ASYMMETRIC AFFINE L_p SOBOLEV INEQUALITIES

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ABSTRACT. A new sharp affine L_p Sobolev inequality for functions on $\mathbb{R}^n$ is established. This inequality strengthens and implies the previously known affine L_p Sobolev inequality which in turn is stronger than the classical L_p Sobolev inequality.

1. Introduction

The sharp L_p Sobolev inequality of Aubin [1] and Talenti [36] is one of the fundamental inequalities of analysis. It plays a central role in a number of different areas such as the theory of partial differential equations, geometric measure theory, and the calculus of variations. In recent years, many variations and generalizations have been obtained, see, e.g., [2, 3, 6, 8, 11, 32, 33, 37] and the references therein.

Recently, Zhang [38] (for $p = 1$) and Lutwak, Yang, and Zhang [27] (for $1 < p < n$) formulated and proved a sharp affine L_p Sobolev inequality. This remarkable inequality is invariant under all affine transformations of $\mathbb{R}^n$ and turned out to be significantly stronger than the classical L_p Sobolev inequality although it does not rely on any Euclidean geometric structure. As was shown in [38], the affine Zhang–Sobolev inequality is equivalent to the extended Petty projection inequality established in [38]. In the Euclidean setting, all the L_p Sobolev inequalities have the classical isoperimetric inequality at their core (for $p = 1$ both inequalities are equivalent as discovered Maz'ya [31] and, independently, by Federer and Fleming [10]). In the affine setting, the situation is more difficult. Here, new geometry is needed to pass from the case $p = 1$ to $p > 1$. To establish the affine L_p Sobolev inequality for $p > 1$, Lutwak, Yang and Zhang [25] had to first establish an L_p Petty projection inequality.

In this article we establish a new sharp affine L_p Sobolev inequality which strengthens and directly implies the previously known sharp affine L_p Sobolev inequality of Lutwak, Yang, and Zhang. The geometry behind this new Sobolev inequality is an L_p affine isoperimetric inequality, stronger than the L_p Petty projection inequality, which was recently established by the authors in [13]. This crucial geometric inequality was made possible by recent advances in valuation theory by Ludwig [17, 19].

We denote by $W^{1,p}(\mathbb{R}^n)$ the space of real-valued L_p functions on $\mathbb{R}^n$ ($n \geq 2$) with weak L_p partial derivatives. Let $|\cdot|$ denote the standard Euclidean norm on $\mathbb{R}^n$ and let $\|f\|_p$ denote the usual L_p norm of f in $\mathbb{R}^n$. The classical sharp L_p Sobolev inequality states that if $f \in W^{1,p}(\mathbb{R}^n)$, with real p satisfying $1 \leq p < n$, then

$$\left(\int_{\mathbb{R}^n} |\nabla f|^p \, dx \right)^{1/p} \geq \hat{c}_{n,p} \, \|f\|_{p^*}, \tag{1.1}$$

where $p^* = np/(n-p)$. The optimal constants $\hat{c}_{n,p}$ in this inequality are due to Federer and Fleming [10] and Maz'ya [31] for $p = 1$ and to Aubin [1] and Talenti [36] for $p > 1$. The extremal functions for inequality (1.1) are the characteristic functions of balls for $p = 1$ and for $p > 1$ equality is attained when

$$f(x) = (a + b|(x - x_0)|^{p/(p-1)})^{1-n/p},$$

with a, $b > 0$, and $x_0 \in \mathbb{R}^n$.

The sharp affine L_p Sobolev inequality of Zhang [38] and Lutwak, Yang, and Zhang [27] states that if $f \in W^{1,p}(\mathbb{R}^n)$, $1 \le p < n$, then

$$\left(\int_{S^{n-1}} \|\mathrm{D}_u f\|_p^{-n}\, du \right)^{-1/n} \ge \tilde{c}_{n,p}\, \|f\|_{p^*}, \tag{1.2}$$

where $\mathrm{D}_u f$ is the directional derivative of f in the direction $u \in S^{n-1}$. The optimal constants $\tilde{c}_{n,p}$ in (1.2) were explicitly computed in [38] (for $p = 1$) and [27]. The determination of $\hat{c}_{n,p}$ and $\tilde{c}_{n,p}$ in (1.1) and (1.2) is in many situations not as important as the identification of extremal functions. The extremals associated with inequality (1.2) for $p = 1$ are the characteristic functions of ellipsoids and for $p > 1$ equality is attained when

$$f(x) = (a + |\phi(x - x_0)|^{p/(p-1)})^{1-n/p},$$

with $a > 0$, $\phi \in \mathrm{GL}(n)$, and $x_0 \in \mathbb{R}^n$.

We emphasize that inequality (1.2) is invariant under affine transformations of $\mathbb{R}^n$, while the classical L_p Sobolev inequality (1.1) is invariant only under rigid motions. That the affine L_p Sobolev inequality is stronger than (1.1) follows from an application of Hölder's inequality (cf. [27, p. 33]):

$$\left(\int_{\mathbb{R}^n} |\nabla f|^p\, dx \right)^{1/p} \ge a_{n,p} \left(\int_{S^{n-1}} \|\mathrm{D}_u f\|_p^{-n}\, du \right)^{-1/n} \ge \hat{c}_{n,p}\, \|f\|_{p^*}.$$

Here, equality in the left inequality holds if and only if $\|\mathrm{D}_u f\|_p$ is independent of $u \in S^{n-1}$. The constant $a_{n,p}$ was computed in [27].

For $u \in S^{n-1}$ and $f \in W^{1,p}(\mathbb{R}^n)$, we denote by

$$\mathrm{D}_u^+ f(x) = \max\{\mathrm{D}_u f(x), 0\}$$

the positive part of the directional derivative of f in the direction u.

