BLASCHKE VALUATIONS

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ABSTRACT. All continuous linearly intertwining symmetric Blaschke valuations on convex bodies are completely classified. It is shown that there is a unique non-trivial such valuation. On symmetric bodies, this valuation is the curvature image operator.

1. INTRODUCTION

A *valuation* is a function $\mathrm{Z} : \mathcal{Q} \to \langle \mathcal{G}, + \rangle$ defined on a class of subsets of $\mathbb{R}^n$ with values in an abelian semigroup $\langle \mathcal{G}, + \rangle$ which satisfies

$$\mathrm{Z}(K \cup L) + \mathrm{Z}(K \cap L) = \mathrm{Z}\, K + \mathrm{Z}\, L, \tag{1}$$

whenever $K, L, K \cup L, K \cap L \in \mathcal{Q}$. Fundamental geometric quantities have been characterized as valuations with natural additional properties. The most famous result in this direction is Hadwiger's celebrated characterization of the intrinsic volumes as continuous and rigid motion invariant real valued valuations. (See e.g. [23], [38], [39] for information on the classical theory of valuations and its applications to integral and stochastic geometry.) Important recent classifications of real or complex valued valuations and related results can be found e.g. in [1]–[7], [10], [20], [21], [27], [31], [32].

In classical convex geometry, the basic additions of convex bodies are Minkowski and Blaschke addition (see Section 2 for precise definitions). Assuming compatibility with the general linear group, Ludwig [26], [29] obtained a complete classification of *Minkowski valuations*, i.e. valuations where addition in (1) is Minkowski addition. Her results establish simple characterizations of fundamental operators like the projection or centroid body operator (see [15]–[18], [28], [30], [41], [43] for related results). This raises the natural question of classifying *Blaschke valuations*, i.e. valuations where addition in (1) is Blaschke addition.

In this paper a classification is established for all continuous symmetric Blaschke valuations on convex bodies which are compatible with the general linear group. It turns out that the only nontrivial example of such a valuation is the curvature image operator on symmetric bodies.

The curvature image is a central notion in the affine geometry of convex bodies; see e.g. [24], [35], [36]. It is important for affine isoperimetric inequalities; see e.g. [33], [37]. In particular, the curvature image is a crucial ingredient in the classical proof of the Blaschke-Santaló inequality. Moreover, the curvature image is closely related to affine normals and affine surface area and thus is an important concept in affine differential geometry (see e.g. [25]).

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In order to state the main result, we collect some notation. A detailed description of the occurring definitions will be given in Sections 2 and 3. Let $\mathcal{K}^n$ denote the space of *convex bodies*, i.e. nonempty compact convex subsets of $\mathbb{R}^n$, endowed with the Hausdorff metric. We write $\mathcal{K}^n_o \subset \mathcal{K}^n$ for convex bodies with nonempty interior which contain the origin. Origin-symmetric bodies in $\mathcal{K}^n_o$ are denoted by $\mathcal{K}^n_c$. Associated with a convex body $K \in \mathcal{K}^n$ is its *surface area measure* $S(K,\cdot)$. If $\omega \subseteq S^{n-1}$ is a Borel set, then $S(K,\omega)$ is defined as the $(n-1)$-dimensional Hausdorff measure of the set of all boundary points of K for which there exists an outward unit normal vector belonging to ω. Suppose that K, $L \in \mathcal{K}^n$ have nonempty interiors. Then Minkowski's existence and uniqueness theorem guarantees the existence of a convex body $K \# L \in \mathcal{K}^n$ with nonempty interior (unique up to translations) such that

$$S(K \# L, \cdot) = S(K,\cdot) + S(L,\cdot).$$

This addition is called *Blaschke addition* and turns the set $\mathcal{K}^n_c$ into an abelian semigroup which we denote by $\langle \mathcal{K}^n_c, \# \rangle$. A valuation with respect to the semigroup $\langle \mathcal{K}^n_c, \# \rangle$ is called *symmetric Blaschke valuation*. Recent results involving Blaschke addition can be found e.g. in [8], [12], [19], [22], [34]–[36], and [42].

A convex body $K \in \mathcal{K}^n$ has *curvature function* $f(K,\cdot) : S^{n-1} \to \mathbb{R}$ if its surface area measure $S(K,\cdot)$ has $f(K,\cdot)$ as a density with respect to spherical Lebesgue measure. The *symmetric curvature image* $\Lambda_c K$ of $K \in \mathcal{K}^n_o$ is defined as the unique body in $\mathcal{K}^n_c$ with curvature function

$$f(\Lambda_c K, \cdot) = \tfrac{1}{2}\, \rho(K,\cdot)^{n+1} + \tfrac{1}{2}\, \rho(-K,\cdot)^{n+1}.$$

Here, $\rho(K,\cdot) : S^{n-1} \to \mathbb{R}$ denotes the *radial function* of K, i.e. $\rho(K,u) = \max\{\lambda \geq 0 :\ \lambda u \in K\}$ for $u \in S^{n-1}$.

The following definition was crucial for previous characterizations of valuations within the affine theory of convex bodies. An operator $\mathrm{Z} : \mathcal{K}^n_o \to \mathcal{K}^n_c$ is called *linearly intertwining* if it is positively homogeneous and satisfies either

$$\mathrm{Z}(\phi K) = \phi\, \mathrm{Z}\, K \tag{2}$$

for every $\phi \in \mathrm{SL}(n)$ and all $K \in \mathcal{K}^n_o$, or

$$\mathrm{Z}(\phi K) = \phi^{-t}\, \mathrm{Z}\, K \tag{3}$$

for every $\phi \in \mathrm{SL}(n)$ and all $K \in \mathcal{K}^n_o$.

Our main result for dimensions $n \geq 3$ is the following.

Theorem 1. *A map $\mathrm{Z} : \mathcal{K}^n_o \to \langle \mathcal{K}^n_c, \# \rangle$ is a nontrivial, continuous, and linearly intertwining symmetric Blaschke valuation if and only if there exists a constant $c > 0$ such that*

$$\mathrm{Z}\, K = c \Lambda_c K$$

for every $K \in \mathcal{K}^n_o$.

Here, a valuation Z is called *trivial*, if $\mathrm{Z}\, K$ is a multiple of the Blaschke sum of K and $-K$ for every $K \in \mathcal{K}^n_o$. Note that the symmetric curvature image satisfies (3) while the trivial Blascke valuation satisfies (2). We are not aware of additional examples of continuous Blaschke valuations which satisfy (2) or (3) and are not homogeneous. We remark that Blaschke addition coincides with Minkowski addition for 2-dimensional bodies. Since characterizations for planar Minkowski valuations were obtained by Ludwig [29], we confine our attention to dimensions greater or equal than three.

2. Preliminaries

We work in Euclidean n-space $\mathbb{R}^n$ with $n \geq 3$. Let e_i, $i = 1, \dots, n$ be the elements of the standard basis of $\mathbb{R}^n$. The usual scalar product of two vectors x and $y \in \mathbb{R}^n$ shall be denoted by $x \cdot y$. For $y \in \mathbb{R}^n \backslash \{o\}$, the halfspace $\{x \in \mathbb{R}^n : \ x \cdot y \leq t\}$ is denoted by $H^-_{y,t}$. Moreover, let $H^+_{y,t} = \{x \in \mathbb{R}^n : \ x \cdot y \geq t\}$ and $H_{y,t} = H^+_{y,t} \cap H^-_{y,t}$. For simplicity, we write $u^- = H^-_{u,0}$ and $u^+ = H^+_{u,0}$ as well as $u^\perp = H_{u,0}$. The convex hull of a set $A \subset \mathbb{R}^n$ will be denoted by $[A]$. To shorten notation we write $[A, \pm x_1, \dots, \pm x_m]$ instead of $[A \cup \{x_1, -x_1, \dots, x_m, -x_m\}]$ for $A \subset \mathbb{R}^n$, $m \in \mathbb{N}$, and $x_1, \dots, x_m \in \mathbb{R}^n$. Let $\|\cdot\|$ be the Euclidean norm and denote by $B^n := \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ the Euclidean unit ball. The boundary of B^n is denoted by S^{n-1}. We write $B(o, r)$ for the Euclidean ball with center at the origin and radius r. Usual Lebesgue measure on $\mathbb{R}^n$ is denoted by V and we abbreviate $V(B^n) = \kappa_n$. For a Borel measurable function $f : S^{n-1} \to \mathbb{R}$, let

$$\|f\|_p = \left(\int_{S^{n-1}} |f(u)|^p \, du \right)^{\frac{1}{p}},$$

where integration is with respect to spherical Lebesgue measure. The function f is said to be in $L^p(S^{n-1})$ or simply in L^p, if $\|f\|_p < \infty$.

General references for the theory of convex bodies are the books by Gardner [11], Gruber [14], and Schneider [40]. Associated with a convex body $K \in \mathcal{K}^n$ is its *support function* $h_K : \mathbb{R}^n \to \mathbb{R}$ defined by

$$h_K(x) = \max\{x \cdot y : \ y \in K\}.$$

The support function is positively homogeneous of degree one. We shall denote by δ the *Hausdorff metric* on $\mathcal{K}^n$: If K, $L \in \mathcal{K}^n$, then $\delta(K, L)$ is defined by

$$\delta(K, L) = \sup_{u \in S^{n-1}} |h_K(u) - h_L(u)|.$$

The *Minkowski sum* $K + L \in \mathcal{K}^n$ of two bodies K, $L \in \mathcal{K}^n$ is the usual vector sum of K and L. Note that

$$h_{K+L} = h_K + h_L.$$

A valuation $\mathrm{Z} : \mathcal{Q} \to \langle \mathcal{K}^n, + \rangle$ is called *Minkowski valuation.* Suppose that $K \in \mathcal{K}^n$ contains the origin in its interior. Then the *polar body* K^* of K is defined by

$$K^* = \{x \in \mathbb{R}^n : \ x \cdot y \leq 1 \ \text{ for all } \ y \in K\}.$$

The support function of K^* and the radial function of K are related by

$$h_{K^*} = \rho_K^{-1} \tag{4}$$

viewed as functions on the sphere.

As explained in the introduction, a body $K \in \mathcal{K}^n$ gives rise to an important finite Borel measure $S(K, \cdot)$ on the sphere called surface area measure. The following basic properties of this measure can be found in [40, Chapter 4]. The connection between surface area measures and the surface area $S(K)$ of a convex body $K \in \mathcal{K}^n$ is provided by

$$S(K, S^{n-1}) = S(K). \tag{5}$$

For every $\lambda > 0$, we have the homogeneity property

$$S(\lambda K, \cdot) = \lambda^{n-1} S(K, \cdot). \tag{6}$$

The measure $S(K,\cdot)$ is rotation covariant, i.e.

$$S(\vartheta K, \vartheta\omega) = S(K,\omega), \qquad \text{for all } \ \vartheta \in \mathrm{O}(n),\ \ \omega \in \mathfrak{B}(S^{n-1}). \tag{7}$$

Here, $\mathfrak{B}(S^{n-1})$ denotes the Borel sets on the sphere S^{n-1}. Moreover, the surface area measure is weakly continuous. To be precise, if $K_j \to K$ with respect to Hausdorff distance, then $S(K_j,\cdot)$ converges weakly to $S(K,\cdot)$.

Surface area measures have their centroids at the origin and cannot be concentrated on any great subsphere. Conversely, Minkowski's existence theorem (see e.g. [40, Theorem 7.1.2]) shows that a finite Borel measure on the sphere which has these two properties is the surface area measure of a certain convex body with nonempty interior. Moreover, Minkowski's uniqueness theorem states that two convex bodies with nonempty interior and the same surface area measure are translates of each other (see e.g. [40, Theorem 7.2.1]). Note that by Minkowski's existence and uniqueness results on surface area measures, Blaschke addition is well defined on the set $\mathcal{K}_c^n$. We need the following stability version due to Diskant [9] (see also [40, Theorem 7.2.2]) of the above uniqueness result. Let $\mathcal{K}^n(r,R)$ denote the set of convex bodies which contain some ball of radius $r > 0$ and are contained in some ball of radius $R > r$.

Theorem 2. *Let $0 < r < R$. There exist numbers $\varepsilon_0 > 0$ and c, depending only on n, r, R with the following property. If K, $L \in \mathcal{K}^n(r,R)$ satisfy*

$$|S(K,\omega) - S(L,\omega)| \leq \varepsilon$$

for every $\omega \in \mathfrak{B}(S^{n-1})$ with some $\varepsilon \in [0,\varepsilon_0]$, then

$$\delta(K,L') \leq c\varepsilon^{\frac{1}{n}}$$

for a suitable translate L' of L.

For arbitrary convex bodies K, $L \in \mathcal{K}^n$ we have

$$\int_{S^{n-1}} h_L(u)\, dS(K,u) = \lim_{\varepsilon \to 0^+} \frac{V(K+\varepsilon L) - V(K)}{\varepsilon},$$

(see e.g. [40, Theorem 5.1.6]). We therefore get

$$\int_{S^{n-1}} h_L(u)\, dS(\phi K,u) = |\det\phi| \int_{S^{n-1}} h_L(\phi^{-t}u)\, dS(K,u). \tag{8}$$

for arbitrary $\phi \in \mathrm{GL}(n)$, where $\det\phi$ denotes the determinant of ϕ. The *cosine transform* of a finite Borel measure μ on S^{n-1} is defined by

$$(\mathrm{C}\,\mu)(x) = \int_{S^{n-1}} |x\cdot v|\, d\mu(v), \qquad x \in \mathbb{R}^n.$$

For non-negative Borel measurable f on S^{n-1}, let $\mathrm{C}\,f$ be the cosine transform of the absolutely continuous measure (with respect to spherical Lebesgue measure) with density f. An important property of this integral transform is the following injectivity behavior. If μ is a signed finite even Borel measure on the sphere, then

$$\int_{S^{n-1}} |u\cdot v|\, d\mu(v) = 0 \quad \text{for all } \ u \in S^{n-1} \Longrightarrow \mu = 0, \tag{9}$$

(see e.g. [11, Theorem C.2.1]). We need the transformation behavior

$$\mathrm{C}\,S(\phi K,\cdot)(x) = |\det\phi|\, \mathrm{C}\,S(K,\cdot)(\phi^{-1}x), \qquad x \in \mathbb{R}^n, \tag{10}$$

of the cosine transform which is the special case of formula (8) for $L = [\pm x]$.

