Finite Automata

Zoltán Ésik[1] * and Werner Kuich[2]

[1] Dept. of Computer Science, University of Szeged, Hungary, and GRLMC, Rovira i Virgili University, Tarragona, Spain
[2] Technische Universität Wien, Wien, Austria

1 Introduction

The purpose of this chapter is to develop the theory of finite automata starting from ideas based on linear algebra over semirings. Many results in the theory of automata and languages depend only on a few equational axioms. For example, Conway [4] has shown that Kleene's fundamental theorem equating the recognizable languages with the regular ones follows from a few simple identities defining Conway semirings. Such semirings are equipped with a star operation subject to the sum star identity and product star identity.

The use of equations has several advantages. Proofs can be separated into two parts, where the first part establishes the equational axioms, and the second is based on simple equational reasoning. Such proofs have a transparent structure and are usually very easy to understand, since manipulating equations is one of the most common ways of mathematical reasoning. Moreover, since many results depend on the same equations, the first part of such proofs usually provides a basis to several results. Finally, the results obtained by equational reasoning have a much broader scope, since many models share the same equations.

This chapter consists of this and two more sections. Section 2 constitutes a generalization of the theory of regular languages over finite words. In this section we define finite automata over power series semirings and over Conway semirings, and prove theorems of the Kleene-Schützenberger type. Moreover, we introduce finite linear systems and show the coincidence of the set of components of the solutions of such finite linear systems with the set of behaviors of finite automata.

Section 3 constitutes a generalization of the Büchi theory on languages over infinite words. In this section we first define the algebraic structures needed for

* Partially supported by grant no. MTM 43622 from the Ministry of Education and Science of Spain.

this generalization: semiring-semimodule pairs and quemirings. Then we define finite automata over quemirings and prove theorems of the Kleene-Büchi type. Moreover, we consider linear systems over quemirings as a generalization of regular grammars with finite and infinite derivations and show the coincidence of the set of components of the solutions of such linear systems with the set of behaviors of finite automata over quemirings.

The presentation of this chapter is influenced by the ideas presented in Bloom, Ésik [2] and Ésik, Kuich [10, 11].

2 Finite automata over semirings

In this section, we deal with finite automata over semirings. Here semirings constitute a generalization of formal languages with finite words. The main results of this section are various generalizations of the Theorem of Kleene-Schützenberger [25].

This section consists of three subsections. In Subsection 1, we define finite automata over a semiring S and an alphabet Σ whose behavior is a formal power series in $S\langle\langle \Sigma^* \rangle\rangle$. The main result of this subsection is the Theorem of Kleene-Schützenberger (Theorem 2.4): A power series of $S\langle\langle \Sigma^* \rangle\rangle$ is rational iff it is the behavior of a cycle-free finite automaton over the semiring S and the alphabet Σ.

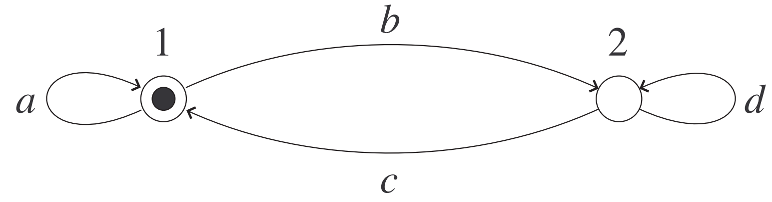
In Subsection 2 we introduce finite S'-automata over a Conway semiring S, where S' is a subset of S. The main result of this subsection is the following generalization of the Theorem of Kleene-Schützenberger (Theorem 2.11): The star semiring generated by S' equals the set of behaviors of finite S'-automata over semirings. The results in Subsections 1 and 2 are similar; but in Subsection 1 we consider cycle-free finite automata over an arbitrary semiring while in Subsection 2 we consider arbitrary finite automata over a Conway semiring.

In Subsection 3 we consider finite linear systems as a generalization of right linear context-free grammars. The main result of this subsection is that a semiring element is a component of the solution of a finite linear system iff it is the behavior of a finite automaton (Theorems 2.17, 2.22).

We now give a typical example which will be helpful for readers with some background in semiring theory (as given in Droste, Kuich [5]) and automata theory. Readers without this background should consult it when finite automata are defined in Subsection 2.2.

Example 2.1. Let $\mathcal{A} = (Q, \Sigma, \delta, 1, \{1\})$ be a finite automaton (a definition is given at the end of Subsection 2.2), where $Q = \{1, 2\}$, $\Sigma = \{a, b, c, d\}$, and $\delta(1, a) = \{1\}$, $\delta(1, b) = \{2\}$, $\delta(2, c) = \{1\}$, $\delta(2, d) = \{2\}$ are the only nonempty images of δ.

The graph of $\mathcal{A}$ is



and the adjacency matrix of this graph is

$$A = \begin{pmatrix} \{a\} & \{b\} \\ \{c\} & \{d\} \end{pmatrix} .$$

(Whenever we use matrices in this example, they are 2×2-matrices and their entries are formal languages over Σ, i. e., elements of the semiring 2^{Σ^*}.)

Consider the powers of A, e. g.,

$$A^2 = \begin{pmatrix} \{aa, bc\} & \{ab, bd\} \\ \{ca, dc\} & \{cb, dd\} \end{pmatrix} , \; A^3 = \begin{pmatrix} \{aaa, abc, bca, bdc\} & \{aab, abd, bcb, bdd\} \\ \{caa, cbc, dca, ddc\} & \{cab, cbd, dcb, ddd\} \end{pmatrix} .$$

It is easily proved by induction on k that $(A^k)_{ij}$ is the language of inscriptions of the paths of length k from state i to state j, $k \geq 0$, $1 \leq i, j \leq 2$. Define A^* by $(A^*)_{ij} = \bigcup_{k \geq 0} (A^k)_{ij}$, $1 \leq i, j \leq 2$. Then $(A^*)_{ij}$ is the language of inscriptions of all the paths from state i to state j, $1 \leq i, j \leq 2$.

We now construct regular expressions for the entries of A^*. Consider the inscriptions of paths from 1 to 1 not passing 1: they are a and bd^nc. Hence, the language of inscriptions of these paths is $\{a\} \cup \{b\}\{d\}^*\{c\}$. Consider now the language of inscriptions of paths from 1 to 1: it is $(\{a\} \cup \{b\}\{d\}^*\{c\})^*$. Hence, $(A^*)_{11} = (\{a\} \cup \{b\}\{d\}^*\{c\})^*$. We obtain $(A^*)_{12}$, if we concatenate $(A^*)_{11}$ with the language of inscriptions of all paths from 1 to 2 not passing through 1: $\{b\}\{d\}^*$. Hence, $(A^*)_{12} = (A^*)_{11}\{b\}\{d\}^*$. By symmetry, we obtain $(A^*)_{22} = (\{d\} \cup \{c\}\{a\}^*\{b\})^*$ and $(A^*)_{21} = (A^*)_{22}\{c\}\{a\}^*$. Hence,

$$A^* = \begin{pmatrix} (\{a\} \cup \{b\}\{d\}^*\{c\})^* & (\{a\} \cup \{b\}\{d\}^*\{c\})^*\{b\}\{d\}^* \\ (\{d\} \cup \{c\}\{a\}^*\{b\})^*\{c\}\{a\}^* & (\{d\} \cup \{c\}\{a\}^*\{b\})^* \end{pmatrix} .$$

The language $||\mathcal{A}||$ accepted by the finite automaton $\mathcal{A}$ is the language of inscriptions of all paths from the initial state 1 to the final state 1, i. e., $||\mathcal{A}|| = (A^*)_{11} = (\{a\} \cup \{b\}\{d\}^*\{c\})^*$. □

2.1 Finite automata over arbitrary power series semirings

In this subsection, S denotes an arbitrary semiring and Σ a finite alphabet.

A *finite automaton*

$$\mathfrak{A} = (Q, R, A, P)$$

(*over the semiring S and the alphabet Σ*) is given by

(i) a finite nonempty *set of states* Q,
(ii) a *transition matrix* $A \in (S\langle \Sigma \cup \{\varepsilon\}\rangle)^{Q \times Q}$,
(iii) an *initial state vector* $R \in (S\langle \Sigma \cup \{\varepsilon\}\rangle)^{1 \times Q}$,
(iv) a *final state vector* $P \in (S\langle \Sigma \cup \{\varepsilon\}\rangle)^{Q \times 1}$.

The finite automaton $\mathfrak{A}$ is called *cycle-free* if the transition matrix A is cycle-free, cf. Droste, Kuich [5]. The *behavior* $||\mathfrak{A}||$ of the cycle-free finite automaton $\mathfrak{A} = (Q, R, A, P)$ is defined by

$$||\mathfrak{A}|| = \sum_{q_1, q_2 \in Q} R_{q_1} (A^*)_{q_1, q_2} P_{q_2} = RA^*P \,.$$

Since $\mathfrak{A}$ is cycle-free, A^* and $||\mathfrak{A}||$ are well defined.

The *(directed) graph* of the cycle-free finite automaton $\mathfrak{A} = (Q, R, A, P)$, where $Q = \{q_1, \ldots, q_n\}$, is constructed in the usual manner. It has nodes $q_1, \ldots, q_n$ and an edge from node q_i to node q_j if $A_{q_i, q_j} \neq 0$. The *weight* of this edge is $A_{q_i, q_j} \in S\langle \Sigma \cup \{\varepsilon\}\rangle$. The *initial* (resp. *final*) *weight* of a node q_i is given by R_i (resp. P_i). A node is called *initial* (resp. *final*) if its initial (resp. final) weight is unequal to 0. The *weight* of a path is the product of the weigths of its edges. It is easily shown that $(A^k)_{q_i, q_j}$ is the sum of the weights of paths of length k from node q_i to node q_j. Since $(A^*)_{q_i, q_j} = \sum_{k \geq 0} (A^k)_{q_i, q_j}$, $(A^*)_{q_i, q_j}$ is the sum of the weights of the paths from node q_i to node q_j. Hence, $R_{q_1} (A^*)_{q_i, q_j} P_{q_2}$ is this sum for nodes q_1 and q_2, multiplied on the left and right by the initial weight of node q_1 and the final weight of node q_2, respectively. Eventually, the behavior of $\mathfrak{A}$ is the sum of all these terms with summation over all initial states q_1 and all final states q_2.

Theorem 2.1 *If $\mathfrak{A}$ is a cycle-free finite automaton then $||\mathfrak{A}||$ is the sum of the weights of all paths from an initial state to a final state multiplied by the initial and final weights of these states.*

Two cycle-free finite automata $\mathfrak{A}$ and $\mathfrak{A}'$ are *equivalent* if $||\mathfrak{A}|| = ||\mathfrak{A}'||$. A finite automaton $\mathfrak{A} = (Q, R, A, P)$ is called *normalized* if $|Q| \geq 2$ and

(i) $R_{q_1} = \varepsilon$ for some $q_1 \in Q$; $R_q = 0$, for all $q \in Q$, $q \neq q_1$;
(ii) $P_{q_n} = \varepsilon$ for some $q_n \in Q$, $q_n \neq q_1$; $P_q = 0$, for all $q \in Q$, $q \neq q_n$;
(iii) $A_{q, q_1} = A_{q_n, q} = 0$ for all $q \in Q$.

Hence, the directed graph of a normalized finite automaton has the unique initial node q_1 and the unique final node q_n, both with weight ε; moreover, no edges are leading to the initial node and no edges are leaving the final node.

Theorem 2.2 *Each cycle-free finite automaton is equivalent to a normalized cycle-free finite automaton.*

Proof. Let $\mathfrak{A} = (Q, R, A, P)$ be a cycle-free finite automaton. Define the finite automaton $\mathfrak{A}'$ by

$$\mathfrak{A}' = (\{q_0\} \cup Q \cup \{q_f\}, (\varepsilon\ 0\ 0), \begin{pmatrix} 0 & R & 0 \\ 0 & A & P \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \varepsilon \end{pmatrix}) .$$

Here q_0 and q_f are new states. Then $\mathfrak{A}'$ is normalized. Moreover, Corollary 5.8 of Droste, Kuich [5] implies that $\mathfrak{A}'$ is cycle-free and $||\mathfrak{A}'|| = ||\mathfrak{A}||$. □

Remark. Usually, a finite automaton $\mathfrak{A} = (Q, R, A, P)$ is defined as above with the exception that the entries of R and P are in $S\langle\{\varepsilon\}\rangle$. We have choosen the more general definitions for two reasons:

(i) This definition is compatible with the definitions of finite S'-automata given in Subsections 2.2 and 3.2.
(ii) This definition is in correspondence with the definition of finite linear systems generalizing right linear grammars. (See Theorems 2.17, 3.17, 3.18 and Corollaries 2.19, 3.21.)

Moreover, by Theorem 2.2, our finite automata are exactly as powerful as the usual finite automata. □

The three operations sum, product and star are customarily referred to as *rational* operations. A power series r belonging to $S\langle\langle \Sigma^* \rangle\rangle$ is termed *rational* (over S and Σ) if r can be obtained from the polynomials of $S\langle \Sigma^* \rangle$ by finitely many applications of the rational operations, where the star is applied only to *proper* power series. The formula telling how a given rational series in $S\langle\langle \Sigma^* \rangle\rangle$ is obtained from the polynomials of $S\langle \Sigma^* \rangle$ by rational operations is referred to as *regular expression* (see Salomaa [23], Kuich, Salomaa [20]). The family of rational power series (over S and Σ) is denoted by $S^{\mathrm{rat}}\langle\langle \Sigma^* \rangle\rangle$.

Observe that $S^{\mathrm{rat}}\langle\langle \Sigma^* \rangle\rangle$ can equivalently be defined as follows: a power series r is in $S^{\mathrm{rat}}\langle\langle \Sigma^* \rangle\rangle$ if r can be obtained from the polynomials of $S\langle \Sigma^* \rangle$ by finitely many applications of the rational operations, where the star is applied only to *cycle-free* power series, see Droste, Kuich [5]. Let $r = r_0 + r_1$ be cycle-free, where $r_0 = (r, \varepsilon)\varepsilon$ and $r_1 = \sum_{w \in \Sigma^+} (r, w)w$. Then by the sum star identity (Corollary 5.4 of Droste, Kuich [5]) we obtain $r^* = (r_0^* r_1)^* r_0^*$. Hence, $r^* \in S^{\mathrm{rat}}\langle\langle \Sigma^* \rangle\rangle$ according to the original definition.

The collection of all behaviors of cycle-free finite automata (over S and Σ) is denoted by $S^{\mathrm{rec}}\langle\langle \Sigma^* \rangle\rangle$. The classical Theorem of Kleene essentially states that $\mathbb{B}^{\mathrm{rat}}\langle\langle \Sigma^* \rangle\rangle$ and $\mathbb{B}^{\mathrm{rec}}\langle\langle \Sigma^* \rangle\rangle$ coincide.