The main result of this article is the following:

Theorem 1. *If $f \in W^{1,p}(\mathbb{R}^n)$, with $1 \le p < n$, then*

$$\left(\int_{S^{n-1}} \|\mathrm{D}_u^+ f\|_p^{-n}\, du \right)^{-1/n} \ge c_{n,p}\, \|f\|_{p^*}, \tag{1.3}$$

where $p^* = np/(n-p)$. For $p > 1$, the optimal constant $c_{n,p}$ is given by

$$c_{n,p} = 2^{-1/p} (\tfrac{n-p}{p-1})^{1-1/p} \left(\frac{\Gamma(\frac{n}{p})\Gamma(n+1-\frac{n}{p})}{\Gamma(n+1)} \right)^{1/n} \left(\frac{n\Gamma(\frac{n}{2})\Gamma(\frac{p+1}{2})}{\sqrt{\pi}\Gamma(\frac{n+p}{2})} \right)^{1/p},$$

and $c_{n,1} = \lim_{p \to 1} c_{n,p}$. If $p = 1$, equality holds in (1.3) for characteristic functions of ellipsoids and for $p > 1$ equality is attained when

$$f(x) = (a + |\phi(x - x_0)|^{p/(p-1)})^{1-n/p},$$

with $a > 0$, $\phi \in \mathrm{GL}(n)$ and $x_0 \in \mathbb{R}^n$.

Note that inequality (1.3) is invariant under affine transformations of $\mathbb{R}^n$. We will show in Section 6 that, for $p \geq 1$,

$$\left(\int_{S^{n-1}} \|\mathrm{D}_u f\|_p^{-n} \, du \right)^{-1/n} \geq 2^{1/p} \left(\int_{S^{n-1}} \|\mathrm{D}_u^+ f\|_p^{-n} \, du \right)^{-1/n}. \tag{1.4}$$

Since $\tilde{c}_{n,p} = 2^{1/p} c_{n,p}$, the new affine L_p Sobolev inequality (1.3) is stronger than the inequality Lutwak, Yang, and Zhang. In particular, inequality (1.3) is also stronger than the classical L_p Sobolev inequality (1.1). It is crucial to observe that while for inequality (1.2) only the even part of the directional derivatives of f contribute, for the new inequality (1.3) also asymmetric parts are accounted for. This is reflected by the fact that equality in (1.4) holds precisely when $\|\mathrm{D}_u^+ f\|_p$ is an even function on S^{n-1}.

The classical L_2 Sobolev inequality has drawn particular attention due to its conformal invariance, see, e.g., [3, 6, 16]. As noted in [27], the affine L_2 Sobolev inequality of Lutwak, Yang, and Zhang is equivalent under an affine transformation to the L_2 Sobolev inequality. The case $p = 2$ of inequality (1.3), however, yields a stronger inequality.

While the geometric inequalities behind the affine Zhang–Sobolev inequality and inequality (1.3) for $p = 1$ are the same, a new affine isoperimetric inequality recently established by the authors [13] is needed to establish inequality (1.3) for $p > 1$. We will apply this inequality to convex bodies (associated with the given function) which occur as solutions to the L_p Minkowski problem for $1 < p < n$. Since the geometric inequality assumes that the convex bodies contain the origin in their interiors, its application is intricate in the asymmetric situation. Here, the origin can lie on the boundary of the convex bodies which occur as a solution to the L_p Minkowski problem. All this geometric background will be discussed in detail in Sections 3 & 4.

2. Background Material

In the following we state some basic facts about convex bodies and compact domains. General references for the theory of convex bodies are the books by Gardner [12] and Schneider [35]. We will also collect background material from real analysis needed in the proof of Theorem 1.

The setting for this article is Euclidean n-space $\mathbb{R}^n$ with $n \geq 2$. A *convex body* is a compact convex set in $\mathbb{R}^n$ with non-empty interior. Let $\mathcal{K}^n$ denote the set of convex bodies in $\mathbb{R}^n$ endowed with the Hausdorff metric. We write $\mathcal{K}_{\mathrm{o}}^n$ for the set of convex bodies containing the origin in their interiors.

A compact convex set K is uniquely determined by its *support function* $h(K, \cdot)$, where $h(K, x) = \max\{x \cdot y : \; y \in K\}$, $x \in \mathbb{R}^n$, and where $x \cdot y$ denotes the usual inner product of x and y in $\mathbb{R}^n$. Note that $h(K, \cdot)$ is positively homogeneous of degree one and subadditive. Conversely, every function with these properties is the support function of a unique compact convex set.

If $K \in \mathcal{K}_{\mathrm{o}}^n$, the polar body K^* of K is defined by

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } y \in K\}.$$

Let $\rho(K, x) = \max\{\lambda \geq 0 : \; \lambda x \in K\}$, $x \in \mathbb{R}^n \backslash \{0\}$, denote the radial function of K. It follows from the definitions of support functions and radial functions, and the definition of the polar body of K, that

$$\rho(K^*, \cdot) = h(K, \cdot)^{-1} \qquad \text{and} \qquad h(K^*, \cdot) = \rho(K, \cdot)^{-1}. \tag{2.1}$$

A *compact domain* is the closure of a bounded open subset of $\mathbb{R}^n$. If M and N are compact domains in $\mathbb{R}^n$, then the Brunn–Minkowski inequality states that

$$V(M+N)^{1/n} \geq V(M)^{1/n} + V(N)^{1/n},$$

where V denotes the usual n-dimensional Lebesgue measure. For a compact domain M and a convex body K in $\mathbb{R}^n$, define

$$nV_1(M,K) = \liminf_{\varepsilon \to 0^+} \frac{V(M+\varepsilon K) - V(M)}{\varepsilon}.$$

If the boundary ∂M of M is a C^1 submanifold of $\mathbb{R}^n$, then

$$V_1(M,K) = \frac{1}{n} \int_{\partial M} h(K, \nu(x))\, d\mathcal{H}^{n-1}(x), \tag{2.2}$$

where $\nu(x)$ is the exterior unit normal vector of ∂M at x and $\mathcal{H}^{n-1}$ denotes $(n-1)$-dimensional Hausdorff measure (cf. [38, Lemma 3.2]).