The cosine transform gives rise to two fundamental operators in convex geometry and geometric tomography. The *projection body* ΠK of $K \in \mathcal{K}^n$ is the convex body with support function

$$h_{\Pi K} = \tfrac{1}{2}\,\mathrm{C}\,S(K,\cdot).$$

We remark that for a unit vector $u \in S^{n-1}$ the value of $h(\Pi K, u)$ is equal to the $(n-1)$-dimensional volume of the projection of K to the hyperplane orthogonal to u, i.e.

$$\tfrac{1}{2}\,\mathrm{C}\,S(K,\cdot)(u) = V_{n-1}(K|u^\perp). \tag{11}$$

For $\phi \in \mathrm{GL}(n)$, we obtain from formula (10) for $L = [\pm x]$ that

$$\Pi(\phi K) = |\det \phi|\phi^{-t}\Pi K. \tag{12}$$

This was first proved by Petty but the proof given here was found by Lutwak [35]. In particular, the projection body operator is positively homogeneous of degree $n-1$. The *centroid body* ΓK of a compact set $K \subset \mathbb{R}^n$ which is star-shaped with respect to the origin is defined by

$$h_{\Gamma K} = \tfrac{1}{n+1}\,\mathrm{C}\,\rho_K^{n+1}.$$

We remark that our definition of the centroid body differs from the usual one by a normalizing factor. A change to polar coordinates proves

$$h(\Gamma K, u) = \int_K |x\cdot u|\,dx,$$

where integration is with respect to Lebesgue measure. This directly yields

$$\Gamma(\phi K) = |\det \phi|\phi\Gamma K, \tag{13}$$

for every $\phi \in \mathrm{GL}(n)$. Therefore, the centroid body operator Γ is positively homogeneous of degree $n+1$.

We conclude this section with the precise definition of linearly intertwining operators. Let $\mathcal{Q}$ be a subset of the power set of $\mathbb{R}^n$ which is closed with respect to the usual action of the general linear group $\mathrm{GL}(n)$. An operator $\mathrm{Z} : \mathcal{Q} \to \mathfrak{P}(\mathbb{R}^n)$, where $\mathfrak{P}(\mathbb{R}^n)$ denotes the power set of $\mathbb{R}^n$, is called $\mathrm{SL}(n)$ *covariant*, if

$$\mathrm{Z}(\phi K) = \phi\,\mathrm{Z}\,K$$

for all $\phi \in \mathrm{SL}(n)$ and every $K \in \mathcal{Q}$. It is called $\mathrm{SL}(n)$ *contravariant*, if

$$\mathrm{Z}(\phi K) = \phi^{-t}\,\mathrm{Z}\,K$$

for all $\phi \in \mathrm{SL}(n)$ and every $K \in \mathcal{Q}$. Here, ϕ^{-t} denotes the inverse of the transpose of ϕ. We call Z *positively homogeneous*, if there exists an $r \in \mathbb{R}$ such that

$$\mathrm{Z}(\lambda K) = \lambda^r\,\mathrm{Z}\,K$$

for every $\lambda > 0$ and every $K \in \mathcal{Q}$. Finally, Z is called *linearly intertwining* if it is co- or contravariant and positively homogeneous on its domain.

3. The curvature image

A compact set $L \subset \mathbb{R}^n$ which is star-shaped with respect to the origin o is uniquely determined by its *radial function* $\rho_L : S^{n-1} \to \mathbb{R}$ defined for $u \in S^{n-1}$ by $\rho_L(u) = \max\{\lambda \geq 0 : \ \lambda u \in L\}$. The set of compact sets which are star-shaped with respect to the origin and whose radial functions are greater than zero on a set of positive spherical Lebesgue measure is denoted by $\mathcal{S}^n$. Let $L \in \mathcal{S}^n$ be origin-symmetric. By Minkowski's existence and uniqueness theorem the *curvature image* ΛL is defined as the unique origin-symmetric convex body with curvature function

$$f(\Lambda L, \cdot) = \rho(L, \cdot)^{n+1}.$$

We emphasize that our definition of the curvature image follows, up to normalization, [37]. The curvature image and the symmetric curvature image are related by the following operator. The $(n+1)$*-chordal symmetral* $\tilde{\nabla} : \mathcal{S}^n \to \mathcal{S}^n$ is defined by

$$\rho_{\tilde{\nabla} L}^{n+1} = \tfrac{1}{2}\, \rho_L^{n+1} + \tfrac{1}{2}\, \rho_{-L}^{n+1}.$$

Clearly, Λ_c equals $\Lambda \circ \tilde{\nabla}$. The aim of this section is to derive the characterizing properties (stated in Theorem 1) of Λ_c. Previously, the curvature image was defined only on star-shaped sets with continuous radial functions. Since in our context the domain of Λ is larger, we provide detailed proofs of our claims.

Lemma 1. *The symmetric curvature image $\Lambda_c : \mathcal{K}_o^n \to \langle \mathcal{K}_c^n, \# \rangle$ is a continuous linearly intertwining symmetric Blaschke valuation.*

Before we start with the proof of this lemma we emphasize that, by definition, ΛL is origin-symmetric, i.e.

$$\Lambda L = -\Lambda L \quad \text{for origin-symmetric } L \in \mathcal{S}^n. \tag{14}$$

Moreover, if we combine the definitions of projection and centroid bodies, then we arrive at the crucial factorization property of the centroid body operator

$$\Gamma L = c \Pi \Lambda L \quad \text{for origin-symmetric } L \in \mathcal{S}^n, \tag{15}$$

with $c = 2/(n+1)$ (see [35, Formula 6.12]).

Proof. Suppose that $L \in \mathcal{S}^n$ is origin-symmetric. Note that $\rho_{\lambda L} = \lambda \rho_L$. From (6) we therefore get

$$\Lambda \lambda L = \lambda^{\frac{n+1}{n-1}} \Lambda L$$

for every $\lambda > 0$. The $\mathrm{SL}(n)$ contravariance of Λ is an immediate consequence of property (15). Indeed, from (13) and (12) we obtain

$$c \Pi \Lambda(\phi L) = \Gamma(\phi L) = \phi \Gamma L = c \phi \Pi \Lambda L = c \Pi \phi^{-t} \Lambda L.$$

The injectivity property (9) and the symmetry relation (14) yield

$$\Lambda(\phi L) = \phi^{-t} \Lambda L \tag{16}$$

for every $\phi \in \mathrm{SL}(n)$. Note that

$$\rho_{\phi L}(u) = \|\phi^{-1} u\|^{-1} \rho_L(\phi^{-1} u / \|\phi^{-1} u\|)$$

for $\phi \in \mathrm{SL}(n)$. Thus we obtain

$$\tilde{\nabla}(\phi L) = \phi \tilde{\nabla} L.$$

Combining this with (16) proves that $\Lambda \circ \tilde{\nabla}$ is $\mathrm{SL}(n)$ contravariant. Similarly, the homogeneity of $\Lambda \circ \tilde{\nabla}$ follows from the homogeneity of Λ and $\tilde{\nabla}$. Consequently, the symmetric curvature image Λ_c is linearly intertwining.

Next, we establish the continuity of $\Lambda \circ \tilde{\nabla}$ on $\mathcal{K}_o^n$. Let $K_m \to K$ with respect to Hausdorff distance as m tends to infinity. Initially, we establish the fact that

$$\|\rho_{K_m} - \rho_K\|_{n+1} \longrightarrow 0. \tag{17}$$

Since $\rho_{K_m} - \rho_K$ is uniformly bounded, it suffices to prove the pointwise convergence $|\rho_{K_m}(u) - \rho_K(u)| \to 0$ for almost every $u \in S^{n-1}$. If the origin is an interior point of K, then (17) follows from [13, Lemma 2.3.2]. Suppose that the origin is a boundary point of K. Let u be a unit vector and $\lambda_{m_j} = \rho_{K_{m_j}}(u)$ be a convergent subsequence of $\rho_{K_m}(u)$ with limit λ. Moreover, denote by $Q(K,o)$ the support cone of K at o, i.e.

$$Q(K,o) = \bigcap_{v \in N(K,o)} H_{v,0}^-$$

where $N(K,o)$ is the set of all outward unit normal vectors of K at o. If $u \notin Q(K,o)$, then there exists a $v \in S^{n-1}$ such that $u \cdot v > 0$ and $K \subset H_{v,0}^-$. Since $\lambda_{m_j} u \cdot v \leq h_{K_{m_j}}(v)$ for every j, we infer $\lambda u \cdot v \leq 0$. Thus $\lambda = 0 = \rho_K(u)$. Next, suppose that $u \in \operatorname{int} Q(K,o)$. As before, it follows from $\lambda_{m_j} u \cdot v \leq h_{K_{m_j}}(v)$ for every j and every v, that $\lambda u \cdot v \leq h_K(v)$. Consequently, $\lambda u \in K$. Moreover, since $\lambda_{m_j} u \in \operatorname{bd} K_{m_j}$ for every j, we have $\lambda u \in \operatorname{bd} K$. By assumption, $u \cdot v < 0$ for every outward unit normal v of K at o and hence $\lambda = \rho_K(u)$. In conclusion, $\rho_{K_{m_j}}(u) \to \rho_K(u)$ for every convergent subsequence of $\rho_{K_m}(u)$ and every $u \notin \operatorname{bd} Q(K,o)$. This finally proves (17) because $\rho_{K_m}(u) \to \rho_K(u)$ for all $u \notin \operatorname{bd} Q(K,o)$ and $\operatorname{bd} Q(K,o)$ has Lebesgue measure zero being the boundary of a convex set.

From (17) we deduce that the $(n+1)$-chordal symmetrals of the involved bodies satisfy

$$\|\rho_{\tilde{\nabla} K_m} - \rho_{\tilde{\nabla} K}\|_{n+1} \longrightarrow 0. \tag{18}$$

Indeed, the reverse triangle inequality for the ℓ_{n+1}^2-norm shows

$$\|\rho_{\tilde{\nabla} K_m} - \rho_{\tilde{\nabla} K}\|_{n+1} \leq \left(\frac{1}{2} \left\| |\rho_{K_m} - \rho_K|^{n+1} + |\rho_{-K_m} - \rho_{-K}|^{n+1} \right\|_1 \right)^{\frac{1}{n+1}}.$$

The invariance of the spherical Lebesgue measure with respect to orthogonal maps now implies

$$\|\rho_{\tilde{\nabla} K_m} - \rho_{\tilde{\nabla} K}\|_{n+1} \leq \|\rho_{K_m} - \rho_K\|_{n+1}.$$

In order to establish the continuity of $\Lambda \circ \tilde{\nabla}$ it is therefore enough to show that $\Lambda L_k \to \Lambda L$ with respect to Hausdorff distance provided that $\|\rho_{L_k} - \rho_L\|_{n+1} \to 0$ for origin-symmetric bodies L_k, $L \in \mathcal{S}^n$. In order to apply Theorem 2, we have to ensure that there exist numbers r and R which are independent of k and satisfy $0 < r < R$ such that $\Lambda L_k \in \mathcal{K}^n(r,R)$ for every $k \in \mathbb{N}$. This will be done in the next paragraphs. The positive constants $c_1, c_2, \ldots$ in the following derivation will be independent of k. By the mean value theorem and Hölder's inequality we deduce

$$\begin{aligned}
\int_{S^{n-1}} |\rho_{L_k}^{n+1} - \rho_L^{n+1}| \, du &\leq (n+1) \int_{S^{n-1}} |\rho_{L_k} - \rho_L| (\rho_{L_k}^n + \rho_L^n) \, du \\
&\leq (n+1) \|\rho_{L_k} - \rho_L\|_{n+1} \left(\|\rho_{L_k}\|_{n+1}^n + \|\rho_L\|_{n+1}^n \right).
\end{aligned}$$

Since $\|\rho_{L_k} - \rho_L\|_{n+1} \to 0$, the reverse triangle inequality shows that $\|\rho_{L_k}\|_{n+1} \to \|\rho_L\|_{n+1}$. Thus there exists a constant c_1 with

$$\int_{S^{n-1}} |\rho_{L_k}^{n+1} - \rho_L^{n+1}|\, du \leq c_1 \|\rho_{L_k} - \rho_L\|_{n+1} \tag{19}$$

for every $k \in \mathbb{N}$. In particular, by (5), the surface areas of ΛL_k are bounded from above, i.e.

$$S(\Lambda L_k) \leq c_2, \qquad \text{for all } k \in \mathbb{N}. \tag{20}$$

Next, we establish the curvature image inequality

$$V(\Lambda L)^{n-1} \geq \kappa_n^{-2} V(L)^{n+1} \tag{21}$$

for origin-symmetric $L \in \mathcal{S}^n$. For star-shaped sets with positive and continuous radial functions this was established by Lutwak [37, Section 9]. The proof of (21) in the general case is similar to that of Lutwak because the involved functionals admit extensions from continuous to L^{n+1} radial functions. Indeed, if we denote by $\mathcal{K}_{co}^n$ the set of n-dimensional convex bodies whose centroids lie at the origin, then by (4) and Hölder's inequality

$$\begin{aligned} \kappa_n^{\frac{1}{n}} G(\Lambda L) &:= \inf \left\{ V(Q)^{\frac{1}{n}} \int_{S^{n-1}} h_{Q^*}(u)\, dS(\Lambda L, u) : Q \in \mathcal{K}_{co}^n \right\} \\ &= \inf \left\{ V(Q)^{\frac{1}{n}} \int_{S^{n-1}} \rho_Q^{-1}(u) \rho_L^{n+1}(u)\, du : Q \in \mathcal{K}_{co}^n \right\} \\ &\geq nV(L)^{\frac{n+1}{n}}. \end{aligned}$$

Here, $G(\Lambda L)$ is *Petty's geominimal surface area* of ΛL. Since

$$G(K)^n \leq n^n \kappa_n V(K)^{n-1}$$

for arbitrary $K \in \mathcal{K}_o^n$ (see [37, Section 8]), we obtain (21). Consequently,

$$V(\Lambda L_k)^{n-1} \geq \kappa_n^{-2} V(L_k)^{n+1}$$

for every $k \in \mathbb{N}$. The polar coordinate formula for volume states

$$nV(L_k) = \|\rho_{L_k}\|_n^n.$$

By Hölders inequality we know that $\|\rho_{L_k} - \rho_L\|_n \to 0$, and hence $\|\rho_{L_k}\|_n \to \|\rho_L\|_n$. This implies that

$$V(\Lambda L_k) \geq c_3 > 0. \tag{22}$$

We recall that for $K \in \mathcal{K}_o^n$ the isoperimetric inequality states

$$\frac{S(K)^n}{V(K)^{n-1}} \geq \frac{S(B^n)^n}{V(B^n)^{n-1}}$$

(see e.g. [14, Theorem 8.7]). If we combine this with (20), then we arrive at

$$V(\Lambda L_k) \leq c_4. \tag{23}$$

Summarizing (20), (22), and (23), we proved the existence of positive constants c_2, c_3, c_4 which are independent of k such that

$$S(\Lambda L_k) \leq c_2, \qquad \text{and } c_3 \leq V(\Lambda L_k) \leq c_4. \tag{24}$$