As a generalization of this Theorem of Kleene we now prove a variant of the Theorem of Kleene-Schützenberger. (See Schützenberger [25], and Conway [4], Eilenberg [6], Salomaa, Soittola [24], Kuich, Salomaa [20], Kuich [19], Berstel, Reutenauer [1], Ésik, Kuich [9]).

Before proving this generalization, we show that each cycle-free finite automaton is equivalent to one with a proper transition matrix. In the following proof and then without mention we use the isomorphism between $(S\langle\langle \Sigma^* \rangle\rangle)^{Q \times Q}$ and $S^{Q \times Q}\langle\langle \Sigma^* \rangle\rangle$ to simplify our notation (see Droste, Kuich [5], before Lemma 4.1).

Theorem 2.3 *Each cycle-free finite automaton is equivalent to a cycle-free finite automaton* $\mathfrak{A}' = (Q', R', A', P')$*, where* $A' \in (S\langle\Sigma\rangle)^{Q\times Q}$*,* $P' \in (S\langle\{\varepsilon\}\rangle)^{Q\times 1}$*, and there exists a* $q_0 \in Q'$ *such that* $R'_{q_0} = \varepsilon$*,* $R'_q = 0$ *for all* $q \neq q_0$*.*

Proof. For each cycle-free finite automaton there exists, by Theorem 2.2, an equivalent normalized cycle-free finite automaton $\mathfrak{A} = (Q, R, A, P)$. Let $A_0 = (A, \varepsilon)\varepsilon$ and $A_1 = \sum_{x\in\Sigma}(A, x)x$, and define the finite automaton $\mathfrak{A}'$ by $Q' = Q$, $A' = A_0^* A_1$, $R' = R$, $P' = A_0^* P$. Then

$$||\mathfrak{A}'|| = R(A_0^* A_1)^* A_0^* P = R(A_0 + A_1)^* P = RA^* P = ||\mathfrak{A}|| \,.$$

Here we have applied in the second equality the sum star identity (Corollary 5.4 of Droste, Kuich [5]). □

We define, for finite automata $\mathfrak{A} = (Q, R, A, P)$ and $\mathfrak{A}' = (Q', R', A', P')$ with $Q \cap Q' = \emptyset$, the finite automata $\mathfrak{A} + \mathfrak{A}'$, $\mathfrak{A} \cdot \mathfrak{A}'$ and $\mathfrak{A}^+$:

$$\begin{aligned}
\mathfrak{A} + \mathfrak{A}' &= (Q \cup Q', (R\ R'), \begin{pmatrix} A & 0 \\ 0 & A' \end{pmatrix}, \begin{pmatrix} P \\ P' \end{pmatrix}) \,, \\
\mathfrak{A} \cdot \mathfrak{A}' &= (Q \cup Q', (R\ 0), \begin{pmatrix} A & PR' \\ 0 & A' \end{pmatrix}, \begin{pmatrix} 0 \\ P' \end{pmatrix}) \,, \\
\mathfrak{A}^+ &= (Q, R, A + PR, P) \,.
\end{aligned}$$

In the construction of $\mathfrak{A} \cdot \mathfrak{A}'$ (resp. $\mathfrak{A}^+$) we assume that the entries of PR' (resp. PR) are in $S\langle\Sigma \cup \{\varepsilon\}\rangle$ or else $\mathfrak{A}$ or $\mathfrak{A}'$ to be normalized.

Theorem 2.4 (Theorem of Kleene-Schützenberger) $S^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle = S^{\mathrm{rec}}\langle\langle\Sigma^*\rangle\rangle$*.*

Proof. (i) An easy proof by induction on $|Q|$ using Corollary 5.9 of Droste, Kuich [5] shows that $A^* \in (S^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle)^{Q\times Q}$ if $A \in (S\langle\Sigma\rangle)^{Q\times Q}$. This implies by Theorem 2.3 immediately that $S^{\mathrm{rec}}\langle\langle\Sigma^*\rangle\rangle \subseteq S^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle$.

(ii) Let $r, r' \in S^{\mathrm{rec}}\langle\langle\Sigma^*\rangle\rangle$ and assume that $r = ||\mathfrak{A}||$ and $r' = ||\mathfrak{A}'||$ for cycle-free finite automata $\mathfrak{A}$ and $\mathfrak{A}'$.

Application of Corollary 5.8 of Droste, Kuich [5] shows that $\mathfrak{A} + \mathfrak{A}'$ and $\mathfrak{A} \cdot \mathfrak{A}'$ are cycle-free and that the equations $||\mathfrak{A} + \mathfrak{A}'|| = ||\mathfrak{A}|| + ||\mathfrak{A}'||$ and $||\mathfrak{A} \cdot \mathfrak{A}'|| = ||\mathfrak{A}|| \cdot ||\mathfrak{A}'||$ are valid. Hence $r + r' = ||\mathfrak{A} + \mathfrak{A}'||$ and $r \cdot r' = ||\mathfrak{A} \cdot \mathfrak{A}'||$ are in $S^{\mathrm{rec}}\langle\langle\Sigma^*\rangle\rangle$.

Let now $r \in S^{\mathrm{rec}}\langle\langle\Sigma^*\rangle\rangle$ be a proper power series. By Theorem 2.3, there exists a finite automaton $\mathfrak{A} = (Q, R, A, P)$ with $A \in (S\langle\Sigma\rangle)^{Q\times Q}$, $R \in (S\langle\{\varepsilon\}\rangle)^{1\times Q}$ and $P \in (S\langle\{\varepsilon\}\rangle)^{Q\times 1}$ such that $r = ||\mathfrak{A}||$. Consider now the finite automaton $\mathfrak{A}^+$. Since $(r, \varepsilon) = (R, \varepsilon)(P, \varepsilon) = 0$, we obtain $((A + PR)^2, \varepsilon) = ((PR)^2, \varepsilon) = (P, \varepsilon)(R, \varepsilon)(P, \varepsilon)(R, \varepsilon) = (P, \varepsilon)(r, \varepsilon)(R, \varepsilon) = 0$ and $\mathfrak{A}^+$ is cycle-free.

We now compute the behavior of $\mathfrak{A}^+$ and obtain

$$||\mathfrak{A}^+|| = R(A + PR)^* P = R(A^* PR)^* A^* P = (RA^* P)(RA^* P)^* = ||\mathfrak{A}|| \cdot ||\mathfrak{A}||^* \,.$$

Here we have applied in the second equality Theorem 5.2 (the sum star identity) and in the third equality Theorem 3.2 of Droste, Kuich [5].

Easy constructions yield $a\varepsilon, x \in S^{\mathrm{rec}}\langle\langle \Sigma^* \rangle\rangle$, $a \in S$, $x \in \Sigma$. Hence $S\langle \Sigma \cup \{\varepsilon\}\rangle \subseteq S^{\mathrm{rec}}\langle\langle \Sigma^* \rangle\rangle$. Moreover, for a proper power series $r \in S^{\mathrm{rec}}\langle\langle \Sigma^* \rangle\rangle$, $r^* = \varepsilon + rr^*$ by Theorem 3.1 of Droste, Kuich [5]. Hence, $r^* \in S^{\mathrm{rec}}\langle\langle \Sigma^* \rangle\rangle$ and we have proved $S^{\mathrm{rat}}\langle\langle \Sigma^* \rangle\rangle \subseteq S^{\mathrm{rec}}\langle\langle \Sigma^* \rangle\rangle$. This implies our theorem. □

2.2 Finite automata over Conway semirings

Recall from Droste, Kuich [5] and Ésik [8] that a *Conway semiring* is a star semiring that satisfies the *sum star identity*

$$(a+b)^* = (a^*b)^*a^*$$

and the *product star identity*

$$(ab)^* = 1 + a(ba)^*b$$

for all semiring elements a, b. It then follows that the *star fixed point identity*

$$a^* = 1 + aa^*$$

and the *simplified product star identity*

$$a(ba)^* = (ab)^*a$$

hold for all semiring elements a, b.

Let S be a star semiring. Then, for $A \in S^{n\times n}$ we define $A^* \in S^{n\times n}$ inductively as follows:

(i) For $n = 1$ and $A = (a)$, $a \in S$, we define $A^* = (a^*)$.

(ii) For $n > 1$ we partition A into blocks $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and define $A^* = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ with $a, \alpha \in S^{1\times 1}$, $b, \beta \in S^{1\times(n-1)}$, $c, \gamma \in S^{(n-1)\times 1}$, $d, \delta \in S^{(n-1)\times(n-1)}$ by

$$\alpha = (a + bd^*c)^*, \quad \beta = \alpha b d^*, \quad \gamma = \delta c a^*, \quad \delta = (d + ca^*b)^* .$$

(See Theorem 3.3 of Conway [4], Theorem 4.21 of Kuich, Salomaa [20], Theorem 2.5 of Kuich [19], Theorem 5.7 of Droste, Kuich [5], Bloom, Ésik [2], Ésik [8], Section 6 and Example 2.1.) If $\langle S, +, \cdot, ^*, 0, 1\rangle$ is a star semiring then the *star operation* in the star semiring $\langle S^{n\times n}, +, \cdot, ^*, 0, E\rangle$ will always be defined as above.

Theorem 2.5 (Conway [4], Bloom, Ésik [2], Ésik, Kuich [12], Ésik [8]**)** *If S is a Conway semiring then, for $n \geq 1$, $S^{n\times n}$ again is a Conway semiring.*

Let A and A^* be given as in (ii) of the definition of A^* above, but with $a, \alpha \in S^{n_1\times n_1}$, $b, \beta \in S^{n_1\times n_2}$, $c, \gamma \in S^{n_2\times n_1}$, $d, \delta \in S^{n_2\times n_2}$, $n_1 + n_2 = n$. Then the *matrix star identity* is valid in the star semiring S if A^* is independent of the partition of n in summands.

Theorem 2.6 (Conway [4], Bloom, Ésik [2], Ésik, Kuich [12], Ésik [8]) *If S is a Conway semiring then the matrix star identity holds.*

Let S be a star semiring. Then for $r \in S\langle\langle \Sigma^* \rangle\rangle$, we define the star $r^* \in S\langle\langle \Sigma^* \rangle\rangle$ of r inductively as follows:

$$(r^*, \varepsilon) = (r, \varepsilon)^*, \quad (r^*, w) = (r, \varepsilon)^* \sum_{uv=w,\ u \neq \varepsilon} (r, u)(r^*, v),\ w \in \Sigma^*,\ w \neq \varepsilon\,.$$

(See Bloom, Ésik [2] and Theorem 3.5 of Kuich, Salomaa [20].) If $\langle S, +, \cdot, ^*, 0, 1\rangle$ is a star semiring then the *star operation* in the star semiring $\langle S\langle\langle \Sigma^* \rangle\rangle, +, \cdot, ^*, 0, \varepsilon\rangle$ will be always defined as above.

Theorem 2.7 (Bloom, Ésik [2], Ésik, Kuich [12]) *If S is a Conway semiring and Σ is an alphabet then $S\langle\langle \Sigma^* \rangle\rangle$ is again a Conway semiring.*

Corollary 2.8 *If S is a Conway semiring, Σ is an alphabet and $n \geq 1$, then $(S\langle\langle \Sigma^* \rangle\rangle)^{n\times n}$ is again a Conway semiring.*

Each complete semiring is a Conway semiring (Kuich [18], Hebisch [15], Bloom, Ésik [2], Droste, Kuich [5], Ésik [8]). Moreover, for a complete semiring S, the star operations in the complete semirings $S^{n\times n}$ and $S\langle\langle \Sigma^* \rangle\rangle$ are the same as the star operations in the Conway semirings $S^{n\times n}$ and $S\langle\langle \Sigma^* \rangle\rangle$, respectively.

For the rest of this subsection, S denotes a Conway semiring and S' denotes a subset of S. We now generalize the finite automata of Subsection 2.1 to finite S'-automata over a Conway semiring S.

A *finite S'-automaton*

$$\mathfrak{A} = (n, R, A, P), \quad n \geq 1\,,$$

(*over the Conway semiring S*) is given by

(i) a *transition matrix* $A \in (S' \cup \{0, 1\})^{n\times n}$,
(ii) an *initial state vector* $R \in (S' \cup \{0, 1\})^{1\times n}$,
(iii) a *final state vector* $P \in (S' \cup \{0, 1\})^{n\times 1}$.

The *behavior* $||\mathfrak{A}||$ of $\mathfrak{A}$ is defined by

$$||\mathfrak{A}|| = \sum_{1\leq i_1, i_2 \leq n} R_{i_1}(A^*)_{i_1,i_2} P_{i_2} = RA^*P\,.$$

The *(directed) graph of* $\mathfrak{A}$ is constructed analogous to that in Subsection 2.1. It has nodes $1, \ldots, n$ and an edge from node i to node j if $A_{ij} \neq 0$. The *weight* of this edge is $A_{ij} \in S' \cup \{1\}$. The *initial* (resp. *final*) *weight* of a node i is given by R_i (resp. P_i). A node is called *initial* (resp. *final*) if its initial (resp. final) weight is unequal to 0. The *weight* of a path is the product of the weigths of its edges. It is easily shown that $(A^k)_{ij}$ is the sum of the weights of paths of length k from node i to node j. If S is a complete semiring and, hence, $(A^*)_{ij} = \sum_{k\geq 0}(A^k)_{ij}$, then $(A^*)_{ij}$ is the sum of the weights of the paths from node i to node j. Hence, $S_{i_1}(A^*)_{i_1,i_2}P_{i_2}$ is this sum for nodes i_1 and i_2, multiplied on the left and right by the initial weight of node i_1 and the final weight of node i_2, respectively. Eventually, the behavior of $\mathfrak{A}$ is the sum of all these terms with summation over all initial states i_1 and all final states i_2.

Theorem 2.9 *Let S be a complete semiring and $S' \subseteq S$. If $\mathfrak{A}$ is a finite S'-automaton then $||\mathfrak{A}||$ is the sum of the weights of all paths from an initial state to a final state multiplied by the initial and final weights of these states.*

Two finite S'-automata $\mathfrak{A}$ and $\mathfrak{A}'$ are *equivalent* if $||\mathfrak{A}|| = ||\mathfrak{A}'||$. A finite S'-automaton $\mathfrak{A} = (n, R, A, P)$ is called *normalized* if $n \geq 2$ and

(i) $R_1 = 1$, $R_i = 0$, for all $2 \leq i \leq n$;
(ii) $P_n = 1$, $P_i = 0$, for all $1 \leq i \leq n-1$;
(iii) $A_{i,1} = A_{n,i} = 0$, for all $1 \leq i \leq n$.

Hence, the directed graph of a normalized finite S'-automaton has the unique initial node 1 and the unique final node n, both with weight 1; moreover, no edges are leading to the initial node and no edges are leaving the final node.

