaling

$$f(x) = \begin{cases} u(|\phi(x - x_0)|) - u(1), & \text{if } |x| \leq 1 \\ 0, & \text{if } |x| \geq 1, \end{cases}$$

for some $\phi \in \mathrm{SL}(n)$ and $x_0 \in \mathbb{R}^n$.

Affine Gagliardo–Nirenberg inequalities

The L^p Sobolev inequality (49), Nash's inequality (60) and the logarithmic Sobolev inequality (45) are special cases and a limiting case, respectively, of the Gagliardo–Nirenberg inequalities

$$\|f\|_r \leq C_n(p,r,s)\, \|\nabla f\|_p^\theta\, \|f\|_s^{1-\theta}, \tag{62}$$

where $1 < p < n$, $s < r \leq p^*$, and $\theta \in (0,1)$ is determined by scaling invariance. While inequality (62) can be deduced from (49) with the help of Hölder's inequality, the computation of the optimal constants $C_n(p,r,s)$ is an open problem in general. A breakthrough was recently achieved by Del Pino and Dolbeault [26, 27] (see also [25] for a different approach). They obtained the following sharp one-parameter family of inequalities: Suppose that $1 < p < n$, $p < q \leq p(n-1)/(n-p)$ and let

$$r = \frac{p(q-1)}{p-1} \qquad \text{and} \qquad \theta = \frac{n(q-p)}{(q-1)(np-(n-p)q)}. \tag{63}$$

Then, for every $f \in D^{p,q}(\mathbb{R}^n)$,

$$\|f\|_r \leq \gamma_{n,p,q}\, \|\nabla f\|_p^\theta\, \|f\|_q^{1-\theta}, \tag{64}$$

where $D^{p,q}$ denotes the completion of the space of smooth compactly supported functions with respect to the norm $\|\cdot\|_{p,q}$ defined by $\|f\|_{p,q} = \|\nabla f\|_p + \|f\|_q$. The optimal constant $\gamma_{p,q}$ is given by

$$\gamma_{n,p,q} = \left(\frac{q-p}{p\sqrt{\pi}}\right)^\theta \left(\frac{pq}{n(q-p)}\right)^{\theta/p} \left(\frac{\delta}{pq}\right)^{1/r} \left(\frac{\Gamma\left(\frac{q(p-1)}{q-p}\right)\Gamma\left(1+\frac{n}{2}\right)}{\Gamma\left(\frac{\delta(p-1)}{p(q-p)}\right)\Gamma\left(1+\frac{n(p-1)}{p}\right)}\right)^{\theta/n},$$

where $\delta = np - q(n-p)$. Equality holds in (64) if and only if for some $a \in \mathbb{R}$, $b > 0$ and $x_0 \in \mathbb{R}^n$,

$$f(x) = a\left(1 + b|x - x_0|^{p/(p-1)}\right)^{-(p-1)/(q-p)}.$$

Observe that for $q = p(n-1)/(n-p)$, we have $\theta = 1$ and inequality (64) becomes the sharp L_p Sobolev inequality (49) of Aubin and Talenti. On the other hand, the logarithmic Sobolev inequality (45) corresponds to the limit $q \to p$ in (64). Thus the Gagliardo–Nirenberg inequalities (64) interpolate between the sharp L^p Sobolev and the logarithmic Sobolev inequalities.

We conclude this final section with a family of strengthened asymmetric affine Gagliardo–Nirenberg inequalities which interpolate between inequalities (52) and (7). These inequalities are obtained as an immediate corollary of Theorem 1 and (64):

Corollary 7. *Let $1 < p < n$, $p < q \leq p(n-1)/(n-p)$ and let r, θ be given by (63). If $f \in C_0^\infty(\mathbb{R}^n)$, then*

$$\|f\|_r \leq \gamma_{n,p,q}\, \mathcal{E}_p^+(f)^\theta\, \|f\|_q^{1-\theta}. \tag{65}$$

Equality is attained in (65) if for some $a \in \mathbb{R}$, $\phi \in \mathrm{GL}(n)$ and $x_0 \in \mathbb{R}^n$,

$$f(x) = a\left(1 + |\phi(x - x_0)|^{p/(p-1)}\right)^{-(p-1)/(q-p)}.$$

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