This implies that $\Lambda L_k \in \mathcal{K}^n(r, R)$ for all k. Indeed, assume that for every $r > 0$ there exists a k such that $B(o, r) \not\subset \Lambda L_k$. Then we can find a sequence x_j, $j \in \mathbb{N}$, of points in $\mathbb{R}^n$ such that $\|x_j\| \leq 1/j$ but $x_j \notin \Lambda L_{k_j}$ for every $j \in \mathbb{N}$. Thus there exists a sequence of unit vectors u_j and positive numbers t_j such that $x_j \in H^+_{u_j,t_j}$ and $\Lambda L_{k_j} \subset H^-_{u_j,t_j}$. Since ΛL_{k_j} is centrally symmetric, we also have $\Lambda L_{k_j} \subset H^+_{u_j,-t_j}$. Denote by $(\Lambda L_{k_j})|u^\perp$ the image of ΛL_{k_j} under the orthogonal projection onto the hyperplane orthogonal to u. Since the orthogonal projection onto a hyperplane is Lipschitzian, there exists a constant c_5 independent of j such that

$$V(\Lambda L_{k_j}) \leq 2t_j V_{n-1}((\Lambda L_{k_j})|u_j^\perp) \leq c_5 t_j S(\Lambda L_{k_j})$$

for every $j \in \mathbb{N}$. But this contradicts the existence of a lower bound of the volumes in (24) since t_j converge to zero as j tends to infinity. Thus there exists an $r > 0$ with $B(o,r) \subset \Lambda L_k$ for all k. Now, it follows directly from this and the upper bound for the volumes $V(\Lambda L_k)$ that every body ΛL_k has to be contained in a ball $B(o,R)$ for some $R > r$ independent of k. Hence we proved $\Lambda L_k \in \mathcal{K}^n(r,R)$ for every $k \in \mathbb{N}$.

Let ε_0 and c be as in Theorem 2 and suppose $\varepsilon \leq c\varepsilon_0^{1/n}$. Choose $N \in \mathbb{N}$ such that $\|\rho_{L_k} - \rho_L\|_{n+1} \leq c_1^{-1} c^{-n} \varepsilon^n$ for every $k \geq N$. Then we deduce from (19) that

$$|S(\Lambda L_k, \omega) - S(\Lambda L, \omega)| \leq c^{-n}\varepsilon^n.$$

Theorem 2 yields vectors $x_k \in \mathbb{R}^n$ with

$$\delta(\Lambda L_k, \Lambda L - x_k) \leq \varepsilon$$

for each $k \geq N$. But ΛL_k as well as ΛL are origin-symmetric, and so the definition of the Hausdorff metric implies

$$\delta(\Lambda L_k, \Lambda L) \leq \delta(\Lambda L_k, \Lambda L - x_k).$$

So we finally proved that the map $\Lambda \circ \tilde{\nabla}$ is continuous.

The fact that Λ_c is a Blaschke valuation is a direct consequence of the relations

$$\rho_{L_1 \cup L_2}(u) = \max\{\rho_{L_1}(u), \rho_{L_2}(u)\}, \quad \rho_{L_1 \cap L_2}(u) = \min\{\rho_{L_1}(u), \rho_{L_2}(u)\}, \tag{25}$$

for two star-shaped sets $L_1, L_2 \in \mathcal{S}^n$. □

4. Proof of the characterization theorem

In order to establish our classification result we proceed as follows. We consider homogeneous $\mathrm{SL}(n)$ co- or contravariant Blaschke valuations Z separately. It turns out that there exists only one possible degree of homogeneity for Z in each case. For those Z we deduce that the Minkowski valuations $\Pi \circ \mathrm{Z}$ can be extended from $\mathcal{K}^n_o$ to all convex bodies containing the origin. Known characterizations of Minkowski valuations will finally prove that Z is of the form stated in Theorem 1.

We remark that an operator $\mathrm{Z} : \mathcal{Q} \to \mathfrak{P}(\mathbb{R}^n)$ is $\mathrm{SL}(n)$ covariant and positively homogeneous of degree q if and only if it satisfies

$$\mathrm{Z}(\phi K) = (\det \phi)^{\frac{q-1}{n}} \phi \, \mathrm{Z}\, K \tag{26}$$

for every $K \in \mathcal{Q}$ and every $\phi \in \mathrm{GL}(n)$ with positive determinant. Similarly, it is $\mathrm{SL}(n)$ contravariant and homogeneous of degree q if and only if it satisfies

$$\mathrm{Z}(\phi K) = (\det \phi)^{\frac{q+1}{n}} \phi^{-t} \, \mathrm{Z}\, K \tag{27}$$

for every $K \in \mathcal{Q}$ and every $\phi \in \mathrm{GL}(n)$ with positive determinant.

4.1. **The covariant case.**

Lemma 2. *If* $\mathrm{Z} : \mathcal{K}_o^n \to \langle \mathcal{K}_c^n, \# \rangle$ *is a continuous* $\mathrm{SL}(n)$ *covariant symmetric Blaschke valuation which is homogeneous of degree* q, *then* $q \geq 1$.

Proof. Suppose $K \in \mathcal{K}_o^n$ is an arbitrary convex body such that $K \cap e_n^+$ and $K \cap e_n^-$ are n-dimensional. For every positive s we have

$$\begin{aligned}
[K \cap e_n^+, \pm s e_n] \cup [K \cap e_n^-, \pm s e_n] &= [K, \pm s e_n], \\
[K \cap e_n^+, \pm s e_n] \cap [K \cap e_n^-, \pm s e_n] &= [K \cap e_n^\perp, \pm s e_n].
\end{aligned}$$

Since Z is a Blaschke valuation we obtain

$$S(\mathrm{Z}[K, \pm s e_n], \cdot) + S(\mathrm{Z}[K \cap e_n^\perp, \pm s e_n], \cdot) = S(\mathrm{Z}[K \cap e_n^+, \pm s e_n], \cdot) + S(\mathrm{Z}[K \cap e_n^-, \pm s e_n], \cdot),$$

and therefore

$$\begin{aligned}
&\int_{S^{n-1}} h_L \, dS(\mathrm{Z}[K, \pm s e_n], \cdot) + \int_{S^{n-1}} h_L \, dS(\mathrm{Z}[K \cap e_n^\perp, \pm s e_n], \cdot) \\
&= \int_{S^{n-1}} h_L \, dS(\mathrm{Z}[K \cap e_n^+, \pm s e_n], \cdot) + \int_{S^{n-1}} h_L \, dS(\mathrm{Z}[K \cap e_n^-, \pm s e_n], \cdot)
\end{aligned} \tag{28}$$

for an arbitrary convex body $L \in \mathcal{K}^n$. For positive s define a linear map ϕ by

$$\phi e_i = e_i, \qquad i = 1, \ldots, n-1, \qquad \phi e_n = s e_n.$$

From the $\mathrm{SL}(n)$ covariance and homogeneity of Z as well as relation (26) we get

$$\int_{S^{n-1}} h_L \, dS(\mathrm{Z}[K \cap e_n^\perp, \pm s e_n], \cdot) = \int_{S^{n-1}} h_L \, dS\left(s^{\frac{q-1}{n}} \phi \, \mathrm{Z}[K \cap e_n^\perp, \pm e_n], \cdot \right).$$

The homogeneity property (6) of surface area measures and formula (8) yield

$$\begin{aligned}
\int_{S^{n-1}} h_L \, dS(\mathrm{Z}[K \cap e_n^\perp, \pm s e_n], \cdot) &= s^\alpha \int_{S^{n-1}} h_L \, dS(\phi \, \mathrm{Z}[K \cap e_n^\perp, \pm e_n], \cdot) \\
&= s^{\alpha+1} \int_{S^{n-1}} h_L \circ \phi^{-t} \, dS(\mathrm{Z}[K \cap e_n^\perp, \pm e_n], \cdot)
\end{aligned}$$

with $\alpha = \frac{(q-1)(n-1)}{n}$. Since support functions are positively homogeneous of degree one, we obtain

$$\begin{aligned}
&\int_{S^{n-1}} h_L(u) \, dS(\mathrm{Z}[K \cap e_n^\perp, \pm s e_n], u) \\
&\qquad = s^\alpha \int_{S^{n-1}} h_L(s u_1, \ldots, s u_{n-1}, u_n) \, dS(\mathrm{Z}[K \cap e_n^\perp, \pm e_n], u).
\end{aligned}$$

If we take $L = B^n$, then $h_L(x) = \|x\|$ and we infer from (5) and (28) that

$$\begin{aligned}
&s^\alpha \int_{S^{n-1}} \|(s u_1, \ldots, s u_{n-1}, u_n)\| \, dS(\mathrm{Z}[K \cap e_n^\perp, \pm e_n], u) \\
&\qquad = S(\mathrm{Z}[K \cap e_n^+, \pm s e_n]) + S(\mathrm{Z}[K \cap e_n^-, \pm s e_n]) - S(\mathrm{Z}[K, \pm s e_n]).
\end{aligned} \tag{29}$$

Relation (11) proves the equality

$$\begin{aligned}
&\lim_{s \to 0^+} \int_{S^{n-1}} \|(s u_1, \ldots, s u_{n-1}, u_n)\| \, dS(\mathrm{Z}[K \cap e_n^\perp, \pm e_n], u) \\
&\qquad = \int_{S^{n-1}} |u_n| \, dS(\mathrm{Z}[K \cap e_n^\perp, \pm e_n], u) \\
&\qquad = 2 V_{n-1}(\mathrm{Z}[K \cap e_n^\perp, \pm e_n] | e_n^\perp).
\end{aligned}$$

Note that the range of Z consists of n-dimensional convex bodies containing the origin in their interiors. Consequently, the last limit is greater than zero. Moreover, since

$$\begin{aligned}\lim_{s\to 0^+}[K\cap e_n^+, \pm s e_n] &= K\cap e_n^+,\\ \lim_{s\to 0^+}[K\cap e_n^-, \pm s e_n] &= K\cap e_n^-,\\ \lim_{s\to 0^+}[K, \pm s e_n] &= K,\end{aligned}$$

the continuity of Z together with the weak continuity of surface area measures implies that the right hand side of (29) converges to a finite number as s tends to zero. Thus by (29) we obtain that s^α converges for $s \to 0^+$. This implies $\alpha \geq 0$. The definition of α finally shows that the degree q of homogeneity has to be greater or equal than one. □

In order to further reduce the possible degrees of homogeneity of continuous and $\mathrm{SL}(n)$ covariant symmetric Blaschke valuations, we investigate their behavior on the standard simplex

$$T^n = [o, e_1, \dots, e_n].$$

For this reason, the following linear transformations on $\mathbb{R}^n$ will be of importance. For $0 < \lambda < 1$ and $\varepsilon \geq 0$ we define

$$\begin{aligned}\phi_\varepsilon e_2 &= (1-\lambda-\varepsilon)e_1 + (\lambda+\varepsilon)e_2, &\qquad \phi_\varepsilon e_k &= e_k \text{ for } k\neq 2,\\ \psi_\varepsilon e_1 &= (1-\lambda+\varepsilon)e_1 + (\lambda-\varepsilon)e_2, &\qquad \psi_\varepsilon e_k &= e_k \text{ for } k\neq 1.\end{aligned}$$

Moreover, let

$$\begin{aligned}\zeta_\varepsilon e_1 &= (1-\lambda+\varepsilon)e_1 + (\lambda-\varepsilon)e_2,\\ \zeta_\varepsilon e_2 &= (1-\lambda-\varepsilon)e_1 + (\lambda+\varepsilon)e_2,\\ \zeta_\varepsilon e_k &= e_k \text{ for } k\neq 1,2.\end{aligned}$$

Although all of the above maps depend on λ, we will not make this explicit in our notation. For fixed λ and $\varepsilon > 0$, these transformations are invertible. Note that

$$\begin{aligned}\phi_\varepsilon^{-1} e_2 &= -\frac{1-\lambda-\varepsilon}{\lambda+\varepsilon}e_1 + \frac{1}{\lambda+\varepsilon}e_2, &\qquad \phi_\varepsilon^{-1} e_k &= e_k \text{ for } k\neq 2,\\ \psi_\varepsilon^{-1} e_1 &= \frac{1}{1-\lambda+\varepsilon}e_1 - \frac{\lambda-\varepsilon}{1-\lambda+\varepsilon}e_2, &\qquad \psi_\varepsilon^{-1} e_k &= e_k \text{ for } k\neq 1,\end{aligned}$$

as well as

$$\begin{aligned}\zeta_\varepsilon^{-1} e_1 &= \tfrac{\lambda+\varepsilon}{2\varepsilon}e_1 + \tfrac{-\lambda+\varepsilon}{2\varepsilon}e_2,\\ \zeta_\varepsilon^{-1} e_2 &= \tfrac{\lambda-1+\varepsilon}{2\varepsilon}e_1 + \tfrac{1-\lambda+\varepsilon}{2\varepsilon}e_2,\\ \zeta_\varepsilon^{-1} e_k &= e_k \text{ for } k\geq 3.\end{aligned}$$

Moreover, we have

$$\phi_\varepsilon T^n \cup \psi_\varepsilon T^n = T^n, \qquad \phi_\varepsilon T^n \cap \psi_\varepsilon T^n = \zeta_\varepsilon T^n, \tag{30}$$

for $0 < \varepsilon < \min\{\lambda, 1-\lambda\}$.

Lemma 3. *If* $\mathrm{Z} : \mathcal{K}_o^n \to \langle \mathcal{K}_c^n, \# \rangle$ *is a continuous* $\mathrm{SL}(n)$ *covariant symmetric Blaschke valuation which is homogeneous of degree* q*, then* $q = 1$.