Theorem 2.10 *Let S be a Conway semiring and $S' \subseteq S$. Then each finite S'-automaton is equivalent to a normalized finite S'-automaton.*

Proof. Let $\mathfrak{A} = (n, R, A, P)$ be a finite S'-automaton. Define the finite S'-automaton $\mathfrak{A}'$ by

$$\mathfrak{A}' = (1+n+1, (1\ 0\ 0), \begin{pmatrix} 0 & R & 0 \\ 0 & A & P \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}) .$$

Then $\mathfrak{A}'$ is normalized. Applying the matrix star identity yields the proof that $||\mathfrak{A}'|| = ||\mathfrak{A}||$. □

The sub star semiring of S that is generated by S' is denoted by $\mathfrak{Rat}(S')$. The collection of all behaviors of finite S'-automata is denoted by $\mathfrak{Rec}(S')$. The classical Theorem of Kleene essentially states that $\mathfrak{Rat}(\mathbb{B}\langle\Sigma\rangle)$ and $\mathfrak{Rec}(\mathbb{B}\langle\Sigma\rangle)$ coincide. As a generalization of the Theorem of Kleene-Schützenberger we show that $\mathfrak{Rat}(S') = \mathfrak{Rec}(S')$. (See Conway [4], Bloom, Ésik [2], Kuich [18, 19], Ésik, Kuich [9]).

We now define, for given finite S'-automata $\mathfrak{A} = (n, R, A, P)$ and $\mathfrak{A}' = (n', R', A', P')$, the finite S-automata $\mathfrak{A} + \mathfrak{A}'$, $\mathfrak{A} \cdot \mathfrak{A}'$ and $\mathfrak{A}^*$:

$$\mathfrak{A} + \mathfrak{A}' = (n+n', (R\ R'), \begin{pmatrix} A & 0 \\ 0 & A' \end{pmatrix}, \begin{pmatrix} P \\ P' \end{pmatrix}) ,$$

$$\mathfrak{A} \cdot \mathfrak{A}' = (n+n', (R\ 0), \begin{pmatrix} A & PR' \\ 0 & A' \end{pmatrix}, \begin{pmatrix} 0 \\ P' \end{pmatrix}) ,$$

$$\mathfrak{A}^* = (1+n, (1\ 0), \begin{pmatrix} 0 & R \\ P & A \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}) .$$

Theorem 2.11 *Let S be a Conway semiring and $S' \subseteq S$. Then $\mathfrak{Rat}(S') = \mathfrak{Rec}(S')$.*

Proof. (i) An easy proof by induction on n using the matrix star identity shows that $A^* \in \mathfrak{Rat}(S')^{n\times n}$ if $A \in (S' \cup \{0,1\})^{n\times n}$. This implies immediately $\mathfrak{Rec}(S') \subseteq \mathfrak{Rat}(S')$.

(ii) Easy constructions yield $S' \cup \{0,1\} \subseteq \mathfrak{Rec}(S')$. Consider now a and a' in $\mathfrak{Rec}(S')$. Then there exist finite S'-automata $\mathfrak{A} = (n, R, A, P)$ and $\mathfrak{A}' = (n', R', A', P')$ such that $||\mathfrak{A}|| = a$ and $||\mathfrak{A}'|| = a'$. Clearly, $\mathfrak{A} + \mathfrak{A}'$ and $\mathfrak{A}^*$ are finite S'-automata. If PR' is in $S' \cup \{0,1\}$ then also $\mathfrak{A} \cdot \mathfrak{A}'$ is a finite S'-automaton. If PR' is not in $S' \cup \{0,1\}$, choose $\mathfrak{A}$ or $\mathfrak{A}'$ to be normalized. This is, by Theorem 2.10, no loss of generality. Then again $\mathfrak{A} \cdot \mathfrak{A}'$ is a finite S'-automaton. Application of the matrix star identity shows that the equations $||\mathfrak{A} + \mathfrak{A}'|| = ||\mathfrak{A}|| + ||\mathfrak{A}'|| = a + a'$, $||\mathfrak{A} \cdot \mathfrak{A}'|| = ||\mathfrak{A}|| \cdot ||\mathfrak{A}'|| = a \cdot a'$ and $||\mathfrak{A}^*|| = ||\mathfrak{A}||^* = a^*$ are valid. □

We now turn to the power series semiring $S\langle\langle \Sigma^* \rangle\rangle$. A finite $S\langle \Sigma \cup \{\varepsilon\}\rangle$-automaton $\mathfrak{A} = (n, R, A, P)$ is called a *finite automaton (over S and Σ) without ε-moves* if $A \in (S\langle \Sigma \rangle)^{n\times n}$, $R \in (S\langle\{\varepsilon\}\rangle)^{1\times n}$ with $R_1 = \varepsilon$, $R_j = 0$ for $2 \leq j \leq n$, $P \in (S\langle\{\varepsilon\}\rangle)^{n\times 1}$. (This definition holds also for arbitrary semirings.) For $S = \mathbb{B}$ this is the usual definition, i.e., such a finite $\mathbb{B}\langle\Sigma\rangle$-automaton is a nondeterministic finite automaton without ε-moves in the classical sense.

We now show that each finite $S\langle \Sigma \cup \{\varepsilon\}\rangle$-automaton is equivalent to a finite automaton without ε-moves.

Theorem 2.12 *Let S be a Conway semiring. Then each finite $S\langle \Sigma \cup \{\varepsilon\}\rangle$-automaton is equivalent to a finite automaton over A and Σ without ε-moves.*

Proof. For each finite $S\langle \Sigma \cup \{\varepsilon\}\rangle$-automaton there exits, by Theorem 2.10, an equivalent normalized finite $S\langle \Sigma \cup \{\varepsilon\}\rangle$-automaton. Let $\mathfrak{A} = (n, R, A, P)$ be such a normalized finite $S\langle \Sigma \cup \{\varepsilon\}\rangle$-automaton. Let $A_0 = (A, \varepsilon)\varepsilon$ and $A_1 = \sum_{x\in\Sigma}(A, x)x$ and define the finite automaton without ε-moves $\mathfrak{A}' = (n, R, A_0^* A_1, A_0^* P)$. Then

$$||\mathfrak{A}'|| = R(A_0^* A_1)^* A_0^* P = R(A_0 + A_1)^* P = RA^* P = ||\mathfrak{A}|| \,.$$

Here we have applied in the second equality the sum star identity. □

Observe that, in case of a Conway semiring S, we have $\mathfrak{Rat}(S\langle \Sigma \cup \{\varepsilon\}\rangle) = S^{\mathrm{rat}}\langle\langle \Sigma^* \rangle\rangle$. Indeed, it is clear that $S^{\mathrm{rat}}\langle\langle \Sigma^* \rangle\rangle \subseteq \mathfrak{Rat}(S\langle \Sigma \cup \{\varepsilon\}\rangle)$. The reverse inclusion is consequence of the fact that if $r \in S^{\mathrm{rat}}\langle\langle \Sigma^* \rangle\rangle$ then $r^* \in S^{\mathrm{rat}}\langle\langle \Sigma^* \rangle\rangle$, which is shown as follows. Given r, write $r = r_0 + r_1$, where $r_0 = (r, \varepsilon)\varepsilon$ and $r_1 = \sum_{w\in\Sigma^+}(r, w)w$. Now r_1 is also in $S^{\mathrm{rat}}\langle\langle \Sigma^* \rangle\rangle$, and by the sum star identity we obtain $r^* = (r_0^* r_1)^* r_0^* \in S^{\mathrm{rat}}\langle\langle \Sigma^* \rangle\rangle$.

Analogously, in case of a Conway semiring, by Theorem 2.12 we have $\mathfrak{Rec}(S\langle \Sigma \cup \{\varepsilon\}\rangle) = S^{\mathrm{rec}}\langle\langle \Sigma^* \rangle\rangle$.

Corollary 2.13 $S^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle = S^{\mathrm{rec}}\langle\langle\Sigma^*\rangle\rangle = \{||\mathfrak{A}|| \mid \mathfrak{A}$ *is a finite automaton over* S *and* Σ *without* ε*-moves*$\}$.

The classical Theorem of Kleene in terms of formal power series essentially states that $\mathbb{B}^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle$ and $\mathbb{B}^{\mathrm{rec}}\langle\langle\Sigma^*\rangle\rangle$ coincide.

Corollary 2.14 $\mathbb{B}^{\mathrm{rat}}\langle\langle\Sigma^*\rangle\rangle = \mathbb{B}^{\mathrm{rec}}\langle\langle\Sigma^*\rangle\rangle = \{||\mathfrak{A}|| \mid \mathfrak{A}$ *is a finite automaton over* $\mathbb{B}$ *and* Σ *without* ε*-moves*$\}$.

Usually, a nondeterministic finite automaton without ε-moves is defined as follows (see Hopcroft, Ullman [16]). A *nondeterministic finite automaton* (in the classical sense)

$$\mathcal{A} = (Q, \Sigma, \delta, q_1, F)$$

is given by

(i) a finite nonempty *set of states* Q,
(ii) an *input alphabet* Σ,
(iii) a *transition function* $\delta : Q \times \Sigma \to 2^Q$,
(iv) an *initial state* $q_1 \in Q$,
(v) a *set of final states* $F \subseteq Q$.

The transition function δ is extended to a mapping $\hat{\delta} : Q \times \Sigma^* \to 2^Q$ by

$$\hat{\delta}(q, \varepsilon) = \{q\}, \quad \hat{\delta}(q, wx) = \{p \mid p \in \delta(r, x) \text{ for some } r \in \hat{\delta}(q, w)\},$$

for $q \in Q$, $w \in \Sigma^*$ and $x \in \Sigma$.

A word $w \in \Sigma^*$ is *accepted* by $\mathcal{A}$ if $\hat{\delta}(q_1, w) \cap F \neq \emptyset$. The *language* $||\mathcal{A}||$ *accepted by* $\mathcal{A}$, is defined by

$$||\mathcal{A}|| = \{w \in \Sigma^* \mid \hat{\delta}(q_1, w) \cap F \neq \emptyset\}.$$

We now connect the notion of a finite automaton $\mathfrak{A}$ over 2^{Σ^*} without ε-moves with the notion of a nondeterministic finite automaton $\mathcal{A}$ as defined above.

Assume that $\mathfrak{A} = (n, R, A, P)$ and $\mathcal{A} = (Q, \Sigma, \delta, q_1, F)$. Then $\mathfrak{A}$ and $\mathcal{A}$ *correspond to each other* if the following conditions are satisfied:

(i) $|Q| = n$; so we may assume $Q = \{q_1, \ldots, q_n\}$, where i corresponds to q_i, $1 \leq i \leq n$.
(ii) $x \in A_{ij} \Leftrightarrow q_j \in \delta(q_i, x)$, $1 \leq i, j \leq n$, $x \in \Sigma$.
(iii) $R_{q_1} = \{\varepsilon\}$, $R_{q_i} = \emptyset$, $2 \leq i \leq n$.
(iv) $P_i = \{\varepsilon\} \Leftrightarrow q_i \in F$, $P_i = \emptyset \Leftrightarrow q_i \notin F$.

It is easily seen that $||\mathfrak{A}|| = ||\mathcal{A}||$ if $\mathfrak{A}$ and $\mathcal{A}$ correspond to each other. This is due to the fact that

$$w \in (A^k)_{ij} \Leftrightarrow q_j \in \hat{\delta}(q_i, w), \quad 1 \leq i, j \leq n,\ k \geq 0,\ w \in \Sigma^*,\ |w| = k,$$

and

$$w \in (A^*)_{ij} \Leftrightarrow q_j \in \hat{\delta}(q_i, w), \quad 1 \le i, j \le n,\ w \in \Sigma^* .$$

(In the complete star semiring $(2^{\Sigma^*})^{n \times n}$ we have $A^* = \bigcup_{n \ge 0} A^n$.) Hence,

$$\begin{aligned} ||\mathfrak{A}|| = RA^*P = \textstyle\bigcup_{1 \le i,j \le n} R_i (A^*)_{ij} P_j = \bigcup_{q_j \in F} (A^*)_{1j} = \\ \textstyle\bigcup_{q_j \in F} \{w \mid q_j \in \hat{\delta}(q_1, w)\} = \{w \mid \hat{\delta}(q_1, w) \cap F \neq \emptyset\} = ||\mathcal{A}|| . \end{aligned}$$

It is clear that $\mathfrak{Rec}(\Sigma)$ coincides with the collection of all languages over Σ accepted by a nondeterministic finite automaton, and $\mathfrak{Rat}(\Sigma)$ coincides with the set of all languages that can be constructed from the finite subsets of Σ^* by the rational operations of union, concatenation and Kleene-iteration.

Corollary 2.15 (Kleene's Theorem [17]) *In the semiring 2^{Σ^*} of formal languages over Σ, $\mathfrak{Rat}(\Sigma) = \mathfrak{Rec}(\Sigma)$.*

2.3 Finite linear systems

A *finite linear system* (*over the semiring* S) is of the form

$$y = Ay + P .$$

Here $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ is a column vector of variables $y_1, \ldots, y_n$, $A \in S^{n \times n}$ and $P \in S^{n \times 1}$. A *solution* to the finite linear system $y = Ay + P$ is given by a column vector $\sigma \in S^{n \times 1}$ such that $\sigma = A\sigma + P$.

A finite linear system $y = Ay + P$ over the semiring $S\langle\langle \Sigma^* \rangle\rangle$ is called *cycle-free* if A is cycle-free.

Theorem 2.16 *The cycle-free finite linear system $y = Ay + P$ over the semiring $S\langle\langle \Sigma^* \rangle\rangle$ has the unique solution $\sigma = A^*P$.*

Proof. Analogous to the proof of Theorem 5.1 of Droste, Kuich [5]. □

The next theorem connects finite automata and finite linear systems. The special form of the cycle-free finite linear system in Theorem 2.17(i) is needed in connection with regular grammars in Theorem 2.19. In the sequel, e_k denotes the k-th row vector of unity.

Theorem 2.17 *Let $r \in S\langle\langle \Sigma^* \rangle\rangle$. Then the following statements are equivalent.*

(i) r is a component of the solution of a cycle-free finite linear system $y = Ay + P$ over the semiring $S\langle\langle \Sigma^* \rangle\rangle$, where $A \in (S\langle \Sigma \rangle)^{n \times n}$, $A_{i,1} = 0$ for all $1 \le i \le n$, and $P_1 \in S\langle \Sigma \cup \{\varepsilon\} \rangle$, $P_i \in S\langle \Sigma \rangle$, $2 \le i \le n$;
(ii) r is in $S^{\mathrm{rec}}\langle\langle \Sigma^* \rangle\rangle$.