We need the following immediate consequence of the Brunn–Minkowski inequality: If M is a compact domain and K is a convex body in $\mathbb{R}^n$, then

$$V_1(M,K)^n \geq V(M)^{n-1} V(K). \tag{2.3}$$

We will frequently apply Federer's co-area formula (see, e.g., [9, p. 258]). For quick reference we state a version which is sufficient for our purposes: If $f : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz and $g : \mathbb{R}^n \to [0, \infty)$ is measurable, then, for any Borel set $A \subseteq \mathbb{R}$,

$$\int_{f^{-1}(A)\cap\{|\nabla f|>0\}} g(x)\,dx = \int_A \int_{f^{-1}\{y\}} \frac{g(x)}{|\nabla f(x)|}\,d\mathcal{H}^{n-1}(x)\,dy. \tag{2.4}$$

Finally, we require the following consequence (cf. [2, Proposition 2.18]) of Bliss' inequality [4]. For an elementary proof we refer to [27, Lemma 4.1]: Let $f : (0, \infty) \to [0, \infty)$ be decreasing and locally absolutely continuous and let $1 < p < n$. If the integrals exist, then

$$\left(\int_0^\infty |f'(x)|^p x^{n-1}\,dx\right)^{1/p} \geq b_{n,p} \left(\int_0^\infty f(x)^{p^*} x^{n-1}\,dx\right)^{1/p^*}, \tag{2.5}$$

where $p^* = np/(n-p)$ and

$$b_{n,p} = n^{1/p^*} (\tfrac{n-p}{p-1})^{1-1/p} \left(\tfrac{\Gamma(\frac{n}{p})\Gamma(n+1-\frac{n}{p})}{\Gamma(n)}\right)^{1/n}.$$

Equality in (2.5) holds if $f(x) = (ax^{p/(p-1)} + b)^{1-n/p}$, with $a, b > 0$.

3. L_p Projection Bodies and the L_p Minkowski Problem

In this section we collect the material which forms the geometric core in the proof of our main result. The critical ingredients are an L_p affine isoperimetric inequality recently established in [13] and the solution (to the discrete data case) of an L_p extension of the classical Minkowski problem obtained in [7].

The *projection body* ΠK of $K \in \mathcal{K}^n$ is the convex body defined by

$$h(\Pi K, u) = \mathrm{vol}_{n-1}(K|u^\perp),$$

where $\mathrm{vol}_{n-1}(K|u^\perp)$ is the $(n-1)$-dimensional volume of the projection of K onto the hyperplane orthogonal to u.

Introduced by Minkowski, projection bodies have become a central notion in convex geometry, see, e.g., [12, 13, 17, 26] and the references therein. A recent result by Ludwig [19] has demonstrated their special place in affine geometry: The projection operator was characterized as the unique valuation which is contravariant with respect to linear transformations.

The fundamental affine isoperimetric inequality for projection bodies is the *Petty projection inequality*: If $K \in \mathcal{K}^n$, then

$$V(K)^{n-1} V(\Pi^* K) \leq \left(\frac{\kappa_n}{\kappa_{n-1}}\right)^n,$$

with equality if and only if K is an ellipsoid. Here $\Pi^* K = (\Pi K)^*$ and κ_n denotes the volume of the Euclidean unit ball in $\mathbb{R}^n$. This inequality turned out to be far stronger than the classical isoperimetric inequality. It is the geometric inequality behind the affine Zhang–Sobolev inequality [38].

Projection bodies are part of the classical Brunn–Minkowski theory. In a series of articles [22, 23], Lutwak showed that merging the notion of volume with Firey's L_p addition of convex sets leads to a Brunn–Minkowski theory for each $p \geq 1$. Since Lutwak's seminal work, the topic has been much studied, see, e.g., [5, 7, 18, 19, 20, 24, 26, 27, 29, 30]. For $p \geq 1$, $K, L \in \mathcal{K}_o^n$ and $\alpha, \beta \geq 0$ (not both zero), the L_p *Minkowski combination* $\alpha \cdot K +_p \beta \cdot L$ is the convex body defined by

$$h(\alpha \cdot K +_p \beta \cdot L, \cdot)^p = \alpha h(K, \cdot)^p + \beta h(L, \cdot)^p.$$

One of the basic notions of the L_p Brunn–Minkowski theory is the L_p *mixed volume* $V_p(K, L)$ of two bodies K, $L \in \mathcal{K}_o^n$. It was defined in [22] by

$$V_p(K, L) = \frac{p}{n} \lim_{\varepsilon \to 0^+} \frac{V(K +_p \varepsilon \cdot L) - V(K)}{\varepsilon}.$$

Clearly, the diagonal form of V_p reduces to ordinary volume, i.e., for $K \in \mathcal{K}_o^n$,

$$V_p(K, K) = V(K). \tag{3.1}$$

It was shown in [22] that corresponding to each convex body $K \in \mathcal{K}_o^n$, there exists a positive Borel measure on S^{n-1}, the L_p *surface area measure* $S_p(K, \cdot)$ of K, such that for every $L \in \mathcal{K}_o^n$,

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h(L, u)^p \, dS_p(K, u). \tag{3.2}$$

The measure $S_1(K, \cdot)$ is just the classical surface area measure $S(K, \cdot)$ of K. Moreover, it was proved in [22], that the L_p surface area measure is absolutely continuous with respect to $S(K, \cdot)$:

$$dS_p(K, u) = h(K, u)^{1-p} dS(K, u), \qquad u \in S^{n-1}. \tag{3.3}$$

Recall that for a Borel set $\omega \subseteq S^{n-1}$, $S(K, \omega)$ is the $(n-1)$-dimensional Hausdorff measure of the set of all boundary points of K for which there exists a normal vector of K belonging to ω. From the homogeneity properties of the surface area measure and the support function of K, one obtains that, for every $\lambda > 0$,

$$S_p(\lambda K, \cdot) = \lambda^{n-p} S_p(K, \cdot). \tag{3.4}$$

For a finite Borel measure μ on S^{n-1}, we define a continuous function $\mathrm{C}_p^+ \mu$ on S^{n-1}, the *asymmetric* L_p *cosine transform* of μ, by

$$(\mathrm{C}_p^+ \mu)(u) = \int_{S^{n-1}} (u \cdot v)_+^p \, d\mu(v), \qquad u \in S^{n-1},$$

where $(u \cdot v)_+ = \max\{u \cdot v, 0\}$. For $f \in C(S^{n-1})$, let $\mathrm{C}_p^+ f$ be the asymmetric L_p cosine transform of the absolutely continuous measure (with respect to spherical Lebesgue measure) with density f. The *asymmetric* L_p *projection body* $\Pi_p^+ K$ of $K \in \mathcal{K}_o^n$, first considered in [23], is the convex body defined by

$$h(\Pi_p^+ K, \cdot)^p = \mathrm{C}_p^+ S_p(K, \cdot). \tag{3.5}$$

For $p > 1$, Ludwig [19] established the L_p analogue of her classification of the projection operator: She showed that the convex bodies

$$c_1 \cdot \Pi_p^+ K +_p c_2 \cdot \Pi_p^- K, \qquad K \in \mathcal{K}_o^n, \tag{3.6}$$

where $\Pi_p^- K = \Pi_p^+(-K)$ and $c_1, c_2 \geq 0$ (not both zero), constitute all natural L_p extensions of projection bodies.