Proof. From Lemma 2 we know that $q \geq 1$. Assume that $q > 1$ and let $0 < \lambda < 1$ as well as $0 < \varepsilon < \min\{\lambda, 1-\lambda\}$. From relation (30) and the valuation property of Z we deduce

$$\mathrm{Z}\, T^n \# \mathrm{Z}(\zeta_\varepsilon T^n) = \mathrm{Z}(\phi_\varepsilon T^n) \# \mathrm{Z}(\psi_\varepsilon T^n).$$

The definition of Blaschke addition therefore yields

$$S(\mathrm{Z}\, T^n, \cdot) + S(\mathrm{Z}(\zeta_\varepsilon T^n), \cdot) = S(\mathrm{Z}(\phi_\varepsilon T^n), \cdot) + S(\mathrm{Z}(\psi_\varepsilon T^n), \cdot),$$

and hence

$$\mathrm{C}\, S(\mathrm{Z}\, T^n, \cdot) + \mathrm{C}\, S(\mathrm{Z}(\zeta_\varepsilon T^n), \cdot) = \mathrm{C}\, S(\mathrm{Z}(\phi_\varepsilon T^n), \cdot) + \mathrm{C}\, S(\mathrm{Z}(\phi_\varepsilon T^n), \cdot).$$

If we combine the homogeneity and $\mathrm{SL}(n)$ covariance of Z with the relations (6) and (26), then we obtain

$$\begin{aligned} &\mathrm{C}\, S(\mathrm{Z}\, T^n, \cdot)(x) + (\det \zeta_\varepsilon)^\alpha\, \mathrm{C}\, S(\mathrm{Z}\, T^n, \cdot)(\zeta_\varepsilon^{-1} x) \\ &\qquad = (\det \phi_\varepsilon)^\alpha\, \mathrm{C}\, S(\mathrm{Z}\, T^n, \cdot)(\phi_\varepsilon^{-1} x) + (\det \psi_\varepsilon)^\alpha\, \mathrm{C}\, S(\mathrm{Z}\, T^n, \cdot)(\psi_\varepsilon^{-1} x) \end{aligned}$$

with $\alpha = (q(n-1)+1)/n$. Calculating the involved determinants proves

$$\begin{aligned} &\mathrm{C}\, S(\mathrm{Z}\, T^n, \cdot)(x) + (2\varepsilon)^\alpha\, \mathrm{C}\, S(\mathrm{Z}\, T^n, \cdot)(\zeta_\varepsilon^{-1} x) \\ &\qquad = (\lambda+\varepsilon)^\alpha\, \mathrm{C}\, S(\mathrm{Z}\, T^n, \cdot)(\phi_\varepsilon^{-1} x) + (1-\lambda+\varepsilon)^\alpha\, \mathrm{C}\, S(\mathrm{Z}\, T^n, \cdot)(\psi_\varepsilon^{-1} x). \end{aligned} \tag{31}$$

Since $q > 1$, we have $\alpha > 1$. Obviously, the cosine transform is positively homogeneous of degree one and hence

$$\lim_{\varepsilon \to 0^+} (2\varepsilon)^\alpha\, \mathrm{C}\, S(\mathrm{Z}\, T^n, \cdot)(\zeta_\varepsilon^{-1} x) = 0$$

for every $x \in \mathbb{R}^n$. Taking the limit $\varepsilon \to 0^+$ in (31) therefore gives

$$\mathrm{C}\, S(\mathrm{Z}\, T^n, \cdot)(x) = \lambda^\alpha\, \mathrm{C}\, S(\mathrm{Z}\, T^n, \cdot)(\phi_0^{-1} x) + (1-\lambda)^\alpha\, \mathrm{C}\, S(\mathrm{Z}\, T^n, \cdot)(\psi_0^{-1} x).$$

In terms of projection bodies this reads as

$$h_{\Pi\, \mathrm{Z}\, T^n}(x) = \lambda^\alpha h_{\Pi\, \mathrm{Z}\, T^n}(\phi_0^{-1} x) + (1-\lambda)^\alpha h_{\Pi\, \mathrm{Z}\, T^n}(\psi_0^{-1} x).$$

for every $x \in \mathbb{R}^n$. Note that the range of Z consists of bodies containing the origin in their interiors. Thus $\Pi\, \mathrm{Z}\, T^n$ contains the origin in its interior, too. We conclude that the support function $h_{\Pi\, \mathrm{Z}\, T^n}$ is strictly positive on the sphere. The standard basis vector e_3 is an eigenvector of ϕ_0^{-1} and ψ_0^{-1}. Evaluating the last equation at e_3 gives $1 = \lambda^\alpha + (1-\lambda)^\alpha$. Since $0 < \lambda < 1$ was arbitrary and $\alpha > 1$ we arrived at a contradiction. We infer that $q = 1$. $\square$

If $\mathrm{Z} : \mathcal{K}_o^n \to \langle \mathcal{K}_c^n, \# \rangle$ is a continuous $\mathrm{SL}(n)$ covariant symmetric Blaschke valuation which is homogeneous of degree one, then we will prove in the sequel that $\Pi \circ \mathrm{Z}$ can be extended as a valuation to all convex bodies containing the origin. In order to show that this extension is $\mathrm{SL}(n)$ contravariant and homogeneous, we need the following result. We denote convex bodies containing the origin by $\bar{\mathcal{K}}_o^n$.

Lemma 4. *If* $\mathrm{Z} : \bar{\mathcal{K}}_o^n \to \langle \mathcal{K}^n, + \rangle$ *is a Minkowski valuation which is* $\mathrm{SL}(n)$ *contravariant and positively homogeneous of degree* q *on* n*-dimensional bodies, then* Z *is* $\mathrm{SL}(n)$ *contravariant and positively homogeneous of degree* q *on its domain* $\bar{\mathcal{K}}_o^n$.

Proof. By (27) we have to show that

$$\mathrm{Z}\, \phi K = (\det \phi)^{\frac{q+1}{n}} \phi^{-t}\, \mathrm{Z}\, K \tag{32}$$

for every $K \in \bar{\mathcal{K}}_o^n$ and every $\phi \in \mathrm{GL}(n)$ with positive determinant. Let $\dim K = n-k$, where $0 \leq k \leq n$. We prove our assertion by induction on k. Indeed, for

$k = 0$, (32) is true by assumption. As usual, denote by dim the dimension and by lin the linear hull. Assume that (32) holds for bodies of dimension $n - k$ and let $\dim K = n - k - 1$. Choose a vector $u \notin \operatorname{lin} K$. Then

$$[K, u] \cup [K, -u] = [K, \pm u], \qquad [K, u] \cap [K, -u] = K.$$

The convex bodies $[K, u]$, $[K, -u]$, $[K, \pm u]$ are of dimension $n-k$. Clearly, we have

$$\phi[K, u] \cup \phi[K, -u] = \phi[K, \pm u], \qquad \phi[K, u] \cap \phi[K, -u] = \phi K,$$

and the convex bodies $\phi[K, u]$, $\phi[K, -u]$, $\phi[K, \pm u]$ are of dimension $n - k$. Consequently, the valuation property of Z combined with the induction assumption shows

$$\begin{aligned} h_{\mathrm{Z}\,\phi K}(x) &= h_{\mathrm{Z}\,\phi[K,u]}(x) + h_{\mathrm{Z}\,\phi[K,-u]}(x) - h_{\mathrm{Z}\,\phi[K,\pm u]}(x) \\ &= (\det \phi)^{\frac{q+1}{n}} \left(h_{\mathrm{Z}[K,u]}(\phi^{-1}x) + h_{\mathrm{Z}[K,-u]}(\phi^{-1}x) - h_{\mathrm{Z}[K,\pm u]}(\phi^{-1}x) \right) \\ &= (\det \phi)^{\frac{q+1}{n}} h_{\mathrm{Z}\,K}(\phi^{-1}x) \end{aligned}$$

for arbitrary $x \in \mathbb{R}^n$. This immediately proves that (32) holds for bodies of dimension $n - k - 1$. □

Lemma 5. *Suppose that* $\mathrm{Z} : \mathcal{K}_o^n \to \langle \mathcal{K}_c^n, \# \rangle$ *is a continuous* $\mathrm{SL}(n)$ *covariant symmetric Blaschke valuation which is homogeneous of degree one. Then there exists an extension* $\bar{\mathrm{Z}} : \bar{\mathcal{K}}_o^n \to \langle \mathcal{K}^n, + \rangle$ *of* $\Pi \circ \mathrm{Z} : \mathcal{K}_o^n \to \langle \mathcal{K}^n, + \rangle$ *which is a Minkowski valuation. Moreover, this extension* $\bar{\mathrm{Z}}$ *is* $\mathrm{SL}(n)$ *contravariant, homogeneous of degree* $n - 1$*, continuous on* $\mathcal{K}_o^n$*, and has origin-symmetric images.*

Proof. We define the map $\bar{\mathrm{Z}} : \bar{\mathcal{K}}_o^n \to \langle \mathcal{K}^n, + \rangle$ by

$$h_{\bar{\mathrm{Z}}K}(x) = \begin{cases} \frac{1}{2}\, \mathrm{C}\, S(\mathrm{Z}\,K, \cdot)(x) & \dim K = n, \\ |x \cdot u| V_{n-1}(\mathrm{Z}[K, \pm u]|u^\perp) & \dim K = n-1, \operatorname{lin} K = u^\perp, \|u\| = 1, \\ 0 & \dim K < n-1. \end{cases}$$

We remark that $\mathrm{Z}[K, \pm u]|u^\perp$ denotes the image of $\mathrm{Z}[K, \pm u]$ under the orthogonal projection onto the hyperplane $u^\perp$. All functions which occur in the above definition are sublinear and therefore support functions of a unique convex body. The definition of projection bodies immediately proves that $\bar{\mathrm{Z}}$ coincides with $\Pi \circ \mathrm{Z}$ on n-dimensional bodies. First, we prove that $\bar{\mathrm{Z}}$ is a valuation. Suppose that K, $L \in \bar{\mathcal{K}}_o^n$ are convex bodies such that $K \cup L \in \bar{\mathcal{K}}_o^n$ and let $0 \le k \le n$. If $\dim(K \cup L) = k$, then one of the following four cases is valid:

- (1_k) $\dim K = k$, $\dim L = k$, $\dim K \cap L = k$,
- (2_k) $\dim K = k$, $\dim L = k$, $\dim K \cap L = k - 1$,
- (3_k) $\dim K = k$, $\dim L < k$,
- (4_k) $\dim K < k$, $\dim L = k$.

The valuation property trivially holds true for the cases (3_k) and (4_k) because in this situation we have $L \subset K$ and $K \subset L$, respectively. Therefore it suffices to prove

$$h_{\bar{\mathrm{Z}}(K \cup L)} + h_{\bar{\mathrm{Z}}(K \cap L)} = h_{\bar{\mathrm{Z}}K} + h_{\bar{\mathrm{Z}}L} \tag{33}$$

for the cases (1_k) and (2_k), $0 \le k \le n$. Suppose that (1_n) holds. From the fact that Z is a symmetric Blaschke valuation we get

$$S(\mathrm{Z}(K \cup L), \cdot) + S(\mathrm{Z}(K \cap L), \cdot) = S(\mathrm{Z}\,K, \cdot) + S(\mathrm{Z}\,L, \cdot),$$

and thus

$$\tfrac{1}{2}\,\mathrm{C}\,S(\mathrm{Z}(K\cup L),\cdot)+\tfrac{1}{2}\,\mathrm{C}\,S(\mathrm{Z}(K\cap L),\cdot)=\tfrac{1}{2}\,\mathrm{C}\,S(\mathrm{Z}\,K,\cdot)+\tfrac{1}{2}\,\mathrm{C}\,S(\mathrm{Z}\,L,\cdot).$$

The definition of $\bar{\mathrm{Z}}$ immediately proves (33) for this case. In order to establish relation (33) for the case (2_n), it is enough to show

$$h_{\bar{\mathrm{Z}}K}+h_{\bar{\mathrm{Z}}(K\cap u^{\perp})}=h_{\bar{\mathrm{Z}}(K\cap u^{+})}+h_{\bar{\mathrm{Z}}(K\cap u^{-})} \tag{34}$$

for arbitrary bodies $K\in\bar{\mathcal{K}}_o^n$ and every $u\in S^{n-1}$ such that $K\cap u^{\pm}$ are both n-dimensional. Let us begin with the special case $u=e_n$. Since Z is a Blaschke valuation on n-dimensional bodies and

$$\begin{aligned}[K\cap e_n^{+},\pm se_n]\cup[K\cap e_n^{-},\pm se_n] &= [K,\pm se_n],\\ [K\cap e_n^{+},\pm se_n]\cap[K\cap e_n^{-},\pm se_n] &= [K\cap e_n^{\perp},\pm se_n],\end{aligned}$$

for every positive s, we obtain

$$\begin{aligned}&\mathrm{C}\,S(\mathrm{Z}[K,\pm se_n],\cdot)+\mathrm{C}\,S(\mathrm{Z}[K\cap e_n^{\perp},\pm se_n],\cdot)\\ &\qquad=\mathrm{C}\,S(\mathrm{Z}[K\cap e_n^{+},\pm se_n],\cdot)+\mathrm{C}\,S(\mathrm{Z}[K\cap e_n^{-},\pm se_n],\cdot).\end{aligned} \tag{35}$$

For positive s define a linear map ϕ by

$$\phi e_i=e_i,\qquad i=1,\dots,n-1,\qquad \phi e_n=se_n.$$

From the $\mathrm{SL}(n)$ covariance and homogeneity of Z as well as relation (26) we get

$$\mathrm{C}\,S(\mathrm{Z}[K\cap e_n^{\perp},\pm se_n],\cdot)=\mathrm{C}\,S(\phi\,\mathrm{Z}[K\cap e_n^{\perp},\pm e_n],\cdot).$$

Formula (10) and the positive homogeneity of the cosine transform therefore yield

$$\begin{aligned}\mathrm{C}\,S(\mathrm{Z}[K\cap e_n^{\perp},\pm se_n],\cdot)(x) &= s\,\mathrm{C}\,S(\mathrm{Z}[K\cap e_n^{\perp},\pm e_n],\cdot)(\phi^{-1}x)\\ &= \mathrm{C}\,S(\mathrm{Z}[K\cap e_n^{\perp},\pm e_n],\cdot)((sx_1,\dots,sx_{n-1},x_n))\end{aligned}$$

for every $x\in\mathbb{R}^n$. Thus we obtain by the homogeneity of C and formula (11) that

$$\begin{aligned}\lim_{s\to0^+}\mathrm{C}\,S(\mathrm{Z}[K\cap e_n^{\perp},\pm se_n],\cdot)(x) &= \mathrm{C}\,S(\mathrm{Z}[K\cap e_n^{\perp},\pm se_n],\cdot)(x_ne_n)\\ &= |x_n|\,\mathrm{C}\,S(\mathrm{Z}[K\cap e_n^{\perp},\pm se_n],\cdot)(e_n)\\ &= 2|x_n|V_{n-1}(\mathrm{Z}[K\cap e_n^{\perp},\pm e_n]|e_n^{\perp}).\end{aligned}$$