Proof. (i) $\Rightarrow$ (ii): The solution of $y = Ay + P$ is $\sigma = A^*P$. We now construct cycle-free finite automata $\mathfrak{A}_k$ such that $||\mathfrak{A}_k|| = \sigma_k$, $1 \le k \le n$: $\mathfrak{A}_k = (\{1, \dots, n\}, e_k, A, P)$. Hence, by Theorem 2.2, $\sigma_k \in S^{\mathrm{rec}}\langle\langle \Sigma^* \rangle\rangle$.

(ii) $\Rightarrow$ (i): Let $r \in S^{\mathrm{rec}}\langle\langle \Sigma^* \rangle\rangle$. Then, by Theorem 2.3, there exists a cycle-free finite automaton $\mathfrak{A} = (\{1, \dots, n\}, e_1, A, P)$, $A \in (S\langle \Sigma \rangle)^{n \times n}$, $P \in (S\langle \{\varepsilon\} \rangle)^{n \times 1}$, such that $||\mathfrak{A}|| = r$. Consider now the finite linear system

$$\begin{pmatrix} y_0 \\ y \end{pmatrix} = \begin{pmatrix} 0 & e_1 A \\ 0 & A \end{pmatrix} \begin{pmatrix} y_0 \\ y \end{pmatrix} + \begin{pmatrix} e_1 P + e_1 AP \\ AP \end{pmatrix} .$$

Here $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$. By Corollary 5.8 of Droste, Kuich [5], this finite linear system is cycle-free and its solution is given by

$$\sigma = \begin{pmatrix} \varepsilon & e_1 AA^* \\ 0 & A^* \end{pmatrix} \begin{pmatrix} e_1 P + e_1 AP \\ AP \end{pmatrix} .$$

Hence, $\sigma_0 = e_1 P + e_1 AP + e_1 AA^* AP = e_1 P + e_1 AP + e_1 AAA^* P = e_1 A^* P = ||\mathfrak{A}|| = r$. Here we have applied Theorems 3.1 and 3.2 of Droste, Kuich [5]. $\square$

Let $G = (\{y_1, \dots, y_n\}, \Sigma, \Pi, y_1)$ be a context-free grammar. Then G is called *right linear* if Π contains only productions of the form $y_i \to xy_j$ or $y_i \to x$, where $x \in \Sigma \cup \{\varepsilon\}$ and $1 \le i, j \le n$. And G is called *regular* if Π contains only productions of the form $y_i \to xy_j$, $y_i \to x$, where $x \in \Sigma$, or $y_1 \to \varepsilon$; but if $y_1 \to \varepsilon \in \Pi$ then y_1 must not appear on the right side of any production of the grammar.

Consider the right linear grammars $G_i = (\{y_1, \dots, y_n\}, \Sigma, \Pi, y_i)$, $1 \le i \le n$. Then define the finite linear system $y = Ay + P$ over the semiring 2^{Σ^*} by $A_{i,j} = \{x \mid y_i \to xy_j \in \Pi\}$ and $P_i = \{x \mid y_i \to x \in \Pi\}$, $1 \le i, j \le n$. Conversely, given a finite linear system $y = Ay + P$ with $A_{i,j}, P_i \in \Sigma \cup \{\varepsilon\}$, $1 \le i, j \le n$, over the semiring 2^{Σ^*}, define the right linear grammars $G_i = (\{y_1, \dots, y_n\}, \Sigma, \Pi, y_i)$ by $y_i \to xy_j \in \Pi$ iff $x \in A_{i,j}$ and $y_i \to x \in \Pi$ iff $x \in P_i$. Whenever we speak of right linear grammars *corresponding* to a finite linear system, or vice versa, then we mean the correspondence in the sense of the above definition.

The next theorem is a special case of a result referred to in Petre, Salomaa [22].

Theorem 2.18 *Let $G_i = (\{y_1, \dots, y_n\}, \Sigma, \Pi, y_i)$, $1 \le i \le n$, be regular grammars and consider the corresponding finite linear system over 2^{Σ^*} with unique solution σ. Then $\sigma_i = L(G_i)$, $1 \le i \le n$.*

Consider a cycle-free finite linear system of the form of Theorem 2.17(i) and the corresponding right linear grammars $G_1, \dots, G_n$. Then G_1 is regular.

Corollary 2.19 *A language is generated by a regular grammar iff it is accepted by a nondeterministic finite automaton.*

More statements on the correspondence of finite automata and right linear grammars can be found in the forthcoming Theorems 3.17, 3.18 and Corollary 3.21 with $k = 0$, and Corollary 3.22.

We now turn to Conway semirings S.

Theorem 2.20 *Let S be a Conway semiring. Then $\sigma = A^*P$ is a solution to the finite linear system $y = Ay + P$ over S.*

Proof. $A\sigma + P = AA^*P + P = (AA^* + E)P = A^*P = \sigma$. Here we have applied Theorem 2.5 and the star fixed point identity, cf. Ésik [8]. □

We call the above solution of the system $y = Ay + P$ the *canonical solution*. Next, we consider inductive star semirings, cf. Ésik, Kuich [9], Ésik [8]. Recall that every inductive star semiring is a Conway semiring (and an iteration semiring).

Theorem 2.21 *Let S be an inductive star semiring. Then the canonical solution is the least solution to the finite linear system $y = Ay + P$ over S.*

Proof. By Corollary 8 of Ésik [8]. □

Let $S' \subseteq S$. A finite linear system $y = Ay + P$ over the semiring S is called *finite S'-linear system* if the entries of A and P are in $S' \cup \{0, 1\}$.

The next theorem connects finite S'-linear systems and finite S'-automata.

Theorem 2.22 *Let S be a Conway semiring and $S' \subseteq S$. Then, for $a \in S$, the following statements are equivalent:*

(i) a is a component of the canonical solution of a finite S'-linear system;
(ii) a is the behavior $||\mathfrak{A}||$ of a finite S'-automaton $\mathfrak{A}$.

Proof. (i) $\Rightarrow$ (ii): Consider the finite S'-linear system $y = Ay + P$ with n variables and the k-th component $e_k A^* P$ of its canonical solution, $1 \le k \le n$. Then the behavior of the finite S'-automaton $\mathfrak{A}_k = (n, e_k, A, P)$ equals $e_k A^* P$ for all $1 \le k \le n$.

(ii) $\Rightarrow$ (i): By Theorem 2.10, a is, without loss of generality, the behavior of a normalized finite S'-automaton $\mathfrak{A} = (n, e_1, A, P)$. The behavior $||\mathfrak{A}|| = e_1 A^* P$ is then the first component of the canonical solution of the finite S'-linear system $y = Ay + P$. □

Corollary 2.23 *Let S be an inductive star semiring and $S' \subseteq S$. Then, for $a \in S$, the following statements are equivalent:*

(i) a is a component of the least solution of a finite S'-linear system;
(ii) a is the behavior $||\mathfrak{A}||$ of a finite S'-automaton $\mathfrak{A}$.

Since every continuous semiring is an inductive star semiring, the last two results also apply for continuous semirings.

3 Finite automata over quemirings

In this section, we deal with semiring-semimodule pairs and finite automata over quemirings. Here semiring-semimodule pairs constitute a generalization of formal languages with finite and infinite words. The semiring models formal languages with finite words while the semimodule models formal languages with infinite words. The main result of this section is a generalization of the Kleene Theorem of Büchi [3] in the setting of semiring-semimodule pairs. (See also Perrin, Pin [21].)

This section consists of three subsections. In Subsection 1 we introduce the algebraic structures used in this section: semiring-semimodule pairs and quemirings.

In Subsection 2 we define finite automata over quemirings. Given a Conway semiring-semimodule pair (S, V), cf. Bloom, Ésik [2], Ésik [8], we prove a Kleene Theorem for S'-finite automata, where S' is a subset of S: the collection of all behaviors of S'-finite automata coincides with the generalized star quemiring generated by S' (Theorem 3.11). A special case of this Kleene Theorem is the result of Büchi [3].

In Subsection 3 we consider finite linear systems over quemirings as a generalization of regular grammars with finite and infinite derivations. We show a connection between certain solutions of these finite linear systems, the weights of finite and infinite derivations with respect to regular grammars and the behaviors of finite automata over quemirings.

We now give a typical example on Büchi automata over infinite words.

Example 3.1. A *(finite) Büchi automaton*

$$\mathcal{A} = (Q, \Sigma, \delta, q, G)$$

is given by

(i) a finite *set of states* $Q = \{q_1, \ldots, q_n\}$, $n \geq 1$,
(ii) an *input alphabet* Σ,
(iii) a *transition function* $\delta : Q \times \Sigma \to 2^Q$,
(iv) an *initial state* $q \in Q$,
(v) a *set of repeated states* $G = \{q_1, \ldots, q_k\}$, $k \geq 0$.

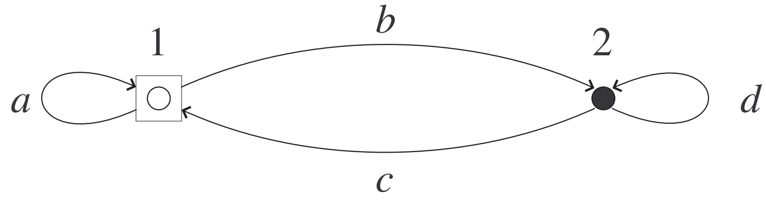
A *run of* $\mathcal{A}$ on an infinite word $w \in \Sigma^\omega$, $w = a_1 a_2 a_3 \ldots$, is an infinite sequence of states $q(0), q(1), q(2), q(3), \ldots$ such that the following conditions are satisfied:

(i) $q(0) = q$,
(ii) $q(i) \in \delta(q(i-1), a_i)$ for $i \geq 1$.

A word $w \in \Sigma^\omega$ is *Büchi accepted by* $\mathcal{A}$ if there exists a run ρ of $\mathcal{A}$ on w and a repeated state in G occuring infinitely often in ρ.

The *behavior* $||\mathcal{A}|| \subseteq \Sigma^\omega$ of $\mathcal{A}$ is defined to be the set of infinite words that are Büchi accepted by $\mathcal{A}$ (see Büchi [3]).

Let now $\mathcal{A} = (Q, \Sigma, \delta, 2, \{1\})$ be a Büchi automaton, where $Q = \{1, 2\}$, $\Sigma = \{a, b, c, d\}$, and $\delta(1, a) = \{1\}$, $\delta(1, b) = \{2\}$, $\delta(2, c) = \{1\}$, $\delta(2, d) = \{2\}$ are the only nonempty images of δ. The graph of $\mathcal{A}$ is



and the adjacency matrix of this graph is

$$A = \begin{pmatrix} \{a\} & \{b\} \\ \{c\} & \{d\} \end{pmatrix} .$$

(See Example 2.1.)

The language of inscriptions of paths from 1 to 1 not passing 1 is given by $\{a\} \cup \{b\}\{d\}^*\{c\}$. Hence, the ω-language of inscriptions of infinite paths starting in 1 and passing infinitely often through 1 is $(\{a\} \cup \{b\}\{d\}^*\{c\})^\omega$ and the ω-language of inscriptions of infinite paths starting in 1, passing finitely often through 1 and infinitely often through 2 is $(\{a\} \cup \{b\}\{d\}^*\{c\})^*\{b\}\{d\}^\omega$. By symmetry, the ω-language of inscriptions of infinite paths starting in 2 and passing infinitely often through 2 (resp. finitely often through 2 and infinitely often through 1) is $(\{d\} \cup \{c\}\{a\}^*\{b\})^\omega$ (resp. $(\{d\} \cup \{c\}\{a\}^*\{b\})^*\{c\}\{a\}^\omega$).

We now define a column vector A^ω by

$$A^\omega = \begin{pmatrix} (\{a\} \cup \{b\}\{d\}^*\{c\})^\omega \cup (\{a\} \cup \{b\}\{d\}^*\{c\})^*\{b\}\{d\}^\omega \\ (\{d\} \cup \{c\}\{a\}^*\{b\})^\omega \cup (\{d\} \cup \{c\}\{a\}^*\{b\})^*\{c\}\{a\}^\omega \end{pmatrix} ,$$

where $(A^\omega)_1$ (resp. $(A^\omega)_2$) is the ω-language of inscriptions of all infinite paths starting in 1 (resp. 2). Observe that

$$(\{a\} \cup \{b\}\{d\}^*\{c\})^\omega \cap (\{a\} \cup \{b\}\{d\}^*\{c\})^*\{b\}\{d\}^\omega = \emptyset$$

and

$$(\{d\} \cup \{c\}\{a\}^*\{b\})^\omega \cap (\{d\} \cup \{c\}\{a\}^*\{b\})^*\{c\}\{a\}^\omega = \emptyset.$$

The ω-language of inscriptions of infinite paths starting in 2 and passing infinitely often through 1 is $\{d\}^*\{c\}(\{a\} \cup \{b\}\{d\}^*\{c\})^\omega$. We define a column vector $A^{\omega,1}$ by

$$A^{\omega,1} = \begin{pmatrix} (\{a\} \cup \{b\}\{d\}^*\{c\})^\omega \\ \{d\}^*\{c\}(\{a\} \cup \{b\}\{d\}^*\{c\})^\omega \end{pmatrix} ,$$

where $(A^{\omega,1})_1$ (resp. $(A^{\omega,1})_2$) is the ω-language of inscriptions of all infinite paths starting in 1 (resp. 2) and passing infinitely often through 1.

The ω-language $||\mathcal{A}||$ is the ω-language of all inscriptions of infinite paths starting in 2 and passing infinitely often through 1, i. e., $||\mathcal{A}|| = (A^{\omega,1})_2$. $\square$

3.1 Semiring-semimodule pairs and quemirings

Suppose that S is a semiring and V is a commutative monoid written additively. We call V a (left) S-*semimodule* if V is equipped with a (left) action

$$\begin{aligned} S \times V &\to V \\ (s, v) &\mapsto sv \end{aligned}$$

subject to the following rules:

$$\begin{aligned} s(s'v) &= (ss')v \\ (s + s')v &= sv + s'v \\ s(v + v') &= sv + sv' \\ 1v &= v \\ 0v &= 0 \\ s0 &= 0, \end{aligned}$$

for all $s, s' \in S$ and $v, v' \in V$. When V is an S-semimodule, we call (S, V) a *semiring-semimodule pair.*

Suppose that (S, V) is a semiring-semimodule pair such that S is a star semiring and S and V are equipped with an omega operation $^\omega : S \to V$. Then we call (S, V) a *star semiring-omega semimodule pair.* Following Bloom, Ésik [2], see also Ésik [8], we call a star semiring-omega semimodule pair (S, V) a *Conway semiring-semimodule pair* if S is a Conway semiring and if the omega operation satisfies the *sum omega identity* and the *product omega identity*:

$$\begin{aligned} (a + b)^\omega &= (a^*b)^\omega + (a^*b)^*a^\omega \\ (ab)^\omega &= a(ba)^\omega, \end{aligned}$$

for all $a, b \in S$. It then follows that the *omega fixed point identity* holds, i.e.,

$$aa^\omega = a^\omega,$$

for all $a \in S$.