The (symmetric) L_p *projection body* $\Pi_p K$ of $K \in \mathcal{K}_o^n$, defined in [26], is

$$\Pi_p K = \tfrac{1}{2} \cdot \Pi_p^+ K +_p \tfrac{1}{2} \cdot \Pi_p^- K.$$

Lutwak, Yang, and Zhang [26] (see also Campi and Gronchi [5]) established an L_p extension of the Petty projection inequality for the (symmetric) L_p projection operator which forms the geometry behind their sharp affine L_p Sobolev inequality: If $K \in \mathcal{K}_o^n$, then

$$V(K)^{n/p-1} V(\Pi_p^* K) \leq \left(\frac{\kappa_n \Gamma\left(\frac{n+p}{2}\right)}{\pi^{(n-1)/2} \Gamma\left(\frac{1+p}{2}\right)} \right)^{n/p}, \tag{3.7}$$

with equality if and only if K is an ellipsoid centered at the origin.

Recently the authors [13] established the L_p Petty projection inequality for each member of the family (3.6) of L_p projection operators. The geometric core of the asymmetric affine L_p Sobolev inequality (1.3) is the following special case of this result:

Theorem 2. *If $p > 1$ and $K \in \mathcal{K}_o^n$, then*

$$V(K)^{n/p-1} V(\Pi_p^{+,*} K) \leq \left(\frac{\kappa_n \Gamma\left(\frac{n+p}{2}\right)}{\pi^{(n-1)/2} \Gamma\left(\frac{1+p}{2}\right)} \right)^{n/p}, \tag{3.8}$$

where equality is attained if K is an ellipsoid centered at the origin.

Although this inequality was formulated in [13] for dimensions $n \geq 3$, we remark that it also holds true in dimension $n = 2$. The proof is verbally the same as the one given in [13]. Since surface area measures have their center of mass at the origin, we have

$$\Pi_1^+ K = \Pi K.$$

Thus, for $p = 1$, inequality (3.8) is the classical Petty projection inequality.

It was also shown in [13] that inequality (3.8), for $p > 1$, is stronger than the L_p Petty projection inequality (3.7) of Lutwak, Yang, and Zhang: If $K \in \mathcal{K}_o^n$, then

$$V(\Pi_p^* K) \leq V(\Pi_p^{+,*} K). \tag{3.9}$$

If p is not an odd integer, equality holds precisely for origin-symmetric K.

We turn now to the second main ingredient in the proof of Theorem 1. The L_p Minkowski problem asks for necessary and sufficient conditions for a Borel measure μ on S^{n-1} to be the L_p surface area measure of a convex body. A solution to this problem for $p > n$ was given by Chou and Wang [7]. Moreover, Chou and Wang [7] established the solution to the discrete-data case of the L_p Minkowski problem for all $p > 1$ (see also [15] for an alternate approach). The following solution to the discrete L_p Minkowski problem due to Chou and Wang will be crucial:

Theorem 3. *If $\alpha_1, \ldots, \alpha_k > 0$ and $u_1, \ldots, u_k \in S^{n-1}$ are not contained in a closed hemisphere, then, for any $p > 1$, $p \neq n$, there exists a unique polytope $P \in \mathcal{K}_o^n$ such that*

$$\sum_{j=1}^{k} \alpha_j \delta_{u_j} = S_p(P, \cdot).$$

Here, δ_u denotes the probability measure with unit point mass at $u \in S^{n-1}$.

We will also apply two auxiliary results [28, Lemma 2.2 & 2.3] concerning the volume normalized L_p Minkowski problem: Let μ be a positive Borel measure on S^{n-1}, and let $K \in \mathcal{K}^n$ contain the origin. Suppose that

$$V(K)h(K,\cdot)^{p-1}\mu = S(K,\cdot),$$

and that for some constant $c > 0$,

$$\int_{S^{n-1}} (u \cdot v)_+^p \, d\mu(v) \geq \frac{n}{c^p} \quad \text{for every } u \in S^{n-1}.$$

Then

$$V(K) \geq \kappa_n \left(\frac{n}{\mu(S^{n-1})}\right)^{n/p} \qquad \text{and} \qquad K \subset cB_n, \tag{3.10}$$

where B_n denotes the Euclidean unit ball in $\mathbb{R}^n$.

4. A Critical Lemma

A crucial part in the proof of our main result is the construction of a family of convex bodies containing the origin in their interiors from a given function. It is essential that the origin is an interior point in order to apply the critical geometric inequality (3.8) afterwards. In [26], this was done by using the solution to the even L_p Minkowski problem. In our case, we have to deal with the solutions to the general L_p Minkowski problem. Here, the bodies can contain the origin in their boundaries (cf. [15]). Therefore, we will associate a two parametric family of convex polytopes with a given function. These polytopes are obtained from the solution to the discrete-date case of the L_p Minkowski problem which ensures that they contain the origin as an interior point. This will allow us to use the relevant geometric inequality.

A function $f \in C^\infty(\mathbb{R}^n)$ is called *smooth*. Suppose f is smooth and has compact support. Then the level set

$$[f]_t = \{x \in \mathbb{R}^n : |f(x)| \geq t\}$$

is compact for every $0 < t \leq \|f\|_\infty$, where $\|f\|_\infty$ denotes the maximum value of $|f|$ over $\mathbb{R}^n$.