From (35) and the convergence behavior

$$\begin{aligned}\lim_{s\to0^+}[K\cap e_n^{+},\pm se_n] &= K\cap e_n^{+},\\ \lim_{s\to0^+}[K\cap e_n^{-},\pm se_n] &= K\cap e_n^{-},\\ \lim_{s\to0^+}[K,\pm se_n] &= K,\end{aligned}$$

we deduce that

$$\begin{aligned}&\mathrm{C}\,S(\mathrm{Z}\,K,\cdot)(x)+2|x\cdot e_n|V_{n-1}(\mathrm{Z}[K\cap e_n^{\perp},\pm e_n]|e_n^{\perp})\\ &\qquad=\mathrm{C}\,S(\mathrm{Z}(K\cap e_n^{+}),\cdot)(x)+\mathrm{C}\,S(\mathrm{Z}(K\cap e_n^{-}),\cdot)(x).\end{aligned}$$

The definition of $\bar{\mathrm{Z}}$ implies that (34) holds for the case (2_n) and $u=e_n^{\perp}$. Now, let $u\in S^{n-1}$ and denote by ϑ a proper rotation with $\vartheta e_n=u$. Note that the $\mathrm{SL}(n)$

covariance of Z and (12) imply the $\mathrm{SL}(n)$ contravariance of $\Pi \circ \mathrm{Z}$ on n-dimensional bodies. Thus $\bar{\mathrm{Z}}$ is $\mathrm{SL}(n)$ contravariant on n-dimensional bodies, too. Hence

$$\begin{aligned}
&h_{\bar{\mathrm{Z}}(K\cap u^+)}(x) + h_{\bar{\mathrm{Z}}(K\cap u^-)}(x)\\
&\qquad = h_{\bar{\mathrm{Z}}(\vartheta^{-1}K\cap e_n^+)}(\vartheta^{-1}x) + h_{\bar{\mathrm{Z}}(\vartheta^{-1}K\cap e_n^+)}(\vartheta^{-1}x)\\
&\qquad = h_{\bar{\mathrm{Z}}(\vartheta^{-1}K)}(\vartheta^{-1}x) + |\vartheta^{-1}x\cdot e_n| V_{n-1}(\mathrm{Z}[\vartheta^{-1}K\cap e_n^\perp, \pm e_n]|e_n^\perp)\\
&\qquad = h_{\bar{\mathrm{Z}}K}(x) + |x\cdot u| h_{\Pi\,\mathrm{Z}[\vartheta^{-1}K\cap e_n^\perp,\pm e_n]}(e_n)\\
&\qquad = h_{\bar{\mathrm{Z}}K}(x) + |x\cdot u| h_{\Pi\,\mathrm{Z}[K\cap u^\perp,\pm u]}(u)\\
&\qquad = h_{\bar{\mathrm{Z}}K}(x) + |x\cdot u| V_{n-1}(\mathrm{Z}[K\cap u^\perp, \pm u]|u^\perp)\\
&\qquad = h_{\bar{\mathrm{Z}}K}(x) + h_{\bar{\mathrm{Z}}(K\cap u^\perp)}(x).
\end{aligned}$$

This settles the proof of the valuation property for the case (2_n). Next, we deal with the case (1_{n-1}). Denote by u the unit normal of the orthogonal complement of $\mathrm{lin}(K\cup L)$. Note that

$$[K,\pm u]\cup[L,\pm u] = [K\cup L,\pm u], \qquad \text{and} \qquad [K,\pm u]\cap[L,\pm u] = [K\cap L,\pm u].$$

All of the above bodies are of dimension n. Since Z is a symmetric Blaschke valuation we infer

$$S(\mathrm{Z}[K\cup L,\pm u],\cdot) + S(\mathrm{Z}[K\cap L,\pm u],\cdot) = S(\mathrm{Z}[K,\pm u],\cdot) + S(\mathrm{Z}[L,\pm u],\cdot).$$

Consequently we have

$$\begin{aligned}
&\tfrac12\,\mathrm{C}\,S(\mathrm{Z}[K\cup L,\pm u],\cdot)(u) + \tfrac12\,\mathrm{C}\,S(\mathrm{Z}[K\cap L,\pm u],\cdot)(u)\\
&\qquad\qquad = \tfrac12\,\mathrm{C}\,S(\mathrm{Z}[K,\pm u],\cdot)(u) + \tfrac12\,\mathrm{C}\,S(\mathrm{Z}[L,\pm u],\cdot)(u),
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
&V_{n-1}(\mathrm{Z}[K\cup L,\pm u]|u^\perp) + V_{n-1}(\mathrm{Z}[K\cap L,\pm u]|u^\perp)\\
&\qquad = V_{n-1}(\mathrm{Z}[K,\pm u]|u^\perp) + V_{n-1}(\mathrm{Z}[L,\pm u]|u^\perp).
\end{aligned}$$

This proves the desired relation (33) for the case (1_{n-1}). In order to establish the valuation property (33) for the case (2_{n-1}), it suffices to prove

$$h_{\bar{\mathrm{Z}}K} = h_{\bar{\mathrm{Z}}(K\cap v^+)} + h_{\bar{\mathrm{Z}}(K\cap v^-)} \tag{36}$$

for $K\subset u^\perp$ and a unit vector v which is orthogonal to u such that $K\cap v^\pm$ are $(n-1)$-dimensional. We begin with the special case $u=e_n$ and $v=e_{n-1}$. Note that

$$\begin{aligned}
[K\cap e_{n-1}^+,\pm s e_{n-1}]\cup[K\cap e_{n-1}^-,\pm s e_{n-1}] &= [K,\pm s e_{n-1}],\\
[K\cap e_{n-1}^+,\pm s e_{n-1}]\cap[K\cap e_{n-1}^-,\pm s e_{n-1}] &= [K\cap e_{n-1}^\perp,\pm s e_{n-1}].
\end{aligned}$$

From the validity of (33) for the case (1_{n-1}) we know that

$$h_{\bar{\mathrm{Z}}[K,\pm s e_{n-1}]} + h_{\bar{\mathrm{Z}}[K\cap e_{n-1}^\perp,\pm s e_{n-1}]} = h_{\bar{\mathrm{Z}}[K\cap e_{n-1}^+,\pm s e_{n-1}]} + h_{\bar{\mathrm{Z}}[K\cap e_{n-1}^-,\pm s e_{n-1}]}.$$

By the definition of $\bar{\mathrm{Z}}$ this implies

$$\begin{aligned}
&V_{n-1}(\mathrm{Z}[K,\pm s e_{n-1},\pm e_n]|e_n^\perp) + V_{n-1}(\mathrm{Z}[K\cap e_{n-1}^\perp,\pm s e_{n-1},\pm e_n]|e_n^\perp)\\
&\quad = V_{n-1}(\mathrm{Z}[K\cap e_{n-1}^+,\pm s e_{n-1},\pm e_n]|e_n^\perp) + V_{n-1}(\mathrm{Z}[K\cap e_{n-1}^-,\pm s e_{n-1},\pm e_n]|e_n^\perp).
\end{aligned}$$

For positive s define a linear map ϕ by

$$\phi e_{n-1} = s e_{n-1}, \quad \text{and} \quad \phi e_i = e_i, \qquad i=1,\dots,n, \quad i\neq n-1.$$

If we denote the orthogonal projection onto a linear subspace H by π_H, then the covariance of Z proves

$$\begin{aligned} V_{n-1}(\mathrm{Z}[K \cap e_{n-1}^{\perp}, \pm s e_{n-1}, \pm e_n] | e_n^{\perp}) &= V_{n-1}(\pi_{e_n^{\perp}} \mathrm{Z}[K \cap e_{n-1}^{\perp}, \pm s e_{n-1}, \pm e_n]) \\ &= V_{n-1}(\pi_{e_n^{\perp}} \phi \, \mathrm{Z}[K \cap e_{n-1}^{\perp}, \pm e_{n-1}, \pm e_n]). \end{aligned}$$

Since $\pi_{e_n^{\perp}} \phi \, \mathrm{Z}[K \cap e_{n-1}^{\perp}, \pm e_{n-1}, \pm e_n]$ converges to $\pi_{e_{n-1}^{\perp} \cap e_n^{\perp}} \mathrm{Z}[K \cap e_{n-1}^{\perp}, \pm e_{n-1}, \pm e_n]$ with respect to Hausdorff distance as s tends to zero and V_{n-1} is a continuous functional on convex bodies contained in $e_n^{\perp}$, we obtain

$$\lim_{s \to 0^+} V_{n-1}(\mathrm{Z}[K \cap e_{n-1}^{\perp}, \pm s e_{n-1}, \pm e_n] | e_n^{\perp}) = 0.$$

Note that the following convergence relations

$$\begin{aligned} \lim_{s \to 0^+} [K, \pm s e_{n-1}, \pm e_n] &= [K, \pm e_n], \\ \lim_{s \to 0^+} [K \cap e_{n-1}^{+}, \pm s e_{n-1}, \pm e_n] &= [K \cap e_{n-1}^{+}, \pm e_n], \\ \lim_{s \to 0^+} [K \cap e_{n-1}^{-}, \pm s e_{n-1}, \pm e_n] &= [K \cap e_{n-1}^{-}, \pm e_n], \end{aligned}$$

hold. Thus

$$V_{n-1}(\mathrm{Z}[K, \pm e_n] | e_n^{\perp}) = V_{n-1}(\mathrm{Z}[K \cap e_{n-1}^{+}, \pm e_n] | e_n^{\perp}) + V_{n-1}(\mathrm{Z}[K \cap e_{n-1}^{-}, \pm e_n] | e_n^{\perp}),$$

and the definition on $\bar{\mathrm{Z}}$ shows

$$h_{\bar{\mathrm{Z}} K} = h_{\bar{\mathrm{Z}}(K \cap e_{n-1}^{+})} + h_{\bar{\mathrm{Z}}(K \cap e_{n-1}^{-})}.$$

This proves (36) for this special case. Next, let ϑ be a proper rotation with $\vartheta e_n = u$ and $\vartheta e_{n-1} = v$. Then the definition of $\bar{\mathrm{Z}}$, the covariance of Z in combination with (12), and the already established special case of the valuation property for (2_{n-1}) yield

$$\begin{aligned} & h_{\bar{\mathrm{Z}}(K \cap v^{+})}(x) + h_{\bar{\mathrm{Z}}(K \cap v^{-})}(x) = \\ & \qquad = |x \cdot u| \left(h_{\Pi \, \mathrm{Z}[K \cap v^{+}, \pm u]}(u) + h_{\Pi \, \mathrm{Z}[K \cap v^{-}, \pm u]}(u) \right) \\ & \qquad = |x \cdot u| \left(h_{\Pi \, \mathrm{Z}[\vartheta^{-1} K \cap e_{n-1}^{+}, \pm e_n]}(e_n) + h_{\Pi \, \mathrm{Z}[\vartheta^{-1} K \cap e_{n-1}^{-}, \pm e_n]}(e_n) \right) \\ & \qquad = |x \cdot u| \, h_{\Pi \, \mathrm{Z}[\vartheta^{-1} K, \pm e_n]}(e_n) \\ & \qquad = |x \cdot u| \, h_{\Pi \, \mathrm{Z}[K, \pm u]}(u) \\ & \qquad = h_{\bar{\mathrm{Z}} K}(x). \end{aligned}$$

In conclusion, it remains to prove (33) for the cases (1_k) and (2_k) where $0 \leq k \leq n-2$. But then the valuation property is trivial because $\bar{\mathrm{Z}}$ vanishes on sets of dimensions less than $n-1$. Thus we finally obtained that $\bar{\mathrm{Z}}$ is a valuation.

Now, we turn to the proof of the $\mathrm{SL}(n)$ contravariance and homogeneity of $\bar{\mathrm{Z}}$. In the above derivation we already used the fact that the $\mathrm{SL}(n)$ covariance of Z and formula (12) imply the $\mathrm{SL}(n)$ contravariance of $\Pi \circ \mathrm{Z}$ on n-dimensional bodies. Consequently, $\bar{\mathrm{Z}}$ is $\mathrm{SL}(n)$ contravariant on n-dimensional bodies. Lemma 4 proves the $\mathrm{SL}(n)$ contravariance on bodies of arbitrary dimension. Similarly, Lemma 4 yields the homogeneity of degree $(n-1)$ of $\bar{\mathrm{Z}}$ on arbitrary bodies containing the origin.

Since Z is continuous on $\mathcal{K}_o^n$, the surface area measures $S(\mathrm{Z}\, K_m, \cdot)$ converge weakly to $S(\mathrm{Z}\, K, \cdot)$ for every sequence $K_m \in \mathcal{K}_o^n$, $m \in \mathbb{N}$, of convex bodies with $K_m \to K \in \mathcal{K}_o^n$. Consequently, we deduce the pointwise convergence of the support functions $\frac{1}{2}\operatorname{C} S(\operatorname{Z} K_m, \cdot)$ to $\frac{1}{2}\operatorname{C} S(\operatorname{Z} K, \cdot)$ on the sphere. But pointwise convergence of support functions implies uniform convergence (see e.g. [40, Theorem 1.8.12]). Hence $\Pi \circ \operatorname{Z}$ is continuous on $\mathcal{K}_o^n$. Since $\bar{\operatorname{Z}}$ coincides with Z on n-dimensional bodies, the proof of this Lemma is finished. □

We need the following classification result of contravariant Minkowski valuations due to Ludwig [29, Theorem 2]. Let $\bar{\mathcal{P}}_o^n$ denote convex polytopes which contain the origin.