Theorem 3.1 (Bloom, Ésik [2], Ésik, Kuich [12], Ésik [8]) *If (S, V) is a Conway semiring-semimodule pair then, for $n \geq 1$, $(S^{n\times n}, V^{n\times 1})$ again is a Conway semiring-semimodule pair.*

Ésik, Kuich [13] define a *complete semiring-semimodule pair* to be a semiring-semimodule pair (S, V) such that S is a complete semiring, V is a complete monoid with

$$\begin{aligned} s(\sum_{i\in I} v_i) &= \sum_{i\in I} sv_i \\ (\sum_{i\in I} s_i)v &= \sum_{i\in I} s_iv, \end{aligned}$$

for all $s \in S$, $v \in V$, and for all families $s_i,\ i \in I$ over S and $v_i,\ i \in I$ over V. Moreover, it is required that an *infinite product operation*

$$(s_1, s_2, \ldots) \mapsto \prod_{j \geq 1} s_j$$

is given mapping infinite sequences over S to V subject to the following three conditions:

$$\prod_{i \geq 1} s_i = \prod_{i \geq 1} (s_{n_{i-1}+1} \cdot \ldots \cdot s_{n_i})$$
$$s_1 (\prod_{i \geq 1} s_{i+1}) = \prod_{i \geq 1} s_i$$
$$\prod_{j \geq 1} \sum_{i_j \in I_j} s_{i_j} = \sum_{(i_1, i_2, \ldots) \in I_1 \times I_2 \times \ldots} \prod_{j \geq 1} s_{i_j},$$

where in the first equation $0_0 \leq n_1 \leq n_2 \leq \ldots$ and $I_1, I_2, \ldots$ are arbitrary index sets. Suppose that (S, V) is complete. Then we define

$$s^* = \sum_{i \geq 0} s^i$$
$$s^\omega = \prod_{i \geq 1} s,$$

for all $s \in S$. This turns (S, V) into a star semiring-omega semimodule pair. By Ésik, Kuich [13], see also Ésik [8], each complete semiring-semimodule pair is a Conway semiring-semimodule pair.

A *star-omega semiring* is a semiring S equipped with unary operations * and $^\omega : S \to S$. A star-omega semiring S is called *complete* if (S, S) is a complete semiring-semimodule pair, i. e., if S is complete and is equipped with an infinite product operation that satisfies the three conditions stated above.

Example 3.2. Suppose that Σ is an alphabet. Let Σ^* denote the set of all finite words over Σ including the empty word ε, and let Σ^ω denote the set of all ω-words over Σ. The set 2^{Σ^*} of all subsets of Σ^*, equipped with the operations of set union as sum and concatenation as product is a semiring, where 0 is the empty set $\emptyset$ and 1 is the set $\{\varepsilon\}$. Moreover, equipped with the usual star operation, 2^{Σ^*} is a Conway semiring. Also, 2^{Σ^ω}, equipped with union as the sum operation and the empty set as 0 is a commutative idempotent monoid. Define an action of 2^{Σ^*} on 2^{Σ^ω} by $KL = \{uv \mid u \in K,\ v \in L\}$, for all $K \subseteq \Sigma^*$ and $L \subseteq \Sigma^\omega$. Moreover, for each sequence $(K_0, K_1, \ldots)$ over 2^{Σ^*}, let $\prod_{j \geq 0} K_j = \{u_0 u_1 \ldots \in \Sigma^\omega \mid u_i \in K_i,\ i \geq 0\}$. Then $(2^{\Sigma^*}, 2^{\Sigma^\omega})$ is a complete semiring-semimodule pair with idempotent module 2^{Σ^ω}. Note that in this example, $1^\omega = 0$, where $1 = \{\varepsilon\}$ and $0 = \emptyset$. □

Example 3.3. Consider the semiring $\mathbb{N}^\infty = N \cup \{\infty\}$, obtained by adjoining a top element ∞ to the semiring of the natural numbers. Note that $\mathbb{N}^\infty$ is a complete semiring where an infinite sum is ∞ iff either a summand is ∞ or the number of nonzero summands is infinite, moreover, ∞ multiplied with a nonzero element on either side gives ∞. Define an infinite product

$$(n_1, n_2, \ldots) \mapsto \prod_{j \geq 1} n_j$$

on $\mathbb{N}^\infty$ as follows. If some n_j is 0, then so is the product. Otherwise, if all but a finite number of the n_j are 1s, then the infinite product is the product of those n_j with $n_j > 1$. In all remaining cases, the infinite product is ∞. Then $\mathbb{N}^\infty$ is a complete star-omega semiring, where * and $^\omega$ are defined as above.

Let Σ denote an alphabet. The semiring $S = \mathbb{N}^\infty\langle\langle \Sigma^* \rangle\rangle$ of all power series over Σ^* with coefficients in $\mathbb{N}^\infty$ is a complete and continuous semiring. Now let $V = \mathbb{N}^\infty\langle\langle \Sigma^\omega \rangle\rangle$ be the collection of all formal power series over Σ^ω with coefficients in $\mathbb{N}^\infty$. Thus, the elements of V are formal sums of the sort

$$s = \sum_{w \in \Sigma^\omega} (s, w) w,$$

where each coefficient (s, w) belongs to $\mathbb{N}^\infty$. Now V can be turned into an S-semimodule by the pointwise sum operation and the action $(r, s) \mapsto rs$ defined by

$$(rs, w) = \sum_{u \in \Sigma^*, v \in \Sigma^\omega,\ uv = w} (r, u)(s, v),$$

where the infinite sum on the right hand side exists since $\mathbb{N}^\infty$ is complete. We may also define an infinite product taking sequences over S to series in V. Given $s_1, s_2, \ldots$ in S, we define $\prod_{j \geq 1} s_j$ to be the series r in V with

$$(r, w) = \sum_{w = w_1 w_2 \ldots \in \Sigma^\omega} \prod_{j \geq 1} (s_j, w_j).$$

Then $(\mathbb{N}^\infty\langle\langle \Sigma^* \rangle\rangle, \mathbb{N}^\infty\langle\langle \Sigma^\omega \rangle\rangle)$ is a complete semiring-semimodule pair and thus a Conway semiring-semimodule pair.

This can be generalized to a large extent. Suppose that S is a complete star-omega semiring. If Σ is a set, consider the complete semiring $S\langle\langle \Sigma^* \rangle\rangle$ and the complete monoid $S\langle\langle \Sigma^\omega \rangle\rangle$ of all series over Σ^ω with coefficients in S equipped with the pointwise sum operation. If we define the action sr of $s \in S\langle\langle \Sigma^* \rangle\rangle$ on $r \in S\langle\langle \Sigma^\omega \rangle\rangle$ by

$$(sr, w) = \sum_{w = uv} (s, u)(r, v),$$

then $(S\langle\langle \Sigma^* \rangle\rangle, S\langle\langle \Sigma^\omega \rangle\rangle)$ becomes a semiring-semimodule pair. Now $S\langle\langle \Sigma^* \rangle\rangle$ is a star semiring, and if we define the infinite product operation

$$(s_1, s_2, \ldots) \mapsto \prod_{j \geq 1} s_j \in S\langle\langle \Sigma^\omega \rangle\rangle$$

by

$$(\prod_{j\geq 1} s_j, w) = \sum_{w=w_1w_2\ldots\in\Sigma^\omega} \prod_{j\geq 1}(s_j, w_j),$$

then $(S\langle\langle\Sigma^*\rangle\rangle, S\langle\langle\Sigma^\omega\rangle\rangle)$ becomes a complete semiring-semimodule pair, hence a Conway semiring-semimodule pair, satisfying $s^\omega = 0$, where each $s \in S$ is identified with a series in the usual way. □

Consider a star semiring-omega semimodule pair (S, V). Following Bloom, Ésik [2], see also Ésik [8] and Example 3.1, we define a matrix operation ${}^\omega : S^{n\times n} \to V^{n\times 1}$ on a star semiring-omega semimodule pair (S, V) as follows. When $n = 1$, so that $A = (a)$, for some $a \in S$, $A^\omega = (a^\omega)$. For $n > 1$ we partition A into blocks

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{1}$$

where $a \in S^{1\times 1}$, $b \in S^{1\times(n-1)}$, $c \in S^{(n-1)\times 1}$, $d \in S^{(n-1)\times(n-1)}$, and define

$$A^\omega = \begin{pmatrix} (a + bd^*c)^\omega + (a + bd^*c)^*bd^\omega \\ (d + ca^*b)^\omega + (d + ca^*b)^*ca^\omega \end{pmatrix}. \tag{2}$$

Let A and A^ω be given as in the definition above, but with $a \in S^{n_1\times n_1}$, $b \in S^{n_1\times n_2}$, $c \in S^{n_2\times n_1}$, $d \in S^{n_2\times n_2}$, where $n_1 + n_2 = n$. Then the *matrix omega identity* is valid in the star semiring-omega semimodule pair if A^ω is independent of the partition of n into summands.

Theorem 3.2 (Bloom, Ésik [2], Ésik, Kuich [12], Ésik [8]) *If (S, V) is a Conway semiring-semimodule pair then the matrix omega identity holds in the Conway semiring-semimodule pair $(S^{n\times n}, V^{n\times 1})$ for all $n \geq 1$.*

Following Ésik, Kuich [11] (see also Example 3.1), we define matrix operations ${}^{\omega,k} : S^{n\times n} \to V^{n\times 1}$, $0 \leq k \leq n$, as follows. Assume that $A \in S^{n\times n}$ is decomposed into blocks a, b, c, d as in (1), but with a of dimension $k \times k$ and d of dimension $(n - k) \times (n - k)$. Then

$$A^{\omega,k} = \begin{pmatrix} (a + bd^*c)^\omega \\ d^*c(a + bd^*c)^\omega \end{pmatrix} \tag{3}$$

Observe that $A^{\omega,0} = 0$ and $A^{\omega,n} = A^\omega$.

Suppose that (S, V) is a semiring-semimodule pair and consider $T = S\times V$. Define on T the operations

$$(s, u) \cdot (s', v) = (ss', u + sv)$$
$$(s, u) + (s', v) = (s + s', u + v)$$

and constants $0 = (0, 0)$ and $1 = (1, 0)$. Equipped with these operations and constants, T satisfies the identities

$$(x+y)+z = x+(y+z) \tag{4}$$
$$x+y = y+x \tag{5}$$
$$x+0 = x \tag{6}$$
$$(x\cdot y)\cdot z = x\cdot(y\cdot z) \tag{7}$$
$$x\cdot 1 = x \tag{8}$$
$$1\cdot x = x \tag{9}$$
$$(x+y)\cdot z = (x\cdot z)+(y\cdot z) \tag{10}$$
$$0\cdot x = 0. \tag{11}$$

Define also the unary operation ¶ on T: $(s,u)\P = (s,0)$. Thus, ¶ selects the "first component" of the pair (s,u), while multiplication with 0 on the right selects the "second component", for $(s,u)\cdot 0 = (0,u)$, for all $u\in V$. The new operation satisfies:

$$x\P\cdot(y+z) = (x\P\cdot y)+(x\P\cdot z) \tag{12}$$
$$x = x\P + (x\cdot 0) \tag{13}$$
$$x\P\cdot 0 = 0 \tag{14}$$
$$(x+y)\P = x\P + y\P \tag{15}$$
$$(x\cdot y)\P = x\P\cdot y\P. \tag{16}$$

Note that when V is idempotent, also

$$x\cdot(y+z) = x\cdot y + x\cdot z$$

holds.

Elgot [7] defined a *quemiring* to be an algebraic structure T equipped with the above operations $\cdot, +, \P$ and constants $0,1$ satisfying the identities (4)–(11) and (12)–(16). A morphism of quemirings is a function preserving the operations and constants. It follows that $x\P\P = x\P$, for all x in a quemiring T. Moreover, $x\P = x$ iff $x\cdot 0 = 0$.

When T is a quemiring, $S = T\P = \{x\P \mid x\in T\}$ is easily seen to be a semiring. Moreover, $V = T0 = \{x\cdot 0 \mid x\in T\}$ contains 0 and is closed under $+$, and, furthermore, $sx\in V$ for all $s\in S$ and $x\in V$. Each $x\in T$ may be written in a unique way as the sum of an element of $T\P$ and of an element of $T0$ as $x = x\P + x\cdot 0$. Sometimes, we will identify $S\times\{0\}$ with S and $\{0\}\times V$ with V. It is shown in Elgot [7] that T is isomorphic to the quemiring $S\times V$ determined by the semiring-semimodule pair (S,V).

Suppose now that (S,V) is a star semiring-omega semimodule pair. Then we define on $T = S\times V$ a *generalized star operation*:

$$(s,v)^{\otimes} = (s^*, s^\omega + s^*v) \tag{17}$$

for all $(s,v)\in T$. Note that the star and omega operations can be recovered from the generalized star operation, since s^* is the first component of $(s,0)^{\otimes}$ and s^ω is the second component. Thus:

$$(s^*, 0) = (s, 0)^{\otimes}¶$$
$$(0, s^{\omega}) = (s, 0)^{\otimes} \cdot 0.$$

Observe that, for $(s, 0) \in S \times \{0\}$, $(s, 0)^{\otimes} = (s^*, 0) + (0, s^{\omega})$.

Suppose now that T is an (abstract) quemiring equipped with a generalized star operation $^{\otimes}$. As explained above, T as a quemiring is isomorphic to the quemiring $S \times V$ associated with the semiring-semimodule pair (S, V), where $S = T¶$ and $V = T0$, an isomorphism being the map $x \mapsto (x¶, x \cdot 0)$. It is clear that a generalized star operation $^{\otimes} : T \to T$ is determined by a star operation $^* : S \to S$ and an omega operation $^{\omega} : S \to V$ by (17) iff

$$x^{\otimes}¶ = (x¶)^{\otimes}¶ \tag{18}$$
$$x^{\otimes} \cdot 0 = (x¶)^{\otimes} \cdot 0 + x^{\otimes}¶ \cdot x \cdot 0 \tag{19}$$

hold. Indeed, these conditions are clearly necessary. Conversely, if (18) and (19) hold, then for any $x¶ \in T¶$ we may define

$$(x¶)^* = (x¶)^{\otimes}¶ \tag{20}$$
$$(x¶)^{\omega} = (x¶)^{\otimes} \cdot 0\,. \tag{21}$$

It follows that (17) holds. The definition of star and omega was forced.

Let us call a quemiring equipped with a generalized star operation $^{\otimes}$ satisfying (18) and (19) a *generalized star quemiring.*

3.2 Finite automata over quemirings and a Kleene Theorem

In this subsection we consider finite automata over quemirings and prove a Kleene Theorem. Throughout this subsection, (S, V) denotes a Conway semiring-semimodule pair and T denotes the generalized star quemiring $S \times V$. Moreover, S' denotes a subset of S.