Lemma 1. *Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is smooth and has compact support. Then, for almost every $t \in (0, \|f\|_\infty)$, there exists a sequence of convex polytopes $P_k^t \in \mathcal{K}_\mathrm{o}^n$, $k \in \mathbb{N}$, such that*

$$\lim_{k\to\infty} P_k^t = K_f^t \in \mathcal{K}^n$$

and

$$V(K_f^t) = \frac{1}{n}\int_{\partial[f]_t} h(K_f^t, \nabla f(x))^p |\nabla f(x)|^{-1} \, d\mathcal{H}^{n-1}(x). \tag{4.1}$$

Moreover, there exists a convex body $L_f^t \in \mathcal{K}_\mathrm{o}^n$ such that

$$\lim_{k\to\infty} \Pi_p^+ P_k^t = L_f^t.$$

Proof : By Sard's theorem, for almost every $t \in (0, \|f\|_\infty)$, the boundary $\partial[f]_t$ of $[f]_t$ is a smooth $(n-1)$-dimensional submanifold with everywhere nonzero normal vector ∇f. Let t be chosen in this way and denote by $\nu(x) = \nabla f(x)/|\nabla f(x)|$ the unit normal of $\partial[f]_t$ at x.

Let μ^t be the finite positive Borel measure on S^{n-1} defined by

$$\int_{S^{n-1}} g(v)\, d\mu^t(v) = \int_{\partial[f]_t} g(\nu(x))|\nabla f(x)|^{p-1}\, d\mathcal{H}^{n-1}(x), \tag{4.2}$$

for $g \in C(S^{n-1})$. Since

$$\{\nu(x) :\ x \in \partial[f]_t\} = S^{n-1}, \tag{4.3}$$

it follows that for every $u \in S^{n-1}$,

$$\int_{S^{n-1}} (u \cdot v)_+\, d\mu^t(v) = \int_{\partial[f]_t} (u \cdot \nu(x))_+ |\nabla f(x)|^{p-1}\, d\mathcal{H}^{n-1}(x) > 0.$$

Therefore, the measure μ^t cannot be concentrated in a closed hemisphere.

As in [35, pp. 392-3], construct a sequence μ_k^t, $k \in \mathbb{N}$, of discrete measures on S^{n-1} whose support is not contained in a closed hemisphere and such that μ_k^t converges weakly to μ^t as $k \to \infty$. By Theorem 3, for each $k \in \mathbb{N}$, there exists a polytope $P_k^t \in \mathcal{K}_{\mathrm{o}}^n$ such that

$$\mu_k^t = S_p(P_k^t, \cdot). \tag{4.4}$$

We want to show that the sequence of polytopes P_k^t is bounded. To this end, define for each $k \in \mathbb{N}$ a new polytope Q_k^t by

$$Q_k^t = V(P_k^t)^{-1/p} P_k^t.$$

By (3.3) and the homogeneity (3.4) of L_p surface area measures, the polytopes Q_k^t, $k \in \mathbb{N}$, form a solution to the volume normalized L_p Minkowski problem

$$V(Q_k^t) h(Q_k^t, \cdot)^{p-1} \mu_k^t = S(Q_k^t, \cdot). \tag{4.5}$$

Moreover, from definition (3.5), relation (4.4) and the weak convergence of the measures μ_k^t, it follows that for every $u \in S^{n-1}$,

$$h(\Pi_p^+ P_k^t, u)^p = \int_{S^{n-1}} (u \cdot v)_+^p\, d\mu_k^t(v) \longrightarrow \int_{S^{n-1}} (u \cdot v)_+^p\, d\mu^t(v) > 0. \tag{4.6}$$

Since pointwise convergence of support functions implies uniform convergence (see, e.g., [35, Theorem 1.8.12]), there exists a $c > 0$ such that for all $k \in \mathbb{N}$,

$$\int_{S^{n-1}} (u \cdot v)_+^p\, d\mu_k^t(v) > c, \quad \text{for every } u \in S^{n-1}. \tag{4.7}$$

From (4.5), (4.7) and (3.10), we deduce that the sequence Q_k^t, $k \in \mathbb{N}$, is bounded. Moreover, by (3.10) and the weak convergence of the measures μ_k^t, the volumes $V(Q_k^t)$ are bounded from below by a constant independent of k. Therefore, the original sequence $P_k^t = V(Q_k^t)^{1/(p-n)} Q_k^t$ is also bounded.

By the Blaschke selection theorem (see, e.g., [35, Theorem 1.8.6]), we can select a subsequence of the P_k^t converging to a convex body K_f^t. After relabeling (if necessary) we may assume that $\lim_{k\to\infty} P_k^t = K_f^t$. From (3.1), (3.2), and relation (4.4), we obtain

$$V(K_f^t) = \lim_{k\to\infty} V(P_k^t) = \lim_{k\to\infty} \frac{1}{n} \int_{S^{n-1}} h(P_k^t, v)^p\, d\mu_k^t(v).$$

Thus, the uniform convergence of the support functions $h(P_k^t,\cdot)$, the weak convergence of the measures μ_k^t, and definition (4.2), yield

$$V(K_f^t)=\frac{1}{n}\int_{\partial[f]_t} h(K_f^t,\nabla f(x))^p|\nabla f(x)|^{-1}\,d\mathcal{H}^{n-1}(x).$$

Finally, we define $h(L_f^t,\cdot)^p=\mathrm{C}_p^+\mu^t$. By definition (4.2), we have

$$h(L_f^t,u)^p=\int_{\partial[f]_t}(u\cdot\nabla f(x))_+^p|\nabla f(x)|^{-1}\,d\mathcal{H}^{n-1}(x),\qquad u\in S^{n-1}. \tag{4.8}$$

From (4.6), we deduce that $h(L_f^t,\cdot)$ is the support function of a convex body $L_f^t\in\mathcal{K}_{\mathrm{o}}^n$ and that $\lim_{k\to\infty}\Pi_p^+P_k^t=L_f^t$. ■

5. Proof of the Main Result

After these preparations, we are now in a position to proof our main result. We want to point out that the approach we use to establish Theorem 1 is based on ideas and techniques of Lutwak, Yang, and Zhang [27].

We will need the *decreasing rearrangement* $\bar{f}$ of a function $f:\mathbb{R}^n\to\mathbb{R}$. It is defined by

$$\bar{f}(x)=\inf\{t>0:\ V([f]_t)<\kappa_n|x|^n\}.$$

Note that the level set $[\bar{f}]_t$ is a dilate of the unit ball B_n and its volume is equal to $V([f]_t)$. Moreover, for all $p\geq 1$,

$$\|f\|_p=\|\bar{f}\|_p. \tag{5.1}$$

We will first reduce the proof of Theorem 1 to the class of smooth functions with compact support.