Theorem 3. *Suppose that $\operatorname{Z} : \bar{\mathcal{P}}_o^n \to \langle \mathcal{K}^n, + \rangle$ is an $\operatorname{SL}(n)$ contravariant Minkowski valuation which is homogeneous of degree $n-1$. Then there exist constants $c_1 \geq 0$, c_2, $c_3 \in \mathbb{R}$ with $c_1 + c_2 + c_3 \geq 0$ such that*

$$\operatorname{Z} P = c_1 \Pi P + c_2 \Pi_o P + c_3(-\Pi_o P) \tag{37}$$

for every $P \in \bar{\mathcal{P}}_o^n$.

Here (as well as in Theorems 4 and 5) formulas like (37) have to be read as

$$h_{\operatorname{Z} P}(x) = c_1 h_{\Pi P}(x) + c_2 h_{\Pi_o P}(x) + c_3 h_{-\Pi_o P}(x)$$

for $x \in \mathbb{R}^n$. The notation is only used if $h_{\operatorname{Z} P}$ is a support function (which is guaranteed by the restrictions on the constants c_i). The operator Π_o is defined as follows. A vector v is a *scaled facet normal* of $P \in \bar{\mathcal{P}}_o^n$, if v is an outer normal vector to a facet of P whose length is equal to the $(n-1)$-dimensional volume of the corresponding facet. If we write $\mathcal{V}_o(P)$ for the set of scaled facet normals of P which correspond to facets containing the origin, then

$$\Pi_o P = \sum_{v \in \mathcal{V}_o(P)} [o, v],$$

where the sum is a finite Minkowski sum. If P contains the origin in its interior, then $\Pi_o P = \{o\}$. As an aside, we mention that the projection body operator has a similar representation for polytopes. Indeed, if $\mathcal{V}(P)$ denotes the set of all scaled facet normals, then

$$\Pi P = \sum_{v \in \mathcal{V}(P)} [o, v].$$

Lemma 6. *If $\operatorname{Z} : \mathcal{K}_o^n \to \langle \mathcal{K}_c^n, \# \rangle$ is a continuous $\operatorname{SL}(n)$ covariant Blaschke valuation which is homogeneous of degree one, then there exists a constant $c > 0$ such that*

$$\operatorname{Z} K = c(K \# (-K))$$

for every $K \in \mathcal{K}_o^n$.

Proof. Lemma 5 implies that we can extend $\Pi \circ \operatorname{Z}$ to an $\operatorname{SL}(n)$ contravariant valuation $\bar{\operatorname{Z}} : \bar{\mathcal{K}}_o^n \to \langle \mathcal{K}^n, + \rangle$ which is homogeneous of degree $n-1$ and continuous on n-dimensional bodies. The restriction of $\bar{\operatorname{Z}}$ to $\bar{\mathcal{P}}_o^n$ satisfies the assumptions of Theorem 3. Thus there exist constants $c_1 \geq 0$, c_2, $c_3 \in \mathbb{R}$ with $c_1 + c_2 + c_3 \geq 0$ such that

$$\bar{\operatorname{Z}} P = c_1 \Pi P + c_2 \Pi_o P + c_3(-\Pi_o P)$$

for every $P \in \bar{\mathcal{P}}_o^n$. Let $t = e_1 + \cdots + e_n$. The continuity of Π, the fact that the polytope $T^n - \varepsilon t$ contains the origin in its interior for sufficiently small ε, and the continuity of $\bar{\mathrm{Z}}$ prove

$$\begin{aligned} c_1 \Pi T^n &= \lim_{\varepsilon \to 0^+} c_1 \Pi (T^n - \varepsilon t) \\ &= \lim_{\varepsilon \to 0^+} \bar{\mathrm{Z}}(T^n - \varepsilon t) \\ &= \bar{\mathrm{Z}} T^n \\ &= c_1 \Pi T^n + c_2 \Pi_o T^n + c_3(-\Pi_o T^n). \end{aligned}$$

Since $h_{\Pi_o T^n}(e_1) \neq 0 = h_{-\Pi_o T^n}(e_1)$ we get $c_2 = c_3 = 0$. By assumption, Z does not contain $\{o\}$ in its range which gives $c_1 > 0$. Consequently, we have $\bar{\mathrm{Z}} P = c_1 \Pi P$ for every polytope $P \in \bar{\mathcal{P}}_o^n$. Since n-dimensional polytopes containing the origin are dense in $\mathcal{K}_o^n$, the continuity of the involved operators yields

$$\Pi \, \mathrm{Z} \, K = \bar{\mathrm{Z}} K = c_1 \Pi K$$

for every $K \in \mathcal{K}_o^n$. Rewriting this in terms of the cosine transform gives

$$\mathrm{C}\, S(\mathrm{Z}\, K, \cdot) = c_1 \,\mathrm{C}\, S(K, \cdot) = \tfrac{c_1}{2} \,\mathrm{C}\, S(K \# (-K), \cdot).$$

But $S(\mathrm{Z}\, K, \cdot)$ and $S(K\#(-K), \cdot)$ are even, and thus the injectivity property (9) shows $\mathrm{Z}\, K = c(K \# (-K))$ with $c = (c_1/2)^{1/(n-1)}$. □

We summarize the results of this subsection in

Lemma 7. *A map* $\mathrm{Z} : \mathcal{K}_o^n \to \langle \mathcal{K}_c^n, \# \rangle$ *is a continuous, homogeneous, and* $\mathrm{SL}(n)$ *covariant Blaschke valuation if and only if there exists a constant* $c > 0$ *such that*

$$\mathrm{Z}\, K = c(K \# (-K))$$

for every $K \in \mathcal{K}_o^n$.

4.2. **The contravariant case.** As in the covariant case, we start by reducing the possible degrees of homogeneity of continuous $\mathrm{SL}(n)$ contravariant symmetric Blaschke valuations.

Lemma 8. *If* $\mathrm{Z} : \mathcal{K}_o^n \to \langle \mathcal{K}_c^n, \# \rangle$ *is a continuous* $\mathrm{SL}(n)$ *contravariant symmetric Blaschke valuation which is homogeneous of degree* q, *then* $q \geq 1/(n-1)$.

Proof. The proof is similar to that of Lemma 2. Suppose $K \in \mathcal{K}_o^n$ is an arbitrary convex body such that $K \cap e_n^+$ and $K \cap e_n^-$ are n-dimensional. For every positive s we have

$$\begin{aligned} [K \cap e_n^+, \pm s e_n] \cup [K \cap e_n^-, \pm s e_n] &= [K, \pm s e_n], \\ [K \cap e_n^+, \pm s e_n] \cap [K \cap e_n^-, \pm s e_n] &= [K \cap e_n^\perp, \pm s e_n]. \end{aligned}$$

Since Z is a Blaschke valuation, we have

$$\begin{aligned} &\int_{S^{n-1}} h_L \, dS(\mathrm{Z}[K, \pm s e_n], \cdot) + \int_{S^{n-1}} h_L \, dS(\mathrm{Z}[K \cap e_n^\perp, \pm s e_n], \cdot) \\ &= \int_{S^{n-1}} h_L \, dS(\mathrm{Z}[K \cap e_n^+, \pm s e_n], \cdot) + \int_{S^{n-1}} h_L \, dS(\mathrm{Z}[K \cap e_n^-, \pm s e_n], \cdot) \end{aligned} \tag{38}$$

for an arbitrary convex body $L \in \mathcal{K}^n$. For positive s define a linear map ϕ by

$$\phi e_i = e_i, \qquad i = 1, \ldots, n-1, \qquad \phi e_n = s e_n.$$

From the $\mathrm{SL}(n)$ contravariance and homogeneity of Z as well as relation (27) we get

$$\int_{S^{n-1}} h_L \, dS(\mathrm{Z}[K\cap e_n^\perp, \pm s e_n], \cdot) = \int_{S^{n-1}} h_L \, dS\left(s^{\frac{q+1}{n}} \phi^{-t}\, \mathrm{Z}[K\cap e_n^\perp, \pm e_n], \cdot\right).$$

The homogeneity property (6) of surface area measures and formula (8) yield

$$\begin{aligned}\int_{S^{n-1}} h_L \, dS(\mathrm{Z}[K\cap e_n^\perp, \pm s e_n], \cdot) &= s^\alpha \int_{S^{n-1}} h_L \, dS(\phi^{-t}\, \mathrm{Z}[K\cap e_n^\perp, \pm e_n], \cdot) \\ &= s^{\alpha-1} \int_{S^{n-1}} h_L \circ \phi \, dS(\mathrm{Z}[K\cap e_n^\perp, \pm e_n], \cdot)\end{aligned}$$

for $\alpha = \frac{(q+1)(n-1)}{n}$. If we take $L = B^n$, then $h_L(x) = \|x\|$ and we infer from (5) and (38) that

$$\begin{aligned} & s^{\alpha-1} \int_{S^{n-1}} \|(u_1,\ldots,u_{n-1}, s u_n)\| \, dS(\mathrm{Z}[K\cap e_{n-1}^\perp, \pm e_n], u) \\ (39) \qquad & = S(\mathrm{Z}[K\cap e_n^+, \pm s e_n]) + S(\mathrm{Z}[K\cap e_n^-, \pm s e_n]) - S(\mathrm{Z}[K, \pm s e_n]).\end{aligned}$$

Since $\|(u_1,\ldots,u_{n-1},0)\| > 0$ for all $u \in S^{n-1}\backslash\{\pm e_n\}$ and surface area measures of n-dimensional bodies are not concentrated on any great subsphere we conclude

$$\lim_{s\to 0^+} \int_{S^{n-1}} \|(u_1,\ldots,u_{n-1}, s u_n)\| \, dS(\mathrm{Z}[K\cap e_n^\perp, \pm e_n], u) > 0.$$

Moreover, we have

$$\begin{aligned}\lim_{s\to 0^+} [K\cap e_n^+, \pm s e_n] &= K\cap e_n^+, \\ \lim_{s\to 0^+} [K\cap e_n^-, \pm s e_n] &= K\cap e_n^-, \\ \lim_{s\to 0^+} [K, \pm s e_n] &= K.\end{aligned}$$

Hence the continuity of Z together with the weak continuity of surface area measures implies that the right hand side of (38) converges to a finite number as s tends to zero. Thus by (39) we obtain that $s^{\alpha-1}$ converges for $s \to 0^+$. This implies $\alpha \geq 1$. The definition of α finally shows that the degree q of homogeneity has to be greater or equal than $1/(n-1)$. $\square$

The next result is the covariant counterpart of Lemma 4.

Lemma 9. *If* $\mathrm{Z} : \bar{\mathcal{K}}_o^n \to \langle \mathcal{K}^n, + \rangle$ *is a Minkowski valuation which is* $\mathrm{SL}(n)$ *covariant and positively homogeneous of degree* q *on* n*-dimensional bodies, then* Z *is* $\mathrm{SL}(n)$ *covariant and positively homogeneous of degree* q *on its domain* $\bar{\mathcal{K}}_o^n$*.*

Proof. We have to show that

$$\mathrm{Z}\, \phi K = (\det \phi)^{\frac{q-1}{n}} \phi\, \mathrm{Z}\, K \qquad (40)$$

for every $K \in \bar{\mathcal{K}}_o^n$ and every $\phi \in \mathrm{GL}(n)$ with positive determinant. Let $\dim K = n-k$, where $0 \leq k \leq n$. We prove our assertion by induction on k. Indeed, for $k = 0$, (40) is true by assumption. Assume that (40) holds for bodies of dimension $n-k$ and let $\dim K = n-k-1$. Choose a vector $u \notin \operatorname{lin} K$. Then

$$[K, u] \cup [K, -u] = [K, \pm u], \qquad [K, u] \cap [K, -u] = K.$$

The convex bodies $[K,u]$, $[K,-u]$, $[K,\pm u]$ are of dimension $n-k$. Clearly, we have

$$\phi[K, u] \cup \phi[K, -u] = \phi[K, \pm u], \qquad \phi[K, u] \cap \phi[K, -u] = \phi K,$$

and the convex bodies $\phi[K,u]$, $\phi[K,-u]$, $\phi[K,\pm u]$ are of dimension $n-k$. Consequently, the valuation property of Z combined with the induction assumption proves

$$\begin{aligned} h_{\mathrm{Z}\,\phi K}(x) &= h_{\mathrm{Z}\,\phi[K,u]}(x) + h_{\mathrm{Z}\,\phi[K,-u]}(x) - h_{\mathrm{Z}\,\phi[K,\pm u]}(x) \\ &= (\det\phi)^{\frac{q-1}{n}} \left(h_{\mathrm{Z}[K,u]}(\phi^t x) + h_{\mathrm{Z}[K,-u]}(\phi^{-1}x) - h_{\mathrm{Z}[K,\pm u]}(\phi^t x)\right) \\ &= (\det\phi)^{\frac{q-1}{n}} h_{\mathrm{Z}\,K}(\phi^t x) \end{aligned}$$

for arbitrary $x \in \mathbb{R}^n$. This immediately proves that (40) holds for bodies of dimension $n-k-1$. $\square$

The following characterization theorem was also established by Ludwig [29, Theorem 1]. We will make use of it in order to prove that the degree of homogeneity of a homogeneous, $\mathrm{SL}(n)$ contravariant, and continuous symmetric Blaschke valuation cannot be equal to $1/(n-1)$.