A *finite S'-automaton* (*over the quemiring T*)

$$\mathfrak{A} = (n, R, A, P, k)$$

is given by

(i) a finite *set of states* $\{1, \ldots, n\}$, $n \geq 1$,
(ii) a *transition matrix* $A \in (S' \cup \{0, 1\})^{n \times n}$,
(iii) an *initial state vector* $R \in (S' \cup \{0, 1\})^{1 \times n}$,
(iv) a *final state vector* $P \in (S' \cup \{0, 1\})^{n \times 1}$,
(v) a *set of repeated states* $\{1, \ldots, k\}$, $k \geq 0$.

The *behavior* of $\mathfrak{A}$ is an element of T and is defined by

$$||\mathfrak{A}|| = RA^*P + RA^{\omega,k}\,.$$

If $\mathfrak{A} = (n, (i_1\ i_2), \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, k)$, where

$$
\begin{aligned}
&i_1 \in (S' \cup \{0,1\})^{1\times k},\ i_2 \in (S' \cup \{0,1\})^{1\times(n-k)},\\
&a \in (S' \cup \{0,1\})^{k\times k},\ b \in (S' \cup \{0,1\})^{k\times(n-k)},\\
&c \in (S' \cup \{0,1\})^{(n-k)\times k},\ d \in (S' \cup \{0,1\})^{(n-k)\times(n-k)},\\
&p_1 \in (S' \cup \{0,1\})^{k\times 1},\ p_2 \in (S' \cup \{0,1\})^{(n-k)\times 1},
\end{aligned}
$$

we write also

$$\mathfrak{A} = (n; i_1, i_2; a, b, c, d; p_1, p_2; k)\,.$$

Theorem 3.3 *If (S, V) is a complete semiring-semimodule pair and $\mathfrak{A}$ is a finite S'-automaton then $||\mathfrak{A}|| = F + I$, where F is the sum of the weights of all finite paths from an initial state to a final state, multiplied by the initial and final weights of these states, and where I is the sum of the weights of all infinite paths starting at an initial state, passing infinitely often through repeated states, and multiplied by the initial weight of this initial state.*

Proof. By formalization of the considerations in Example 3.1 (see the proof of Theorem 3.1 of Ésik, Kuich [11]). □

Note that the finite S'-automata $\mathfrak{A} = (n, R, A, P)$ and $\mathfrak{A}' = (n, R, A, P, 0)$ over a Conway semiring S and a star semiring-omega semimodule pair (S, V), respectively, have the same behavior $||\mathfrak{A}|| = ||\mathfrak{A}'|| = RA^*P$.

By definition, ω-$\mathfrak{Rat}(S')$ is the Conway semiring-semimodule pair generated by S'.

We now will prove a Kleene Theorem: Let $a \in S \times V$. Then $a \in \omega$-$\mathfrak{Rat}(S')$ iff a is the behavior of a finite S'-automaton. To achieve this result we need a few theorems and corollaries. In our arguments, we will use Theorem 3.2.

Let $\mathfrak{A} = (n, R, A, P, k)$ be a finite S'-automaton. It is called *normalized* if

(i) $n \geq 2$ and $k \leq n - 2$;
(ii) $R_{n-1} = 1$ and $R_j = 0$ for $j \neq n - 1$;
(iii) $P_n = 1$ and $P_j = 0$ for $j \neq n$;
(iv) $A_{i,n-1} = 0$ and $A_{n,i} = 0$ for all $1 \leq i \leq n$.

Two finite S'-automata $\mathfrak{A}$ and $\mathfrak{A}'$ are *equivalent* if $||\mathfrak{A}|| = ||\mathfrak{A}'||$.

Theorem 3.4 *Each finite S'-automaton $\mathfrak{A} = (n, R, A, P, k)$ is equivalent to a normalized finite S'-automaton $\mathfrak{A}' = (n + 1 + 1, R', A', P', k)$.*

Proof. We define $R' = (0\ 1\ 0)$, $A' = \begin{pmatrix} A & 0 & P \\ R & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $P' = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Let now $\mathfrak{A} = (n; i_1, i_2; a, b, c, d; p_1, p_2; k)$. Then

$$A' = \begin{pmatrix} a & b & 0 & p_1 \\ c & d & 0 & p_2 \\ i_1 & i_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and the first k entries of $A'^{\omega,k}$ are equal to

$$\begin{aligned}
&\left(a + (b\ 0\ p_1)\begin{pmatrix} d & 0 & p_2 \\ i_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^* \begin{pmatrix} c \\ i_1 \\ 0 \end{pmatrix}\right)^\omega \\
&= \left(a + (b\ 0\ p_1)\begin{pmatrix} d^* & 0 & d^*p_2 \\ i_2d^* & 1 & i_2d^*p_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c \\ i_1 \\ 0 \end{pmatrix}\right)^\omega \\
&= (a + bd^*c)^\omega .
\end{aligned}$$

Hence, the last $n - k + 2$ entries of $A'^{\omega,k}$ are equal to

$$\begin{pmatrix} d & 0 & p_2 \\ i_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^* \begin{pmatrix} c \\ i_1 \\ 0 \end{pmatrix} (a + bd^*c)^\omega = \begin{pmatrix} d^*c \\ i_2d^*c + i_1 \\ 0 \end{pmatrix} (a + bd^*c)^\omega$$

and we obtain

$$\begin{aligned}
||\mathfrak{A}'|| &= R'A'^*P' + R'A'^{\omega,k} = (A'^*)_{n+1,n+2} + (A'^{\omega,k})_{n+1} \\
&= RA^*P + (i_2d^*c + i_1)(a + bd^*c)^\omega = RA^*P + RA^{\omega,k} = ||\mathfrak{A}|| .
\end{aligned}$$

□

Lemma 3.5 *If* $\mathfrak{A} = (n; i_1, i_2; a, b, c, d; p_1, p_2; k)$ *is a finite* S'*-automaton then*

$$\begin{aligned}
||\mathfrak{A}|| = {} & i_1(a + bd^*c)^*(p_1 + bd^*p_2) + i_2d^*c(a + bd^*c)^*(p_1 + bd^*p_2) + \\
& i_2d^*p_2 + i_1(a + bd^*c)^\omega + i_2d^*c(a + bd^*c)^\omega .
\end{aligned}$$

Let $\mathfrak{A} = (n; i_1, i_2; a, b, c, d; f, g; m)$ and $\mathfrak{A}' = (n'; h, i; a', b', c', d'; p_1, p_2; k)$ be finite S'-automata. Then we define the finite S'-automata $\mathfrak{A} + \mathfrak{A}'$ and $\mathfrak{A} \cdot \mathfrak{A}'$ to be

$$\begin{aligned}
\mathfrak{A} + \mathfrak{A}' = (n + n'; &(i_1\ h), (i_2\ i); \\
&\begin{pmatrix} a & 0 \\ 0 & a' \end{pmatrix}, \begin{pmatrix} b & 0 \\ 0 & b' \end{pmatrix}, \begin{pmatrix} c & 0 \\ 0 & c' \end{pmatrix}, \begin{pmatrix} d & 0 \\ 0 & d' \end{pmatrix}; \\
&\begin{pmatrix} f \\ p_1 \end{pmatrix}, \begin{pmatrix} g \\ p_2 \end{pmatrix}, m + k)
\end{aligned}$$

and

$$\begin{aligned}
\mathfrak{A} \cdot \mathfrak{A}' = (n + n'; &(i_1\ 0), (i_2\ 0); \\
&\begin{pmatrix} a & fh \\ 0 & a' \end{pmatrix}, \begin{pmatrix} b & fi \\ 0 & b' \end{pmatrix}, \begin{pmatrix} c & gh \\ 0 & c' \end{pmatrix}, \begin{pmatrix} d & gi \\ 0 & d' \end{pmatrix}; \\
&\begin{pmatrix} 0 \\ p_1 \end{pmatrix}, \begin{pmatrix} 0 \\ p_2 \end{pmatrix}, m + k) .
\end{aligned}$$

For the definition of $\mathfrak{A}\cdot\mathfrak{A}'$ we assume that either $\begin{pmatrix} f \\ g \end{pmatrix}(h\ i) \in (S'\cup\{0,1\})^{n\times n'}$ or $\mathfrak{A}'$ is normalized. Observe that the definitions of $\mathfrak{A}+\mathfrak{A}'$ and $\mathfrak{A}\cdot\mathfrak{A}'$ (and of $\mathfrak{A}^{\otimes}$ which is defined below) are the usual ones except that certain rows and columns are permuted. These permutations are needed since the set of repeated states of a finite S'-automaton is always a set $\{1,\ldots,k\}$.

Theorem 3.6 *Let $\mathfrak{A}$ and $\mathfrak{A}'$ be finite S'-automata. Then $||\mathfrak{A}+\mathfrak{A}'|| = ||\mathfrak{A}|| + ||\mathfrak{A}'||$ and $||\mathfrak{A}\cdot\mathfrak{A}'|| = ||\mathfrak{A}||\cdot||\mathfrak{A}'||$.*

Proof. Let $\mathfrak{A}$ and $\mathfrak{A}'$ be defined as above. We first show $||\mathfrak{A}+\mathfrak{A}'|| = ||\mathfrak{A}||+||\mathfrak{A}'||$ and compute $||\mathfrak{A}+\mathfrak{A}'||\cdot 0$. The transition matrix of $\mathfrak{A}+\mathfrak{A}'$ is given by

$$A = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a' & 0 & b' \\ c & 0 & d & 0 \\ 0 & c' & 0 & d' \end{pmatrix}.$$

We now compute the first $m+k$ entries of $A^{\omega,m+k}$. This column vector of dimension $m+k$ is given by

$$\left(\begin{pmatrix} a & 0 \\ 0 & a' \end{pmatrix} + \begin{pmatrix} b & 0 \\ 0 & b' \end{pmatrix}\begin{pmatrix} d & 0 \\ 0 & d' \end{pmatrix}^*\begin{pmatrix} c & 0 \\ 0 & c' \end{pmatrix}\right)^\omega =$$

$$\begin{pmatrix} a+bd^*c & 0 \\ 0 & a'+b'd'^*c' \end{pmatrix}^\omega = \begin{pmatrix} (a+bd^*c)^\omega \\ (a'+b'd'^*c')^\omega \end{pmatrix}.$$

The last $n+n'-(m+k)$ entries of $A^{\omega,m+k}$ are given by the product of

$$\begin{pmatrix} d & 0 \\ 0 & d' \end{pmatrix}^*\begin{pmatrix} c & 0 \\ 0 & c' \end{pmatrix} = \begin{pmatrix} d^*c & 0 \\ 0 & d'^*c' \end{pmatrix}$$

with the column vector computed above. Hence, we obtain by Lemma 3.5

$$||\mathfrak{A}+\mathfrak{A}'||\cdot 0 = (i_1\ h\ i_2\ i)A^{\omega,m+k} = i_1(a+bd^*c)^\omega + h(a'+b'd'^*c')^\omega + i_2d^*c(a+bd^*c)^\omega + id'^*c'^*(a'+b'd'^*c')^\omega = (||\mathfrak{A}||+||\mathfrak{A}'||)\cdot 0\,.$$

We now compute $||\mathfrak{A}+\mathfrak{A}'||\P$. If, in the transition matrix A of $\mathfrak{A}+\mathfrak{A}'$ we commute the $m+1,\ldots,m+k$ row and column with the $m+k+1,\ldots,n+k$ row and column, and do the same with the initial and final vector we obtain by the star permutation identity (see Conway [4], Bloom, Ésik [2], Ésik, Kuich [9] and Ésik [8]).

$$||\mathfrak{A}+\mathfrak{A}'||\P = (i_1\ i_2\ h\ i)\begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a' & b' \\ 0 & 0 & c' & d' \end{pmatrix}^*\begin{pmatrix} f \\ g \\ p_1 \\ p_2 \end{pmatrix}$$

$$= (i_1\ i_2)\begin{pmatrix} a & b \\ c & d \end{pmatrix}^*\begin{pmatrix} f \\ g \end{pmatrix} + (h\ i)\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}^*\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

$$= (||\mathfrak{A}||+||\mathfrak{A}'||)\P\,.$$

Hence, $||\mathfrak{A} + \mathfrak{A}'|| = ||\mathfrak{A}|| + ||\mathfrak{A}'||$.

We now show $||\mathfrak{A} \cdot \mathfrak{A}'|| = ||\mathfrak{A}|| \cdot ||\mathfrak{A}'||$ and compute $||\mathfrak{A} \cdot \mathfrak{A}'|| \cdot 0$. The transition matrix of $\mathfrak{A} \cdot \mathfrak{A}'$ is given by

$$A = \begin{pmatrix} a & fh & b & fi \\ 0 & a' & 0 & b' \\ c & gh & d & gi \\ 0 & c' & 0 & d' \end{pmatrix} .$$

We now compute the first $m + k$ entries of $A^{\omega, m+k}$. This column vector of dimension $m + k$ is given by

$$\begin{aligned}
&\left(\begin{pmatrix} a & fh \\ 0 & a' \end{pmatrix} + \begin{pmatrix} b & fi \\ 0 & b' \end{pmatrix} \begin{pmatrix} d^* & d^* g i d'^* \\ 0 & d'^* \end{pmatrix} \begin{pmatrix} c & gh \\ 0 & c' \end{pmatrix} \right)^\omega \\
&= \begin{pmatrix} a + bd^*c & (f + bd^*g)(h + id'^*c') \\ 0 & a' + b'd'^*c' \end{pmatrix}^\omega \\
&= \begin{pmatrix} (a + bd^*c)^\omega + (a + bd^*c)^*(f + bd^*g)(h + id'^*c')(a' + b'd'^*c')^\omega \\ (a' + b'd'^*c')^\omega \end{pmatrix} .
\end{aligned}$$

The last $n + n' - (m + k)$ entries of $A^{\omega, m+k}$ are given by the product of

$$\begin{pmatrix} d & gi \\ 0 & d' \end{pmatrix}^* \begin{pmatrix} c & gh \\ 0 & c' \end{pmatrix} = \begin{pmatrix} d^*c & d^*g(h + id'^*c') \\ 0 & d'^*c' \end{pmatrix}$$

with the column vector computed above. Hence, we obtain

$$\begin{aligned}
&||\mathfrak{A} \cdot \mathfrak{A}'|| \cdot 0 \\
&= (i_1 \ 0 \ i_2 \ 0) A^{\omega, m+k} \\
&= i_1(a + bd^*c)^\omega + i_1(a + bd^*c)^*(f + bd^*g)(h + id'^*c)(a' + b'd'^*c')^\omega \\
&\quad + i_2 d^*c(a + bd^*c)^\omega + i_2 d^*c(a + bd^*c)^*(f + bd^*g)(h + id'^*c')(a' + b'd'^*c')^\omega \\
&\quad + i_2 d^*g(h + id'^*c')(a' + b'd'^*c')^\omega .
\end{aligned}$$

On the other side, we obtain by Lemma 3.5

$$\begin{aligned}
&||\mathfrak{A}|| \cdot ||\mathfrak{A}'|| \cdot 0 \\
&= ||\mathfrak{A}|| \cdot 0 + ||\mathfrak{A}||\P \cdot ||\mathfrak{A}'|| \cdot 0 \\
&= i_1(a + bd^*c)^\omega + i_2 d^*c(a + bd^*c)^\omega + (i_1(a + bd^*c)^*(f + bd^*g) \\
&\quad + i_2 d^*c(a + bd^*c)^*(f + bd^*g) + i_2 d^*g)(h + id'^*c')(a' + b'd'^*c')^\omega .
\end{aligned}$$

Hence, $||\mathfrak{A} \cdot \mathfrak{A}'|| \cdot 0 = ||\mathfrak{A}|| \cdot ||\mathfrak{A}'|| \cdot 0$.