Lemma 2. *In order to prove Theorem 1, it is sufficient to verify the following assertion: If $f\in C^\infty(\mathbb{R}^n)$ has compact support and $1\leq p<n$, then*

$$\left(\int_{S^{n-1}}\|\mathrm{D}_u^+f\|_p^{-n}\,du\right)^{-1/n}\geq c_{n,p}\,\|f\|_{p^*}. \tag{5.2}$$

Proof: Assume that (5.2) holds for smooth functions with compact support and let $f\in W^{1,p}$. We may assume that the set $\{x\in\mathbb{R}^n: f(x)\neq 0\}$ has positive measure. First, we will show that $\|\mathrm{D}_u^+f\|_p>0$ for every $u\in S^{n-1}$.

We may assume that $u=e_n$ is the last canonical basis vector. We denote the indicator function of a set $A\subseteq\mathbb{R}^n$ by $\mathbb{I}_A$. Since for each $N\in\mathbb{N}$, almost all points in $\mathbb{R}^n$ are Lebesgue points of $f\cdot\mathbb{I}_{[-N,N]^n}$ (see, e.g., [34, Theorem 7.7]), there exists an n-box $P=[a_1,b_1]\times\cdots\times[a_n,b_n]$ such that $\int_P f\neq 0$.

If $\int_P f>0$, then, since $f\in L_p(\mathbb{R}^n)$, there exist real $a<b<c$ such that

$$\int_{P'}\int_a^b f<\int_{P'}\int_b^c f,$$

where P' denotes the $(n-1)$-box $[a_1,b_1]\times\cdots\times[a_{n-1},b_{n-1}]$. Let $0<\varepsilon<1$ and let $g_i:\mathbb{R}\to[0,1]$, $i=1,\ldots,n-1$, be smooth functions with $g_i=1$ on $[a_i,b_i]$ and $g_i=0$ on $(a_i-\varepsilon,b_i+\varepsilon)^c$. Furthermore, define $g_n:\mathbb{R}\to\mathbb{R}$, by $g_n(x)=\int_{-\infty}^x h_n(x)\,dx$, where h_n is a smooth function which is equal to ε on $[a+\varepsilon,b-\varepsilon]$, $-\varepsilon$ on $[b+\varepsilon,c-\varepsilon]$, and zero on $[a,c]^c$.

If we set $\phi(x) = g_1(x_1) \cdot \ldots \cdot g_n(x_n)$, then ϕ is a non-negative, smooth, and compactly supported function such that $\int_{\mathbb{R}^n} f \partial_n \phi < 0$ for sufficiently small ε. Here, $\partial_n \phi$ denotes the n-th partial derivative of ϕ.

If $\int_P f < 0$, then the above argument applied to $x \mapsto -f(-x)$ yields a non-positive, smooth, and compactly supported function ϕ with $\int_{\mathbb{R}^n} f \partial_n \phi > 0$.

Now suppose that $\|\mathrm{D}_{e_n}^+ f\|_p = 0$. This implies, by the definition of weak derivatives, that $\int_{\mathbb{R}^n} f \partial_n \phi \geq 0$ (≤ 0) for every smooth and compactly supported ϕ which is non-negative (non-positive). This is a contradiction to the above construction. Thus $\|\mathrm{D}_u^+ f\|_p > 0$ for every $u \in S^{n-1}$.

Since $f \in W^{1,p}$, we can find a sequence $f_k, k \in \mathbb{N}$, of smooth functions with compact support such that

$$\|f_k - f\|_p \to 0 \quad \text{and} \quad \|\partial_i f_k - \partial_i f\|_p \to 0$$

for $i = 1, \ldots, n$. By Minkowski's inequality we have

$$c_{n,p}\|f_l - f_m\|_{p^*} \leq \left(\int_{S^{n-1}} \|\mathrm{D}_u^+(f_l - f_m)\|_p^{-n} \, du \right)^{-1/n} \leq \frac{1}{\omega_n^{1/n}} \sum_{i=1}^n \|\partial_i f_l - \partial_i f_m\|_p$$

for all $l, m \in \mathbb{N}$, where ω_n denotes the surface area of the Euclidean unit ball in $\mathbb{R}^n$. Consequently, the sequence $f_k, k \in \mathbb{N}$, is a Cauchy sequence in $L_{p^*}(\mathbb{R}^n)$.

By the completeness of $L_{p^*}(\mathbb{R}^n)$, there exists a function g such that $\|f_k - g\|_{p^*} \to 0$. Since sequences of functions converging in L_q, $q > 0$, posess a subsequence converging almost everywhere, we can find $f_{k_j}, j \in \mathbb{N}$, such that $f_{k_j} \to f$ and $f_{k_j} \to g$ almost everywhere. We conclude that $f = g$ almost everywhere and hence $f_k \to f$ also in $L_{p^*}(\mathbb{R}^n)$.

By the first part of the proof, $\lim_{k\to\infty} \|\mathrm{D}_u^+ f_k\|_p^{-n} = \|\mathrm{D}_u^+ f\|_p^{-n}$ for every unit vector $u \in S^{n-1}$. Thus an application of Fatou's Lemma yields

$$\begin{aligned}
\int_{S^{n-1}} \|\mathrm{D}_u^+ f\|_p^{-n} \, du &= \int_{S^{n-1}} \lim_{k\to\infty} \|\mathrm{D}_u^+ f_k\|_p^{-n} \, du \\
&\leq \liminf_{k\to\infty} \int_{S^{n-1}} \|\mathrm{D}_u^+ f_k\|_p^{-n} \, du \\
&\leq \lim_{k\to\infty} c_{n,p}^{-n} \|f_k\|_{p^*}^{-n} = c_{n,p}^{-n} \|f\|_{p^*}^{-n}. \qquad \blacksquare
\end{aligned}$$