Theorem 4. *Suppose that* $\mathrm{Z} : \bar{\mathcal{P}}_o^n \to \langle \mathcal{K}^n, + \rangle$ *is an* $\mathrm{SL}(n)$ *covariant Minkowski valuation which is homogeneous of degree* 1*. Then there exist non-negative constants* c_1 *and* c_2 *such that*

$$\mathrm{Z}\,P = c_1 P + c_2(-P)$$

for every $P \in \bar{\mathcal{P}}_o^n$*.*

Lemma 10. *If* $\mathrm{Z} : \mathcal{K}_o^n \to \langle \mathcal{K}_c^n, \# \rangle$ *is a continuous* $\mathrm{SL}(n)$ *contravariant symmetric Blaschke valuation which is homogeneous of degree* q*, then* $q > 1/(n-1)$*.*

Proof. Suppose that $q = 1/(n-1)$. Then we are able to define an extension $\bar{\mathrm{Z}}$ of $\Pi \circ \mathrm{Z}$ by

$$h_{\bar{\mathrm{Z}}K}(x) = \begin{cases} \frac{1}{2}\,\mathrm{C}\,S(\mathrm{Z}\,K,\cdot)(x) & \dim K = n, \\ \frac{1}{2}\,\mathrm{C}\,S(\mathrm{Z}[K,\pm b_{k+1},\ldots,\pm b_n],\cdot)(\pi_K x) & \dim K = k < n, \end{cases}$$

where the $b_{k+1},\ldots,b_n$ are an orthonormal basis of the orthogonal complement of $\operatorname{lin} K$ and π_K denotes the orthogonal projection onto $\operatorname{lin} K$. In order to show that this extension is well defined, suppose that $\dim K = k < n$ and $b_{k+1},\ldots,b_n$ as well as $c_{k+1},\ldots,c_n$ are two different orthonormal bases of $(\operatorname{lin} K)^\perp$. Fix an orthonormal basis $b_1,\ldots,b_k$ in $\operatorname{lin} K$. Denote by ϑ a proper rotation with $\vartheta b_i = b_i$, $i = 1,\ldots,k$ and $\vartheta b_i \in \{\pm c_i\}$, $i = k+1,\ldots,n$. Then the contravariance of Z and relation (10) prove

$$\begin{aligned} \mathrm{C}\,S(\mathrm{Z}[K,\pm c_{k+1},\ldots,\pm c_n],\cdot)(\pi_K x) &= \mathrm{C}\,S(\mathrm{Z}\,\vartheta[K,\pm b_{k+1},\ldots,\pm b_n],\cdot)(\pi_K x) \\ &= \mathrm{C}\,S(\vartheta\,\mathrm{Z}[K,\pm b_{k+1},\ldots,\pm b_n],\cdot)(\pi_K x) \\ &= \mathrm{C}\,S(\mathrm{Z}[K,\pm b_{k+1},\ldots,\pm b_n],\cdot)(\vartheta^{-1}\pi_K x) \\ &= \mathrm{C}\,S(\mathrm{Z}[K,\pm b_{k+1},\ldots,\pm b_n],\cdot)(\pi_K x). \end{aligned}$$

Next, we show that $\bar{\mathrm{Z}}$ is a valuation on $\bar{\mathcal{K}}_o^n$. We use the same notation as for the proof of Lemma 5. Let us start with the easy case (1_n). The valuation property of Z implies

$$S(\mathrm{Z}(K\cup L),\cdot) + S(\mathrm{Z}(K\cap L),\cdot) = S(\mathrm{Z}\,K,\cdot) + S(\mathrm{Z}\,L,\cdot),$$

and thus

$$\tfrac{1}{2}\,\mathrm{C}\,S(\mathrm{Z}(K\cup L),\cdot) + \tfrac{1}{2}\,\mathrm{C}\,S(\mathrm{Z}(K\cap L),\cdot) = \tfrac{1}{2}\,\mathrm{C}\,S(\mathrm{Z}\,K,\cdot) + \tfrac{1}{2}\,\mathrm{C}\,S(\mathrm{Z}\,L,\cdot).$$

The definition on $\bar{\mathrm{Z}}$ immediately proves the assertion. Next, assume that (1_k), $0 \le k < n$, holds. Note that

$$\begin{aligned} [K, \pm b_{k+1}, \dots, \pm b_n] \cup [L, \pm b_{k+1}, \dots, \pm b_n] &= [K \cup L, \pm b_{k+1}, \dots, \pm b_n], \\ [K, \pm b_{k+1}, \dots, \pm b_n] \cap [L, \pm b_{k+1}, \dots, \pm b_n] &= [K \cap L, \pm b_{k+1}, \dots, \pm b_n]. \end{aligned}$$

From the assumption that Z is a symmetric Blaschke valuation we infer

$$\begin{aligned} &\mathrm{C}\, S(\mathrm{Z}[K \cup L, \pm b_{k+1}, \dots, \pm b_n], \cdot) + \mathrm{C}\, S(\mathrm{Z}[K \cap L, \pm b_{k+1}, \dots, \pm b_n], \cdot) \\ &\qquad = \mathrm{C}\, S(\mathrm{Z}[K, \pm b_{k+1}, \dots, \pm b_n], \cdot) + \mathrm{C}\, S(\mathrm{Z}[L, \pm b_{k+1}, \dots, \pm b_n], \cdot). \end{aligned}$$

Since $\operatorname{lin} K = \operatorname{lin} L = \operatorname{lin}(K \cup L) = \operatorname{lin}(K \cap L)$, the corresponding projections π_K, π_L, $\pi_{K \cup L}$, and $\pi_{K \cap L}$ are equal and we are done. Now, we consider the case (2_k). It is enough to prove

$$h_{\bar{\mathrm{Z}} K} + h_{\bar{\mathrm{Z}}(K \cap v^\perp)} = h_{\bar{\mathrm{Z}}(K \cap v^+)} + h_{\bar{\mathrm{Z}}(K \cap v^-)} \tag{41}$$

for a unit vector $v \in \operatorname{lin} K$ such that $K \cap v^\pm$ are both k-dimensional. Assume initially that $\operatorname{lin} K = \operatorname{lin}\{e_1, \dots, e_k\}$, and $v = e_k$. We already established the valuation property for the case (1_k). Hence

$$\begin{aligned} &\mathrm{C}\, S(\mathrm{Z}[K, \pm s e_k, \pm e_{k+1}, \dots, \pm e_n], \cdot)(\pi_K x) \\ &\qquad + \mathrm{C}\, S(\mathrm{Z}[K \cap e_k^\perp, \pm s e_k, \pm e_{k+1}, \dots, \pm e_n], \cdot)(\pi_K x) \\ &= \mathrm{C}\, S(\mathrm{Z}[K \cap e_k^+, \pm s e_k, \pm e_{k+1}, \dots, \pm e_n], \cdot)(\pi_K x) \\ &\qquad + \mathrm{C}\, S(\mathrm{Z}[K \cap e_k^-, \pm s e_k, \pm e_{k+1}, \dots, \pm e_n], \cdot)(\pi_K x). \end{aligned} \tag{42}$$

For positive s define a linear map ϕ by

$$\phi e_k = s e_k, \quad \text{and} \quad \phi e_i = e_i, \qquad i = 1, \dots, n, \quad i \neq k.$$

The definition of ϕ, the homogeneity property (6), and relations (27) as well as (10) give

$$\begin{aligned} &\mathrm{C}\, S(\mathrm{Z}[K \cap e_k^\perp, \pm s e_k, \pm e_{k+1}, \dots, \pm e_n], \cdot)(\pi_K x) \\ &\quad = \mathrm{C}\, S(\mathrm{Z}\, \phi[K \cap e_k^\perp, \pm e_k, \pm e_{k+1}, \dots, \pm e_n], \cdot)(\pi_K x) \\ &\quad = \mathrm{C}\, S((\det \phi)^{\frac{1}{n-1}} \phi^{-t}\, \mathrm{Z}[K \cap e_k^\perp, \pm e_k, \pm e_{k+1}, \dots, \pm e_n], \cdot)(\pi_K x) \\ &\quad = \mathrm{C}\, S(\mathrm{Z}[K \cap e_k^\perp, \pm e_k, \pm e_{k+1}, \dots, \pm e_n], \cdot)(\phi \pi_K x). \end{aligned}$$

Note that $\lim_{s \to 0^+} \phi \pi_K x = \pi_{K \cap v^\perp} x$. So if s tends to zero in (42), then we immediately obtain (41) in this situation. In order to prove (41) for the general case (2_n), denote by ϑ a proper rotation with $\vartheta e_n = v$. Then the contravariance of Z, the relation $\vartheta^{-1} \pi_{K \cap v^\perp} x = \pi_{\vartheta^{-1} K \cap e_n^\perp} \vartheta^{-1} x$, and the already established valuation property (41) for $v = e_n$ show

$$\begin{aligned} &\mathrm{C}\, S(\mathrm{Z}[K \cap v^+, \pm v], \cdot)(x) + \mathrm{C}\, S(\mathrm{Z}[K \cap v^-, \pm v], \cdot)(x) \\ &\quad = \mathrm{C}\, S(\mathrm{Z}[\vartheta^{-1} K \cap e_n^+, \pm e_n], \cdot)(\vartheta^{-1} x) + \mathrm{C}\, S(\mathrm{Z}[\vartheta^{-1} K \cap e_n^-, \pm e_n], \cdot)(\vartheta^{-1} x) \\ &\quad = \mathrm{C}\, S(\mathrm{Z}[\vartheta^{-1} K, \pm e_n], \cdot)(\vartheta^{-1} x) + \mathrm{C}\, S(\mathrm{Z}[\vartheta^{-1} K \cap e_n^\perp, \pm e_n], \cdot)(\pi_{\vartheta^{-1} K \cap e_n^\perp} \vartheta^{-1} x) \\ &\quad = \mathrm{C}\, S(\mathrm{Z}[\vartheta^{-1} K, \pm e_n], \cdot)(\vartheta^{-1} x) + \mathrm{C}\, S(\mathrm{Z}[\vartheta^{-1} K \cap e_n^\perp, \pm e_n], \cdot)(\vartheta^{-1} \pi_{K \cap v^\perp} x) \\ &\quad = \mathrm{C}\, S(\mathrm{Z}[K, \pm v], \cdot)(x) + \mathrm{C}\, S(\mathrm{Z}[K \cap v^\perp, \pm v], \cdot)(\pi_{K \cap v^\perp} x). \end{aligned}$$

For the case (2_k) with $k < n$, let $b_1, \dots, b_k$ be an orthonormal basis of $\operatorname{lin} K$ such that $v = b_k$. Extend this basis by $b_{k+1}, \dots, b_n$ to a basis of $\mathbb{R}^n$. Denote by ϑ a proper rotation with $\vartheta e_i = b_i$, $i = 1, \dots, n-1$, and $\vartheta e_n = \pm b_n$. Since $\vartheta^{-1}\pi_K x = \pi_{\vartheta^{-1}K}\vartheta^{-1}x$ and $\vartheta^{-1}\pi_{K\cap v^\perp}x = \pi_{\vartheta^{-1}K\cap e_k^\perp}\vartheta^{-1}x$ we obtain

$$\begin{aligned}
&\mathrm{C}\,S(\mathrm{Z}[K\cap v^+, \pm b_{k+1}, \dots, \pm b_n], \cdot)(\pi_K x)\\
&\qquad + \mathrm{C}\,S(\mathrm{Z}[K\cap v^-, \pm b_{k+1}, \dots, \pm b_n], \cdot)(\pi_K x)\\
&\quad = \mathrm{C}\,S(\mathrm{Z}[\vartheta^{-1}K\cap e_k^+, \pm e_{k+1}, \dots, \pm e_n], \cdot)(\vartheta^{-1}\pi_K x)\\
&\qquad + \mathrm{C}\,S(\mathrm{Z}[\vartheta^{-1}K\cap e_k^-, \pm e_{k+1}, \dots, \pm e_n], \cdot)(\vartheta^{-1}\pi_K x)\\
&\quad = \mathrm{C}\,S(\mathrm{Z}[\vartheta^{-1}K\cap e_k^+, \pm e_{k+1}, \dots, \pm e_n], \cdot)(\pi_{\vartheta^{-1}K}\vartheta^{-1}x)\\
&\qquad + \mathrm{C}\,S(\mathrm{Z}[\vartheta^{-1}K\cap e_k^-, \pm e_{k+1}, \dots, \pm e_n], \cdot)(\pi_{\vartheta^{-1}K}\vartheta^{-1}x)\\
&\quad = \mathrm{C}\,S(\mathrm{Z}[\vartheta^{-1}K, \pm e_{k+1}, \dots, \pm e_n], \cdot)(\pi_{\vartheta^{-1}K}\vartheta^{-1}x)\\
&\qquad + \mathrm{C}\,S(\mathrm{Z}[\vartheta^{-1}K\cap e_k^\perp, \pm e_k, \dots, \pm e_n], \cdot)(\pi_{\vartheta^{-1}K\cap e_k^\perp}\vartheta^{-1}x)\\
&\quad = \mathrm{C}\,S(\mathrm{Z}[\vartheta^{-1}K, \pm e_{k+1}, \dots, \pm e_n], \cdot)(\vartheta^{-1}\pi_K x)\\
&\qquad + \mathrm{C}\,S(\mathrm{Z}[\vartheta^{-1}K\cap e_k^\perp, \pm e_k, \dots, \pm e_n], \cdot)(\vartheta^{-1}\pi_{K\cap v^\perp}x)\\
&\quad = \mathrm{C}\,S(\mathrm{Z}[K, \pm b_{k+1}, \dots, \pm b_n], \cdot)(\pi_K x)\\
&\qquad + \mathrm{C}\,S(\mathrm{Z}[K\cap v^\perp, \pm b_k, \dots, \pm b_n], \cdot)(\pi_{K\cap v^\perp}x).
\end{aligned}$$

This proves the validity of (41) for the case (2_k). Again, the remaining cases (3_k) and (4_k) are trivial. Hence we proved that $\bar{\mathrm{Z}}$ is a valuation on $\bar{\mathcal{K}}_o^n$. Moreover, it is $\mathrm{SL}(n)$ covariant and positively homogeneous of degree 1 on n-dimensional bodies. Thus Lemma 9 implies that $\bar{\mathrm{Z}}$ is $\mathrm{SL}(n)$ covariant and positively homogeneous of degree 1 on $\bar{\mathcal{K}}_o^n$. From Theorem 4 we infer the existence of two constants $c_1, c_2 \geq 0$ such that

$$\bar{\mathrm{Z}}P = c_1 P + c_2(-P)$$

for every polytope $P \in \bar{\mathcal{P}}_o^n$. In particular, we have

$$\Pi\,\mathrm{Z}\,T^n = c_1 T^n + c_2(-T^n).$$

Since $\Pi\,\mathrm{Z}\,T^n$ is origin-symmetric we deduce

$$c_1 = c_1 h_{T^n}(e_1) = h_{\Pi\,\mathrm{Z}\,T^n}(e_1) = h_{\Pi\,\mathrm{Z}\,T^n}(-e_1) = c_2 h_{-T^n}(-e_1) = c_2,$$

and hence

$$\Pi\,\mathrm{Z}\,T^n = c_1(T^n + (-T^n)).$$

Being the cosine transform of an even measure, $\Pi\,\mathrm{Z}\,T^n$ is (by definition) a zonoid with center at o. Hence each support set of $\Pi\,\mathrm{Z}\,T^n$ is centrally symmetric (see e.g. [40, Corollary 3.5.6]). Thus $[c_1 e_1, \dots, c_1 e_n]$ is centrally symmetric, a contradiction. In conclusion, the degree q of homogeneity is not equal to $1/(n-1)$. Lemma 8 implies that $q > 1/(n-1)$. □

Lemma 11. *If* $\mathrm{Z} : \mathcal{K}_o^n \to \langle \mathcal{K}_c^n, \# \rangle$ *is a continuous* $\mathrm{SL}(n)$ *contravariant symmetric Blaschke valuation which is homogeneous of degree* q*, then* $q = (n+1)/(n-1)$.