We now compute $||\mathfrak{A} \cdot \mathfrak{A}'||\P$. If, in the transition matrix A of $\mathfrak{A} \cdot \mathfrak{A}'$ we commute the $m+1, \ldots, m+k$ row and column with the $m+k+1, \ldots, n+k$ row and column, and do the same with the initial and final vector we obtain by the star permutation identity (see Conway [4], Bloom, Ésik [2], Ésik, Kuich [9] and Ésik [8]),

$$||\mathfrak{A}\cdot\mathfrak{A}'||\P = (i_1\ i_2\ 0\ 0)\begin{pmatrix} a & b & fh & fi \\ c & d & gh & gi \\ 0 & 0 & a' & b' \\ 0 & 0 & c' & d' \end{pmatrix}^*$$
$$= (i_1\ i_2)\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* \begin{pmatrix} f \\ g \end{pmatrix} (h\ i) \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}^* \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$
$$= ||\mathfrak{A}||\P \cdot ||\mathfrak{A}'||\P = ||\mathfrak{A}|| \cdot ||\mathfrak{A}'||\P\,.$$

Hence, $||\mathfrak{A}\cdot\mathfrak{A}'|| = ||\mathfrak{A}|| \cdot ||\mathfrak{A}'||$. □

Let $\mathfrak{A} = (n; h, i; a, b, c, d; f, g; k)$ be a finite S'-automaton and write $R = (h\quad i)$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $P = \begin{pmatrix} f \\ g \end{pmatrix}$. Then we define the finite S'-automaton $\mathfrak{A}^{\otimes}$ to be

$$\mathfrak{A}^{\otimes} = (1+n+n; (1\ 0), (0\ 0); \begin{pmatrix} 0 & h \\ 0 & a \end{pmatrix}, \begin{pmatrix} i & R \\ b & 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & c \\ P & 0 \end{pmatrix}, \begin{pmatrix} d & 0 \\ 0 & A \end{pmatrix}; \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}; 1+k)\,.$$

Theorem 3.7 *Let $\mathfrak{A}$ be a finite S'-automaton. Then $||\mathfrak{A}^{\otimes}|| = ||\mathfrak{A}||^{\otimes}$.*

Proof. Let $\mathfrak{A}$ be defined as above. Let

$$A' = \begin{pmatrix} 0 & h & i & R \\ 0 & a & b & 0 \\ 0 & c & d & 0 \\ P & 0 & 0 & A \end{pmatrix}\,.$$

We first compute $||\mathfrak{A}^{\otimes}||\P$. Observe that A' can be written as

$$A' = \begin{pmatrix} 0 & R & R \\ 0 & A & 0 \\ P & 0 & A \end{pmatrix}$$

and that $||\mathfrak{A}^{\otimes}||\P = (A'^*)_{11}$. We obtain

$$(A'^*)_{11} = \left((R\ R) \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}^* \begin{pmatrix} 0 \\ P \end{pmatrix} \right)^*$$
$$= (RA^*P)^* = (||\mathfrak{A}||\P)^* = ||\mathfrak{A}||^{\otimes}\P\,.$$

We now compute the first $1+k$ entries of $A^{\omega,1+k}$. This column vector of dimension $1+k$ is given by

$$\left(\begin{pmatrix} 0 & h \\ 0 & a \end{pmatrix} + \begin{pmatrix} i & R \\ b & 0 \end{pmatrix} \begin{pmatrix} d & 0 \\ 0 & A \end{pmatrix}^* \begin{pmatrix} 0 & c \\ P & 0 \end{pmatrix} \right)^{\omega}$$
$$= \begin{pmatrix} RA^*P & h+id^*c \\ 0 & a+bd^*c \end{pmatrix}^{\omega}\,.$$

Hence,

$$||\mathfrak{A}^{\otimes}|| \cdot 0 = (A^{\omega,1+k})_1 = (RA^*P)^\omega + (RA^*P)^*(h + id^*c)(a + bd^*c)^\omega .$$

By definition, $||\mathfrak{A}||^{\otimes} \cdot 0 = (||\mathfrak{A}||\P)^\omega + (||\mathfrak{A}||\P)^*||\mathfrak{A}|| \cdot 0$. Thus

$$||\mathfrak{A}||^{\otimes} \cdot 0 = (RA^*P)^\omega + (RA^*P)^*(h + id^*c)(a + bd^*c)^\omega = ||\mathfrak{A}^{\otimes}|| \cdot 0$$

and we obtain $||\mathfrak{A}^{\otimes}|| = ||\mathfrak{A}||^{\otimes}$. □

Theorem 3.8 *Let $\mathfrak{A} = (n, R, A, P, k)$ be a finite S'-automaton. Then there exists a finite S'-automaton $\mathfrak{A}\P$ such that $||\mathfrak{A}\P|| = ||\mathfrak{A}||\P$.*

Proof. $||\mathfrak{A}||\P = (n, R, A, P, 0)$. □

Theorem 3.9 *Let $a \in S' \cup \{0, 1\}$. Then there exists a finite S'-automaton $\mathfrak{A}_a$ such that $||\mathfrak{A}_a|| = a$.*

Proof. Let $\mathfrak{A}_a = (2, (1\ 0), \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, 0)$. Then

$$||\mathfrak{A}_a|| = (1\ 0) \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = a .$$

□

Corollary 3.10 *The behaviors of finite S'-automata form a generalized star quemiring that contains S'.*

Theorem 3.11 (Kleene Theorem**)** *Let (S, V) be a Conway semiring-semi-semimodule pair. Then the following statements are equivalent for $(s, v) \in S \times V$:*

(i) $(s, v) = ||\mathfrak{A}||$, where $\mathfrak{A}$ is a finite S'-automaton,
(ii) $(s, v) \in \omega\text{-}\mathfrak{Rat}(S')$,
(iii) $s \in \mathfrak{Rat}(S')$ and v is of the form $\sum_{1 \le i \le m} s_i t_i^\omega$ with $s_i, t_i \in \mathfrak{Rat}(S')$.

Proof. (i) ⇒ (iii): Each entry in A^* and $A^{\omega,k}$ is of the form s and $\sum_{1 \le i \le m} s_i t_i^\omega$, respectively, with $s, s_i, t_i \in \mathfrak{Rat}(S')$.

(iii) ⇒ (ii): $(s, v) = (s, 0) + (0, v)$. Since $(s, 0)$ is in $\mathfrak{Rat}(S') \subseteq \omega\text{-}\mathfrak{Rat}(S')$ and $(0, v) = (0, \sum_{1 \le i \le m} s_i t_i^\omega)$ is in $\omega\text{-}\mathfrak{Rat}(S')$, (s, v) is in $\omega\text{-}\mathfrak{Rat}(S')$.

(ii) ⇒ (i): By Corollary 3.10. □

We now consider finite $S\langle \Sigma \cup \{\varepsilon\}\rangle$-automata over the Conway semiring-semimodule pair $(S\langle\langle \Sigma^* \rangle\rangle, S\langle\langle \Sigma^\omega \rangle\rangle)$, where S is a complete star-omega semiring, and will delete ε-moves in these finite $S\langle \Sigma \cup \{\varepsilon\}\rangle$-automata without changing their behavior.

Theorem 3.12 *Let $(S\langle\langle \Sigma^* \rangle\rangle, S\langle\langle \Sigma^\omega \rangle\rangle)$ be a Conway semiring-semimodule pair satisfying $(s\varepsilon)^\omega = 0$ for all $s \in S$. Let $\mathfrak{A} = (n, R, A, P, k)$ be a finite $S\langle \Sigma \cup \{\varepsilon\}\rangle$-automaton. Then there exists a finite $S\langle \Sigma \cup \{\varepsilon\}\rangle$-automaton $\mathfrak{A}' = (n, R', A', P', k)$ with $||\mathfrak{A}'|| = ||\mathfrak{A}||$ satisfying the following conditions:*

(i) $A' \in (S\langle \Sigma \rangle)^{n \times n}$,
(ii) $R' \in (S\langle \{\varepsilon\} \rangle)^{1 \times n}$,
(iii) $P' \in (S\langle \{\varepsilon\} \rangle)^{n \times 1}$.

Proof. Without loss of generality we assume by Theorem 3.4 that $R \in (S\langle \{\varepsilon\} \rangle)^{1 \times n}$ and $P \in (S\langle \{\varepsilon\} \rangle)^{n \times 1}$. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a is a $k \times k$-matrix and d is a $(n-k) \times (n-k)$-matrix. Let $a = a_0 + a_1$, $b = b_0 + b_1$, $c = c_0 + c_1$, $d = d_0 + d_1$, such that the supports of the entries of a_0, b_0, c_0, d_0 (resp. a_1, b_1, c_1, d_1) are subsets of $\{\varepsilon\}$ (resp. Σ). By assumption, $(a_0 + b_0 d_0^* c_0)^\omega = 0$.

Define the matrices A_{01}, A_{02} and A_1 to be $A_{01} = \begin{pmatrix} 0 & b_0 \\ 0 & d_0 \end{pmatrix}$, $A_{02} = \begin{pmatrix} a_0 & 0 \\ c_0 & 0 \end{pmatrix}$ and $A_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$. We now specify the finite $S\langle \Sigma \cup \{\varepsilon\} \rangle$-automaton $\mathfrak{A}'$: $R' = R(A_{01}^* A_{02})^*$, $A' = A_{01}^* A_1 (A_{01}^* A_{02})^*$ and $P' = A_{01}^* P$. The behavior of $\mathfrak{A}'$ is then given by

$$
\begin{aligned}
||\mathfrak{A}'|| &= R'A'^*P' + R'A'^{\omega,k} \\
&= R(A_{01}^*A_{02})^*(A_{01}^*A_1(A_{01}^*A_{02})^*)^*A_{01}^*PR(A_{01}^*A_{02})^*(A_{01}^*A_1(A_{01}^*A_{02})^*)^{\omega,k} \\
&= R(A_{01} + A_{02} + A_1)^*P + R(A_{01} + A_{02} + A_1)^{\omega,k} \\
&= RA^*P + RA^{\omega,k} = ||\mathfrak{A}|| \,.
\end{aligned}
$$

Here we have applied Theorem 2.10 of Ésik, Kuich [11] in the third equality. □

In the next theorem we construct a finite $S\langle \Sigma \cup \{\varepsilon\} \rangle$-automaton without ε-moves with a unique initial state of initial weight ε.

Theorem 3.13 *Let $(S\langle\langle \Sigma^* \rangle\rangle, S\langle\langle \Sigma^\omega \rangle\rangle)$ be a Conway semiring-semimodule pair as in Theorem 3.12, and consider a finite $S\langle \Sigma \cup \{\varepsilon\} \rangle$-automaton $\mathfrak{A} = (n, R, A, P, k)$. Then there exists a finite $S\langle \Sigma \cup \{\varepsilon\} \rangle$-automaton $\mathfrak{A}' = (n+1, R', A', P', k)$ with $||\mathfrak{A}'|| = ||\mathfrak{A}||$ satisfying the following conditions:*

(i) $A' \in (S\langle \Sigma \rangle)^{(n+1) \times (n+1)}$,
(ii) $R'_j = 0$, $1 \le j \le n$, *and* $R'_{n+1} = \varepsilon$,
(iii) $P' \in (S\langle \{\varepsilon\} \rangle)^{(n+1) \times 1}$.

Proof. We assume that $\mathfrak{A}$ satisfies the conditions of Theorem 3.12. We specify $\mathfrak{A}'$ by $R' = (0 \ \varepsilon)$, $A' = \begin{pmatrix} A & 0 \\ RA & 0 \end{pmatrix}$ and $P' = \begin{pmatrix} P \\ RP \end{pmatrix}$. We compute $A'^* = \begin{pmatrix} A^* & 0 \\ RAA^* & \varepsilon \end{pmatrix}$ and, for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $R = (i_1 \ i_2)$,

$$
A'^{\omega,k} = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ i_1 a + i_2 c & i_1 b + i_2 d & 0 \end{pmatrix}^{\omega,k}
$$

$$= \begin{pmatrix} (a+bd^*c)^\omega \\ d^*c(a+bd^*c)^\omega \\ (i_1(a+bd^*c)+i_2d^*c)(a+bd^*c)^\omega \end{pmatrix}$$
$$= \begin{pmatrix} A^{\omega,k} \\ RA^{\omega,k} \end{pmatrix} .$$

Hence, $||\mathfrak{A}'|| = RAA^*P + RP + RA^{\omega,k} = ||\mathfrak{A}||$. □

In the case of the Boolean semiring, the finite $\mathbb{B}\langle\Sigma\cup\{\varepsilon\}\rangle$-automata of Theorem 3.13 with $P' = 0$ are nothing else than the finite automata introduced by Büchi [3].

In the case of the semiring $\mathbb{N}^\infty$ we get the following result.

Theorem 3.14 *The constructions of Theorems 3.12 and 3.13 do not change, for $w \in \Sigma^*$ (resp. for $w \in \Sigma^\omega$), in the digraphs of the finite automata, the number of finite paths with label w from an initial state to a final state (resp. the number of infinite paths with label w starting in an initial state and passing infinitely often through repeated states).*

3.3 Finite linear systems over quemirings

In this subsection we consider finite linear systems over quemirings as a generalization of regular grammars with finite and infinite derivations.

A *finite S'-linear system* (*with variables $y_1, \ldots, y_n$, over the quemiring $S \times V$*) is a system of equations

$$y = Ay + P \tag{22}$$

where $A \in (S' \cup \{0,1\})^{n\times n}$, $P \in (S' \cup \{0,1\})^{n\times 1}$, $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$. A column vector $\sigma \in (S \times V)^{n\times 1}$ is called a *solution* to the system (22) if

$$\sigma = A\sigma + P .$$

Theorem 3.15 *Let (S, V) be a Conway semiring-semimodule pair. Consider a finite S'-linear system*

$$y = Ay + P ,$$

where $A \in (S' \cup \{0,1\})^{n\times n}$, $P \in (S' \cup \{0,1\})^{n\times 1}$, and $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ is a column vector of variables. Then, for each $0 \le k \le n$, $A^{\omega,k} + A^*P$ is a solution of $y = Ay + P$.