Proof of Theorem 1. In the following let $p > 1$. By Lemma 2 we may assume that f is a smooth function with compact support which is not identically zero. An application of the co-area formula (2.4) shows that

$$\|\mathrm{D}_u^+ f\|_p^p = \int_{\mathbb{R}^n} (u \cdot \nabla f(x))_+^p \, dx = \int_0^{\|f\|_\infty} \int_{\partial[f]_t} \frac{(u \cdot \nabla f(x))_+^p}{|\nabla f(x)|} \, d\mathcal{H}^{n-1}(x) \, dt.$$

By Lemma 1 and (4.8), there exists a convex body $L_f^t \in \mathcal{K}_o^n$ such that

$$\left(\int_{S^{n-1}} \|\mathrm{D}_u^+ f\|_p^{-n} \, du \right)^{-p/n} = \left(\int_{S^{n-1}} \left(\int_0^{\|f\|_\infty} h(L_f^t, u)^p \, dt \right)^{-n/p} du \right)^{-p/n}.$$

Since $h(L_f^t, \cdot)$ is positive, we can apply a consequence of Minkowski's integral inequality (see, e.g., [14, p. 148]), to obtain

$$\left(\int_{S^{n-1}} \|\mathrm{D}_u^+ f\|_p^{-n} \, du \right)^{-p/n} \geq \int_0^{\|f\|_\infty} \left(\int_{S^{n-1}} h(L_f^t, u)^{-n} \, du \right)^{-p/n} dt.$$

Using (2.1) and the polar coordinate formula for volume, we deduce

$$\left(\int_{S^{n-1}} \|\mathrm{D}_u^+ f\|_p^{-n}\, du\right)^{-p/n} \geq \int_0^{\|f\|_\infty} \left(nV(L_f^{t,*})\right)^{-p/n} dt. \tag{5.3}$$

By Lemma 1, there exists a sequence of convex polytopes $P_k^t \in \mathcal{K}_\mathrm{o}^n$ such that $\lim_{k\to\infty} P_k^t = K_f^t \in \mathcal{K}^n$ and $\lim_{k\to\infty} \Pi_p^+ P_k^t = L_f^t$. Thus, from an application of Theorem 2, we obtain

$$(nV(L_f^{t,*}))^{-p/n} = \lim_{k\to\infty} (nV(\Pi_p^{+,*} P_k^t))^{-p/n} \geq e_{n,p} V(K_f^t)^{(n-p)/n}, \tag{5.4}$$

where

$$e_{n,p} = \frac{\pi^{(n-1)/2}\Gamma\left(\frac{1+p}{2}\right)}{n^{p/n}\kappa_n\Gamma\left(\frac{n+p}{2}\right)}.$$

From (5.3) and (5.4), we deduce

$$\left(\int_{S^{n-1}} \|\mathrm{D}_u^+ f\|_p^{-n}\, du\right)^{-p/n} \geq e_{n,p} \int_0^{\|f\|_\infty} V(K_f^t)^{(n-p)/n}\, dt. \tag{5.5}$$

An application of Hölder's integral inequality to volume formula (4.1), yields

$$V(K_f^t)^{(n-p)/np} \geq n^{1-1/p} \left(\int_{\partial[f]_t} \frac{d\mathcal{H}^{n-1}(x)}{|\nabla f(x)|}\right)^{(1-p)/p} V(K_f^t)^{-1/n} V_1([f]_t, K_f^t),$$

where we have used integral representation (2.2). From inequality (2.3), we deduce further that

$$V(K_f^t)^{(n-p)/n} \geq n^{p-1} \left(\int_{\partial[f]_t} \frac{d\mathcal{H}^{n-1}(x)}{|\nabla f(x)|}\right)^{1-p} V([f]_t)^{(n-1)p/n}. \tag{5.6}$$

Another application of the co-area formula (2.4), yields

$$\int_t^{\|f\|_\infty} \int_{\partial[f]_s} \frac{d\mathcal{H}^{n-1}(x)}{|\nabla f(x)|}\, ds = V([f]_t \cap \{|\nabla f| > 0\}).$$

Using Sard's theorem, it is not hard to show that for almost every t satisfying $0 < t < \|f\|_\infty$, there exists a neighborhood U_t of t such that

$$V(f^{-1}(U_t) \cap \{|\nabla f| > 0\}) = V(f^{-1}(U_t)).$$

Therefore, we obtain for almost every t with $0 < t < \|f\|_\infty$,

$$\int_{\partial[f]_t} \frac{d\mathcal{H}^{n-1}(x)}{|\nabla f(x)|} = -V([f]_t)'. \tag{5.7}$$

Combining (5.5), (5.6), and (5.7), we obtain

$$\left(\int_{S^{n-1}} \|\mathrm{D}_u^+ f\|_p^{-n}\, du\right)^{-p/n} \geq \frac{e_{n,p}}{n^{1-p}} \int_0^{\|f\|_\infty} \frac{V([f]_t)^{(n-1)p/n}}{(-V([f]_t)')^{p-1}}\, dt. \tag{5.8}$$

In order to estimate the right integral in (5.8), define $\widehat{f} : (0,\infty) \to \mathbb{R}$, by

$$\bar{f}(x) = \widehat{f}(1/|x|).$$

Since the decreasing rearrangement $\bar{f}(x)$ depends only on the Euclidean norm of x, the function $\widehat{f}$ is well defined and increasing. Since $\widehat{f}$ is locally Lipschitz, a substitution yields

$$\int_0^{\|f\|_\infty} \frac{V([f]_t)^{(n-1)p/n}}{(-V([f]_t)')^{p-1}}\, dt = n^{1-p}\kappa_n^{1-p/n} \int_0^\infty \widehat{f}\,'(s)^p s^{2p-n-1}\, ds.$$

Hence, we can rewrite (5.8) as

$$\left(\int_{S^{n-1}} \|\mathrm{D}_u^+ f\|_p^{-n}\, du\right)^{-p/n} \geq e_{n,p}\kappa_n^{1-p/n} \int_0^\infty \widehat{f}\,'(s)^p s^{2p-n-1}\, ds. \tag{5.9}$$

Using polar coordinates and (5.1), we see that

$$\|\bar{f}\|_{p^*}^{p^*} = n\kappa_n \int_0^\infty \widehat{f}(s)^{p^*} s^{-n-1}\, ds = \|f\|_{p^*}^{p^*}.$$