Proof. From the last lemma we know that $q > 1/(n-1)$. Fix a number λ with $0 < \lambda < 1$. From relation (30) and the valuation property of Z we deduce

$$\mathrm{Z}\,T^n \# \mathrm{Z}(\zeta_\varepsilon T^n) = \mathrm{Z}(\phi_\varepsilon T^n) \# \mathrm{Z}(\psi_\varepsilon T^n),$$

and hence

$$\mathrm{C}\,S(\mathrm{Z}\,T^n, \cdot) + \mathrm{C}\,S(\mathrm{Z}(\zeta_\varepsilon T^n), \cdot) = \mathrm{C}\,S(\mathrm{Z}(\phi_\varepsilon T^n), \cdot) + \mathrm{C}\,S(\mathrm{Z}(\phi_\varepsilon T^n), \cdot)$$

for $0 < \varepsilon < \min\{\lambda, 1-\lambda\}$. If we combine the homogeneity and $\mathrm{SL}(n)$ contravariance of Z with the relations (6) and (27), then we obtain

$$\begin{aligned}&\mathrm{C}\,S(\mathrm{Z}\,T^n,\cdot)(x) + (\det \zeta_\varepsilon)^\alpha\, \mathrm{C}\,S(\mathrm{Z}\,T^n,\cdot)(\zeta_\varepsilon^t x)\\ &\qquad = (\det \phi_\varepsilon)^\alpha\, \mathrm{C}\,S(\mathrm{Z}\,T^n,\cdot)(\phi_\varepsilon^t x) + (\det \psi_\varepsilon)^\alpha\, \mathrm{C}\,S(\mathrm{Z}\,T^n,\cdot)(\psi_\varepsilon^t x)\end{aligned}$$

with $\alpha = (q+1)(n-1)/n - 1$. Calculating the involved determinants proves

$$\begin{aligned}&\mathrm{C}\,S(\mathrm{Z}\,T^n,\cdot)(x) + (2\varepsilon)^\alpha\, \mathrm{C}\,S(\mathrm{Z}\,T^n,\cdot)(\zeta_\varepsilon^t x)\\ &\qquad = (\lambda+\varepsilon)^\alpha\, \mathrm{C}\,S(\mathrm{Z}\,T^n,\cdot)(\phi_\varepsilon^t x) + (1-\lambda+\varepsilon)^\alpha\, \mathrm{C}\,S(\mathrm{Z}\,T^n,\cdot)(\psi_\varepsilon^t x).\end{aligned} \tag{43}$$

Since $q > 1/(n-1)$, we have $\alpha > 0$ which yields

$$\lim_{\varepsilon\to 0^+} (2\varepsilon)^\alpha\, \mathrm{C}\,S(\mathrm{Z}\,T^n,\cdot)(\zeta_\varepsilon^t x) = 0$$

for every $x \in \mathbb{R}^n$. So taking the limit $\varepsilon \to 0^+$ in (43) gives

$$\mathrm{C}\,S(\mathrm{Z}\,T^n,\cdot)(x) = \lambda^\alpha\, \mathrm{C}\,S(\mathrm{Z}\,T^n,\cdot)(\phi_0^t x) + (1-\lambda)^\alpha\, \mathrm{C}\,S(\mathrm{Z}\,T^n,\cdot)(\psi_0^t x).$$

In terms of projection bodies this can be written as

$$h_{\Pi\,\mathrm{Z}\,T^n}(x) = \lambda^\alpha h_{\Pi\,\mathrm{Z}\,T^n}(\phi_0^{-1} x) + (1-\lambda)^\alpha h_{\Pi\,\mathrm{Z}\,T^n}(\psi_0^{-1} x).$$

for every $x \in \mathbb{R}^n$. Note that the range of Z consists of bodies containing the origin in their interiors. Thus $\Pi\,\mathrm{Z}\,T^n$ contains the origin in its interior, too. We conclude that the support function $h_{\Pi\,\mathrm{Z}\,T^n}$ is strictly positive on the sphere. The standard basis vector e_3 is an eigenvector of ϕ_0^t and ψ_0^t. Evaluating the last equation at e_3 gives $1 = \lambda^\alpha + (1-\lambda)^\alpha$ for every λ between zero and one. This implies $\alpha = 1$ and hence $q = (n+1)/(n-1)$. □

Lemma 12. *Suppose that* $\mathrm{Z} : \mathcal{K}_o^n \to \langle \mathcal{K}_c^n, \# \rangle$ *is a continuous* $\mathrm{SL}(n)$ *contravariant symmetric Blaschke valuation which is homogeneous of degree* $(n+1)/(n-1)$. *Then there exists an extension* $\bar{\mathrm{Z}} : \bar{\mathcal{K}}_o^n \to \langle \mathcal{K}^n, + \rangle$ *of* $\Pi \circ \mathrm{Z} : \mathcal{K}_o^n \to \langle \mathcal{K}^n, + \rangle$ *which is a Minkowski valuation. Moreover, this extension* $\bar{\mathrm{Z}}$ *is* $\mathrm{SL}(n)$ *covariant, homogeneous of degree* $n-1$, *and continuous on* $\mathcal{K}_o^n$.

Proof. The extension $\bar{\mathrm{Z}}$ is defined by

$$h_{\bar{\mathrm{Z}}K}(x) = \begin{cases} \frac{1}{2}\,\mathrm{C}\,S(\mathrm{Z}\,K,\cdot) & \dim K = n,\\ 0 & \dim K \le n-1.\end{cases}$$

The proof that $\bar{\mathrm{Z}}$ is a valuation is much easier than before. Indeed, we just have to verify the valuation property for the case (2_n). (The case (1_n) follows as in the proof of Lemma 5.) Again, it suffices to prove

$$h_{\bar{\mathrm{Z}}K} + h_{\bar{\mathrm{Z}}(K\cap u)} = h_{\bar{\mathrm{Z}}(K\cap u^+)} + h_{\bar{\mathrm{Z}}(K\cap u^-)} \tag{44}$$

for an arbitrary unit vector u such that $\dim(K \cap u^\pm) = n$. Assume first that $u = e_n$. Since Z is a Blaschke valuation on n-dimensional bodies and

$$\begin{aligned}{[K\cap e_n^+, \pm s e_n] \cup [K \cap e_n^-, \pm s e_n]} &= [K, \pm s e_n],\\ {[K\cap e_n^+, \pm s e_n] \cap [K \cap e_n^-, \pm s e_n]} &= [K\cap e_n^\perp, \pm s e_n],\end{aligned}$$

for every positive s, we obtain

$$\begin{aligned}&\mathrm{C}\,S(\mathrm{Z}[K, \pm s e_n],\cdot) + \mathrm{C}\,S(\mathrm{Z}[K\cap e_n^\perp, \pm s e_n],\cdot)\\ &\qquad = \mathrm{C}\,S(\mathrm{Z}[K\cap e_n^+, \pm s e_n],\cdot) + \mathrm{C}\,S(\mathrm{Z}[K\cap e_n^-, \pm s e_n],\cdot).\end{aligned} \tag{45}$$

For positive s define a linear map ϕ by

$$\phi e_i = e_i, \qquad i = 1, \dots, n-1, \qquad \phi e_n = s e_n.$$

From the $\mathrm{SL}(n)$ contravariance and homogeneity of Z as well as relation (27) we get

$$\mathrm{C}\,S(\mathrm{Z}[K \cap e_n^{\perp}, \pm s e_n], \cdot) = \mathrm{C}\,S\left(s^{\frac{2}{n-1}} \phi^{-t}\, \mathrm{Z}[K \cap e_n^{\perp}, \pm e_n], \cdot\right).$$

Formula (10) yields

$$\mathrm{C}\,S(\mathrm{Z}[K \cap e_n^{\perp}, \pm s e_n], \cdot)(x) = s\,\mathrm{C}\,S(\mathrm{Z}[K \cap e_n^{\perp}, \pm e_n], \cdot)(\phi^t x),$$

for every $x \in \mathbb{R}^n$ and therefore

$$\lim_{s \to 0^+} \mathrm{C}\,S(\mathrm{Z}[K \cap e_n^{\perp}, \pm s e_n], \cdot)(x) = 0.$$

From (45) and the convergence behavior

$$\begin{aligned}
\lim_{s \to 0^+} [K \cap e_n^{+}, \pm s e_n] &= K \cap e_n^{+}, \\
\lim_{s \to 0^+} [K \cap e_n^{-}, \pm s e_n] &= K \cap e_n^{-}, \\
\lim_{s \to 0^+} [K, \pm s e_n] &= K,
\end{aligned}$$

we deduce that

$$\mathrm{C}\,S(\mathrm{Z}\,K, \cdot)(x) = \mathrm{C}\,S(\mathrm{Z}(K \cap e_n^{+}), \cdot)(x) + \mathrm{C}\,S(\mathrm{Z}(K \cap e_n^{-}), \cdot)(x).$$

The definition of $\bar{\mathrm{Z}}$ implies that (44) holds for the case (2_n) and $u = e_n^{\perp}$. Next, let $u \in S^{n-1}$ be arbitrary. Denote by ϑ a proper rotation with $\vartheta e_n = u$. The definition of $\bar{\mathrm{Z}}$, the covariance of $\Pi \circ \mathrm{Z}$, and the already established valuation property gives

$$\begin{aligned}
h_{\bar{\mathrm{Z}}(K \cap u^+)}(x) + h_{\bar{\mathrm{Z}}(K \cap u^-)}(x) &= h_{\Pi\,\mathrm{Z}(K \cap u^+)}(x) + h_{\Pi\,\mathrm{Z}(K \cap u^-)}(x) \\
&= h_{\Pi\,\mathrm{Z}(\vartheta^{-1} K \cap e_n^+)}(\vartheta^{-1} x) + h_{\Pi\,\mathrm{Z}(\vartheta^{-1} K \cap e_n^-)}(\vartheta^{-1} x) \\
&= h_{\Pi\,\mathrm{Z}(\vartheta^{-1} K)}(\vartheta^{-1} x) \\
&= h_{\bar{\mathrm{Z}} K}(x).
\end{aligned}$$

This settles the proof of the valuation property in general for the case (2_n). In order to see that $\bar{\mathrm{Z}}$ is $\mathrm{SL}(n)$ covariant and homogeneous of degree $n+1$, one just has to apply Lemma 9. □

The following characterization of the centroid body operator due to Ludwig [29, Theorem 1] will be crucial for our purpose.

Theorem 5. *Suppose that* $\mathrm{Z} : \bar{\mathcal{P}}_o^n \to \langle \mathcal{K}^n, + \rangle$ *is an* $\mathrm{SL}(n)$ *covariant Minkowski valuation which is homogeneous of degree* $n+1$*. Then there exist constants* $c_1 \in \mathbb{R}$ *and* $c_2 \geq 0$ *such that*

$$\mathrm{Z}\,P = c_1 m(P) + c_2 \Gamma P$$

for every $P \in \bar{\mathcal{P}}_o^n$.

Here, $m(P)$ denotes the *moment vector* of P, i.e.

$$m(P) = \int_P x\, dx.$$

Lemma 13. *If* $\mathrm{Z} : \mathcal{K}_o^n \to \langle \mathcal{K}_c^n, \# \rangle$ *is a continuous* $\mathrm{SL}(n)$ *contravariant symmetric Blaschke valuation which is homogeneous of degree* $(n+1)/(n-1)$*, then there exists a constant* $c > 0$ *such that*

$$\mathrm{Z}\, K = c\Lambda \tilde{\nabla} K$$

for every $K \in \mathcal{K}_o^n$*.*

Proof. By Lemma 12 we can extend $\Pi \circ \mathrm{Z}$ to a valuation $\bar{\mathrm{Z}}$ defined on $\bar{\mathcal{K}}_o^n$ which is $\mathrm{SL}(n)$ covariant and homogeneous of degree $n+1$. Thus Theorem 5 proves the existence of two constants $c_1 \in \mathbb{R}$ and $c_2 \geq 0$ such that

$$\bar{\mathrm{Z}} P = c_1 m(P) + c_2 \Gamma P$$

for every $P \in \bar{\mathcal{P}}_o^n$. Since $\bar{\mathrm{Z}} P$ and ΓP is origin-symmetric on n-dimensional bodies, we obtain

$$c_1 m(P) \cdot x + c_2 h_{\Gamma P}(x) = h_{\bar{\mathrm{Z}} P}(x) = h_{\bar{\mathrm{Z}} P}(-x) = -c_1 m(P) \cdot x + c_2 h_{\Gamma P}(x)$$

for every $x \in \mathbb{R}^n$ and every $P \in \bar{\mathcal{P}}_o^n$. Consequently, we have $c_1 = 0$. This implies $\Pi\, \mathrm{Z}\, P = c_2 \Gamma P$ for all n-dimensional polytopes. Obviously, $c_2 > 0$. Since $\Pi \circ \mathrm{Z}$ and Γ are continuous on $\mathcal{K}_o^n$, we obtain

$$\Pi\, \mathrm{Z}\, K = c_2 \Gamma K$$

for every $K \in \mathcal{K}_o^n$. By rewriting this in terms of the cosine transform we obtain

$$\mathrm{C}\, S(\mathrm{Z}\, K, \cdot) = \tfrac{2c_2}{n+1}\, \mathrm{C}\, \rho_K^{n+1} = \tfrac{2c_2}{n+1}\, \mathrm{C}\, \rho_{\tilde{\nabla} K}^{n+1}.$$

The injectivity property (9) finally shows $\mathrm{Z}\, K = c\Lambda \tilde{\nabla} K$ with $c = (2c_2/(n+1))^{1/(n-1)}$. $\square$

We summarize the results of this subsection in

Lemma 14. *A map* $\mathrm{Z} : \mathcal{K}_o^n \to \langle \mathcal{K}_c^n, \# \rangle$ *is a continuous, homogeneous, and* $\mathrm{SL}(n)$ *contravariant valuation if and only if there exists a constant* $c > 0$ *such that*

$$\mathrm{Z}\, K = c\Lambda_c K$$

for every $K \in \mathcal{K}_o^n$*.*

If we combine this with Lemma 7, we finally proved Theorem 1.

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