Proof. We obtain, by Theorem 3.1 and the star and omega fixed point identities (cf. Ésik [8]), for each $0 \le k \le n$,

$$A(A^{\omega,k} + A^*P) + P = A^{\omega,k} + A^*P\,.$$

□

Let $k \in \{0, \ldots, n\}$ and $\mathfrak{A}_i = (n, e_i, A, P, k)$, $1 \le i \le n$, be finite S'-automata, where e_i is the i-th vector of unity. Then $||\mathfrak{A}_i||$ is the i-th component of a solution given in Theorem 3.15 of the finite S'-linear system $y = Ay + P$. Therefore, we call the solution $\begin{pmatrix} ||\mathfrak{A}_1|| \\ \vdots \\ ||\mathfrak{A}_n|| \end{pmatrix} = A^{\omega,k} + A^*P$ of $y = Ay + P$ the *k-th automata-theoretic solution* of $y = Ay + P$.

Theorem 3.16 *Let (S, V) be a Conway semiring-semimodule pair and $S' \subseteq S$. Let $\mathfrak{A} = (n, R, A, P, k)$ be a finite S'-automaton. Then $||\mathfrak{A}|| = R\sigma$, where σ is the k-th automata-theoretic solution of the finite S'-linear system $y = Ay + P$.*

Bi-inductive semiring-semimodule pairs were defined in Ésik, Kuich [13], see also Ésik [8]. As shown in these papers, every bi-inductive semiring-semimodule pair is a Conway semiring-semimodule pair. Suppose that (S, V) is a bi-inductive semiring-semimodule pair and consider the quemiring $S \times V$, which is equipped with the pointwise partial order inherited from the orders on S and V. Moreover, consider the above finite S'-linear system $y = Ay + P$ and write it in more detail as

$$y' = ay' + by'' + p' \tag{23}$$
$$y'' = cy' + dy'' + p'' \tag{24}$$

where y' denotes the matrix of the first k entries of y, etc. We point out that the first k components of the above k-th automata theoretic solution of $y = Ay + P$ over $S \times V$ can be obtained by first taking the least solution $d^*(cy' + p'')$ of (24) and substituting this least solution into (23), and then by taking the greatest solution $(a+bd^*c)^\omega + (a+bd^*c)^*(p' + bd^*p'')$ of the resulting equation. The last k componets of the k-th automata theoretic solution are then obtained by substituting this greatest solution into $d^*(cy' + p'')$, the least solution of (24). Thus we obtain the k-th automata theoretic solution

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\omega,k} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}^* \begin{pmatrix} p' \\ p'' \end{pmatrix}.$$

Let S be a complete star-omega semiring and consider a finite S'-linear system $y = Ay + P$ over the quemiring $S\langle\langle \Sigma^* \rangle\rangle \times S\langle\langle \Sigma^\omega \rangle\rangle$ as defined before Theorem 3.15 for $S' = S\langle \Sigma \cup \{\varepsilon\} \rangle$. Write this system in the form

$$y_i = \sum_{1 \le j \le n} \sum_{x \in \Sigma \cup \{\varepsilon\}} (A_{ij}, x) x y_j + \sum_{x \in \Sigma \cup \{\varepsilon\}} (P_i, x) x \,, \quad 1 \le i \le n \,.$$

Analogous to the correspondence stated below Theorem 2.17, the right linear grammars $G_i = (\{y_1, \ldots, y_n\}, \Sigma, \Pi, y_i)$, $1 \le i \le n$, with weights in the semiring S, where $\Pi = \{y_i \to (A_{ij}, x) x y_j \mid 1 \le j \le n,\ x \in \Sigma \cup \{\varepsilon\}\} \cup \{y_i \to (P_i, x) x \mid x \in \Sigma \cup \{\varepsilon\}\}$ *correspond to* this finite $S\langle \Sigma \cup \{\varepsilon\} \rangle$-linear system. Here (A_{ij}, x) and (P_i, x) are the weights of the productions $y_i \to x y_j$ and $y_i \to x$, respectively. Furthermore, let $\mathfrak{A}_i^k = (n, e_i, A, P, k)$ be finite S'-automata, $1 \le i \le n$, for some fixed $k \in \{0, \ldots, n\}$, where e_i is the i-th row vector of unity.

Consider now a finite derivation with respect to G_i:

$$y_i \Rightarrow (A_{i,i_1}, x_1) x_1 y_{i_1} \Rightarrow \ldots \Rightarrow (A_{i,i_1}, x_1) \ldots (A_{i_{m-1},i_m}, x_m) x_1 \ldots x_m y_{i_m} \Rightarrow$$
$$(A_{i,i_1}, x_1) \ldots (A_{i_{m-1},i_m}, x_m)(P_{i_m}, x_{m+1}) x_1 \ldots x_m x_{m+1}$$

generating the word $x_1 \ldots x_m x_{m+1}$ with weight

$$(A_{i,i_1}, x_1) \ldots (A_{i_{m-1},i_m}, x_m)(P_{i_m}, x_{m+1}) \,.$$

This finite derivation corresponds to the following finite path in the directed graph of $\mathfrak{A}_i^k$:

$$(y_i, x_1, y_{i_1}), \ldots, (y_{i_{m-1}}, x_m, y_{i_m})$$

with weight

$$(A_{i,i_1}, x_1) \ldots (A_{i_{m-1},i_m}, x_m) \,,$$

initial weight 1 and final weight $(P_{i_m}, x_{m+1}) x_{m+1}$.

Consider now an infinite derivation with respect to G_i:

$$y_i \Rightarrow (A_{i,i_1}, x_1) x_1 y_{i_1} \Rightarrow \ldots \Rightarrow (A_{i,i_1}, x_1) \ldots (A_{i_{m-1},i_m}, x_m) x_1 \ldots x_m y_{i_m} \Rightarrow \ldots$$

generating the infinite word $x_1 x_2 \ldots x_m \ldots$ with weight

$$(A_{i,i_1}, x_1) \ldots (A_{i_{m-1},i_m}, x_m) \ldots \,.$$

This infinite derivation corresponds to the following infinite path in the directed graph of $\mathfrak{A}_i^k$:

$$(y_i, x_1, y_{i_1}), \ldots, (y_{i_{m-1}}, x_m, y_{i_m}), \ldots$$

with weight $(A_{i,i_1}, x_1) \ldots (A_{i_{m-1},i_m}, x_m) \ldots$ and initial weight 1.

If S is a complete star-omega semiring then $(S\langle\langle \Sigma^* \rangle\rangle, S\langle\langle \Sigma^\omega \rangle\rangle)$ is a complete semiring-semimodule pair by Ésik, Kuich [13], see also Ésik [8]. Hence, we obtain, by Theorem 3.3, the following result for G_i and $\mathfrak{A}_i^k$ as defined above.

Theorem 3.17 *If S is a complete star-omega semiring and $1 \le i \le n$, $0 \le k \le n$, then, for $w \in \Sigma^*$, $(||\mathfrak{A}_i^k||, w) = ((A^* P)_i, w)$ is the sum of the weights of all finite derivations of w with respect to G_i; and for $w \in \Sigma^\omega$, $(||\mathfrak{A}_i^k||, w) = ((A^{\omega,k})_i, w)$ is the sum of the weights of all infinite derivations of w with respect to G_i such that at least one of the variables of $\{y_1, \ldots, y_k\}$ appears infinitely often in these infinite derivations.*

In particular, if $S = \mathbb{N}^\infty$ and $(A_{ij}, x), (P_i, x) \in \{0,1\}$, $x \in \Sigma \cup \{\varepsilon\}$, $1 \le i, j \le n$, then we get the following result.

Theorem 3.18 *For $w \in \Sigma^*$, $(||\mathfrak{A}_i^k||, w) = ((A^*P)_i, w)$ is the number of finite derivations of w with respect to G_i; and for $w \in \Sigma^\omega$, $(||\mathfrak{A}_i^k||, w) = ((A^{\omega,k})_i, w)$ is the number of all infinite derivations of w with respect to G_i such that at least one of the variables of $\{y_1, \dots, y_k\}$ appears infinitely often in these infinite derivations.*

Given a right linear grammar $G_i = (\{y_1, \dots, y_n\}, \Sigma, \Pi, y_i)$, $1 \le i \le n$, with weights as above, and $k \in \{0, \dots, n\}$, $L(G_i)_k$ is defined to be the weighted language

$$L(G_i)_k = \{((A^*P)_i, w)w \mid w \in \Sigma^*\} \cup \{((A^{\omega,k})_i, w)w \mid w \in \Sigma^\omega\}\,.$$

The next theorem, Theorem 3.19, shows that such weighted languages can be generated by right linear grammars with weights in the semiring S which have only two types of productions:

$$y_i \to axy_j \quad \text{and} \quad y_i \to a\varepsilon\,,$$

where $a \in S$ and $x \in \Sigma$. Hence, in such right linear grammars there are no productions $y_i \to ay_j$. Corollary 3.20 shows then, that the two types of productions can be chosen as

$$y_i \to axy_j \quad \text{and} \quad y_i \to ax\,,$$

where $a \in S$ and $x \in \Sigma$. (Of course, ε is no longer derived. Compare this with the definition of a regular grammar below Theorem 2.17. Moreover, compare the forthcoming Corollary 3.20 with Theorem 2.17.)

Theorem 3.19 *Let $(S\langle\langle\Sigma^*\rangle\rangle, S\langle\langle\Sigma^\omega\rangle\rangle)$ be a Conway semiring-semimodule pair where $(s\varepsilon)^\omega = 0$ for all $s \in S$, consider a finite $S\langle\Sigma \cup \{\varepsilon\}\rangle$-linear system $y = Ay + P$, where $A \in (S\langle\Sigma \cup \{\varepsilon\}\rangle)^{n\times n}$, $P \in (S\langle\Sigma \cup \{\varepsilon\}\rangle)^{n\times 1}$, and $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ and let $i \in \{1, \dots, n\}$. Then there exists a finite $S\langle\Sigma \cup \{\varepsilon\}\rangle$-linear system $y' = A'y' + P'$, where $A' \in (S\langle\Sigma\rangle)^{(n+1)\times(n+1)}$, $P' \in (S\langle\{\varepsilon\}\rangle)^{(n+1)\times 1}$, and $y' = \begin{pmatrix} y \\ y_{n+1} \end{pmatrix}$ such that, for all $0 \le k \le n$, the $n+1$-st component of the k-th automata-theoretic solution of $y = Ay + P$ coincides with the i-th component of the k-th automata theoretic solution of $y' = A'y' + P'$.*

Proof. Consider the finite $S\langle\Sigma \cup \{\varepsilon\}\rangle$-automaton $\mathfrak{A}_i^k = (n, e_i, A, P, k)$, whose behavior is $||\mathfrak{A}_i^k|| = (A^*P)_i + (A^{\omega,k})_i$. Starting with $\mathfrak{A}_i^k$, perform the constructions of Theorems 3.12 and 3.13. This yields a finite $S\langle\Sigma \cup \{\varepsilon\}\rangle$-automaton $\mathfrak{A}' = (n+1, e_{n+1}, A', P', k)$ with behavior $||\mathfrak{A}'|| = (A'^*P')_{n+1} + (A'^{\omega,k})_{n+1} = ||\mathfrak{A}_i^k||$. $\square$

Corollary 3.20 *Let $(S\langle\langle \Sigma^*\rangle\rangle, S\langle\langle \Sigma^\omega\rangle\rangle)$ be a Conway semiring-semimodule pair where $(s\varepsilon)^\omega = 0$ for all $s \in S$, consider a finite $S\langle \Sigma \cup \{\varepsilon\}\rangle$-linear system $y = Ay + P$, where $A \in (S\langle \Sigma \cup \{\varepsilon\}\rangle)^{n\times n}$, $P \in (S\langle \Sigma \cup \{\varepsilon\}\rangle)^{n\times 1}$, and $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ and let $i \in \{1,\ldots,n\}$. Then there exists a finite $S\langle \Sigma \cup \{\varepsilon\}\rangle$-linear system $y' = A'y' + P'$, where $A' \in (S\langle \Sigma\rangle)^{(n+1)\times(n+1)}$, $P' \in (S\langle \Sigma\rangle)^{(n+1)\times 1}$, and $y' = \begin{pmatrix} y \\ y_{n+1} \end{pmatrix}$ such that $(A'^{\omega,k} + A'^* A' P'')_{n+1} = (A^{\omega,k} + AA^*P)_i$.*

Proof. Let $y' = A'y' + P''$ be the finite $S\langle \Sigma \cup \{\varepsilon\}\rangle$-linear system constructed according to Theorem 3.19 from $y = Ay + P$. Consider the finite $S\langle \Sigma \cup \{\varepsilon\}\rangle$-linear system $y' = A'y' + P'$, where $P' = A'P''$. Then $(A'^{\omega,k} + A'^*P')_{n+1} = (A'^{\omega,k} + A'^*A'P'')_{n+1} = (A^{\omega,k} + AA^*P)_i$. □

Corollary 3.21 *Let S be a complete star-omega semiring and consider, for some $i \in \{1,\ldots,n\}$, the right linear grammar $G_i = (\{y_1,\ldots,y_n\}, \Sigma, \Pi, y_i)$ with weights in S.*

Then there exists a right linear grammar $G^{(i)} = (\{y_1,\ldots,y_n,y_{n+1}\}, \Sigma, \Pi^{(i)}, y_{n+1})$ with weights, which has only the two types of productions

$$y_i \to axy_i \text{ and } y_i \to a\varepsilon$$
$$(\text{resp. } y_i \to axy_j \text{ and } y_i \to ax),$$

$a \in S$, $x \in \Sigma$, such that, for all $0 \le k \le n$,

$$L(G^{(i)}_{n+1})_k = L(G_i)_k$$
$$(\text{resp. } L(G^{(i)}_{n+1})_k = L(G_i)_k - \{(L(G_i)_k, \varepsilon)\varepsilon\}).$$

If we consider finite $\mathbb{N}^\infty\langle \Sigma \cup \{\varepsilon\}\rangle$-linear systems we obtain the following result about the derivations with respect to the right linear grammars G_i defined above.

Corollary 3.22 *The constructions of Theorem 3.19 and Corollary 3.20 do not change, for $w \in \Sigma^+$ (resp. for $w \in \Sigma^\omega$), the number of finite derivations of w with respect to G_i (resp. the number of infinite derivations of w with respect to G_i such that at least one of the variables of $\{y_1,\ldots,y_n\}$ appears infinitely often in these infinite derivations).*

Hence, the constructions transform unambiguous grammars into unambiguous grammars.

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Index