The substitution $t = 1/s$ and an application of inequality (2.5), therefore yields

$$\left(\int_0^\infty \widehat{f}\,'(s)^p s^{2p-n-1}\, ds\right)^{1/p} \geq \frac{b_{n,p}}{n^{1/p^*}\kappa_n^{1/p^*}} \|f\|_{p^*}. \tag{5.10}$$

Finally, combine inequalities (5.9) and (5.10), to obtain the desired result

$$\left(\int_{S^{n-1}} \|\mathrm{D}_u^+ f\|_p^{-n}\, du\right)^{-1/n} \geq c_{n,p}\, \|f\|_{p^*}. \tag{5.11}$$

In order to see that inequality (1.3) is sharp, take for smooth $K \in \mathcal{K}_\mathrm{o}^n$,

$$f(x) = \left(1 + \rho(K,x)^{p/(1-p)}\right)^{1-n/p}. \tag{5.12}$$

Then, a straightforward (but tedious) calculation shows that inequality (1.3) reduces to the L_p affine isoperimetric inequality (3.8), where equality holds if K is an ellipsoid centered at the origin.

Clearly, the case $p = 1$ of inequality (1.3) can be obtained from a limit of inequality (5.11) as $p \to 1$:

$$\left(\frac{1}{n}\int_{S^{n-1}} \|\mathrm{D}_u^+ f\|_1^{-n}\, du\right)^{-1/n} \geq \frac{\kappa_{n-1}}{\kappa_n}\, \|f\|_{1^*}. \tag{5.13}$$

Noting that $\Pi_1^+ = \Pi$, one can show (cf. [38]) that for characteristic functions of convex bodies, inequality (5.13) reduces to the Petty projection inequality, where equality is attained for ellipsoids. ■

We remark that for $p > 1$ the affine L_p Sobolev inequality (1.2) of Lutwak, Yang, and Zhang reduces to the L_p Petty projection inequality (3.7) if we take f as in (5.12). Thus, it follows from (3.9) that the new inequality (1.3) is in general stronger than (1.2). We will make this fact even more explicit in the next section.

6. A Stronger Inequality

In this last section we show that Theorem 1 provides a stronger result than the affine L_p Sobolev inequality (1.2) of Zhang and Lutwak, Yang, and Zhang. The basic concept behind this observation is a convex body associated with a given function f.

For $p \geq 1$ and $f \in W^{1,p}(\mathbb{R}^n)$, let $B_p^+(f)$ be the convex body defined by

$$h(B_p^+(f), u) = \left(\int_{\mathbb{R}^n} (\mathrm{D}_u^+ f(x))^p \, dx \right)^{1/p} = \left(\int_{\mathbb{R}^n} (u \cdot \nabla f(x))_+^p \, dx \right)^{1/p}.$$

From Minkowski's integral inequality, we deduce that $h(B_p^+(f), \cdot)$ is sublinear and therefore the support function of a unique convex body $B_p^+(f)$. Moreover, by Lemma 2, this body contains the origin in its interior. By (2.1) and the polar coordinate formula for volume, the volume of its polar body is given by

$$V(B_p^{+,*}(f)) = \frac{1}{n} \int_{S^{n-1}} \|\mathrm{D}_u^+ f\|_p^{-n} \, du.$$

Therefore, we can rewrite our main theorem as follows:

Theorem 1′ *If $f \in W^{1,p}(\mathbb{R}^n)$, with $1 \leq p < n$, then*

$$V(B_p^{+,*}(f))^{-1/n} \geq k_{n,p} \, \|f\|_{p^*}.$$

The optimal constant $k_{n,p}$ is given by

$$k_{n,p} = 2^{-1/p} \left(\tfrac{n-p}{p-1}\right)^{1-1/p} \left(\tfrac{\Gamma(\frac{n}{p})\Gamma(n+1-\frac{n}{p})}{\Gamma(n)} \right)^{1/n} \left(\tfrac{n\Gamma(\frac{n}{2})\Gamma(\frac{p+1}{2})}{\sqrt{\pi}\Gamma(\frac{n+p}{2})} \right)^{1/p}.$$

From the definition of L_p Minkowski addition, it follows that

$$h(B_p^+(f) +_p B_p^+(-f), u) = \left(\int_{\mathbb{R}^n} |\mathrm{D}_u f(x)|^p \, dx \right)^{1/p}. \tag{6.1}$$

Thus, the following reformulation of inequality (1.4) shows that Theorem 1 is stronger than inequality (1.2):

Theorem 4. *If $p \geq 1$ and $f \in W^{1,p}(\mathbb{R}^n)$, then*

$$V((B_p^+(f) +_p B_p^+(-f))^*) \leq 2^{-n/p} V(B_p^{+,*}(f)),$$

with equality if and only if $B_p^+(f)$ is origin symmetric.

In order to prove this theorem, we need a result from the dual L_p Brunn–Minkowski theory. The basis of this theory is the following addition on convex bodies. For $\alpha, \beta \geq 0$ (not both zero), Firey's L_p *harmonic radial combination* $\alpha \cdot K \,\widetilde{+}_p\, \beta \cdot L$ of $K, L \in \mathcal{K}_{\mathrm{o}}^n$ is the convex body defined by

$$\rho(\alpha \cdot K \,\widetilde{+}_p\, \beta \cdot L, \cdot)^{-p} = \alpha \rho(K, \cdot)^{-p} + \beta \rho(L, \cdot)^{-p}.$$

Firey started investigations of harmonic L_p combinations in the 1960's which were continued by Lutwak leading to a dual L_p Brunn–Minkowski theory. A cornerstone of this theory is the dual L_p Brunn–Minkowski inequality [23]: If $K, L \in \mathcal{K}_{\mathrm{o}}^n$, then

$$V(K \,\widetilde{+}_p\, L)^{-p/n} \geq V(K)^{-p/n} + V(L)^{-p/n}, \tag{6.2}$$

with equality if and only if K and L are dilates.

Proof of Theorem 4 : From (2.1), (6.1) and the definition of L_p harmonic radial addition, it follows that

$$(B_p^+(f) +_p B_p^+(-f))^* = B_p^{+,*}(f) \,\widetilde{+}_p\, B_p^{+,*}(-f).$$

Since $V(B_p^{+,*}(f)) = V(B_p^{+,*}(-f))$, an application of (6.2) yields the desired inequality along with its equality conditions. ■

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