Clopen sets in products: CLP-compactness and h-homogeneity

Andrea Medini

Department of Mathematics
University of Wisconsin - Madison

March 2, 2012

Preliminaries

Space means topological space. Recall that a subset C of a space X is *clopen* if it is closed and open. A space X is *zero-dimensional* if it has a basis consisting of clopen sets. Every open set O in a product is the union of open rectangles:

$$O = \bigcup_{i \in I} U_i \times V_i.$$

(This is just the definition of product topology.)
The same thing holds, mutatis mutandis, for infinite products.

Question (Šostak, 1990s)

Can every clopen set in a product be written as the union of clopen rectangles?

CLP-rectangularity

Definition (Steprāns and Šostak, 2000)

A product $X = \prod_{i \in I} X_i$ is *CLP-rectangular* if every clopen subset of X is the union of clopen rectangles.

Observe that every zero-dimensional product space is CLP-rectangular, because it has a base consisting of clopen rectangles.

Theorem (Buzyakova, 2001)

There exist $X \subseteq \mathbb{R}$ and $Y \subseteq \mathbb{R}^2$ such that $X \times Y$ is not CLP-rectangular.

CLP-compactness

Definition (Šostak, 1976)

A space X is *CLP-compact* if every clopen cover of X has a finite subcover.

Examples:

- Any connected space.
- Any compact space.
- Any product (CLP-compact) $\times$ (connected) or (CLP-compact) $\times$ (compact).
- The Knaster-Kuratowski fan.
  (A non-compact, totally disconnected but not zero-dimensional subset of $\mathbb{R}^2$.)

CLP-compactness has been studied by Šostak, Sondore, Steprāns and Dikranjan (for topological groups).

Products of CLP-compact spaces

Theorem (Steprāns and Šostak, 2000; Dikranjan 2007)

Let $X = \prod_{i \in I} X_i$ be a product of CLP-compact spaces. Then X is CLP-compact if and only if X is CLP-rectangular.

We will begin by discussing the proof of the following result.

Theorem (Šostak, 1994; Steprāns and Šostak, 2000)

There exist CLP-compact spaces X_1 and X_2 such that $X_1 \times X_2$ is not CLP-compact.

Notice that X_1 and X_2 cannot be zero-dimensional.

Theorem (Steprāns, 2006)

Let $X = X_1 \times \cdots \times X_n$. If each X_i is CLP-compact and sequential then X is CLP-compact.

What about infinite products?

Question (Steprāns and Šostak, 2000)

Is there, for every infinite cardinal κ, a collection of spaces X_ξ for $\xi \in \kappa$ such that $\prod_{\xi \in F} X_\xi$ is CLP-compact for every finite $F \subseteq \kappa$, while $\prod_{\xi \in \kappa} X_\xi$ is not? Does the answer depend on κ?

Theorem (Medini, 2010)

There exists a Hausdorff space X such that X^κ is CLP-compact if and only if κ is finite.

Question (Steprāns and Šostak, 2000)

Suppose that X_i is CLP-compact and second-countable for every $i \in \omega$. Is $\prod_{i \in \omega} X_i$ CLP-compact?

Stone-Čech reminder

Let $\mathbb{N}$ be the discrete space of natural numbers. It will be useful to view the Stone-Čech remainder $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$ as the space of non-principal ultrafilters on $\mathbb{N}$. A collection $\mathcal{U}$ of subsets of $\mathbb{N}$ is a *non-principal ultrafilter* if the following conditions hold.

- If $A, B \in \mathcal{U}$ then $A \cap B \in \mathcal{U}$.
- If $A \in \mathcal{U}$ and $B \supseteq A$ the $B \in \mathcal{U}$.
- (Ultra) For every $A \subseteq \mathbb{N}$, either $A \in \mathcal{U}$ or $\mathbb{N} \setminus A \in \mathcal{U}$.
- (Non-principal) No finite set belongs to $\mathcal{U}$.

Given $A \subseteq \mathbb{N}$, define

$$A^* = \{\mathcal{U} \in \mathbb{N}^* : A \in \mathcal{U}\}.$$

The collection of all sets in this form is a clopen basis for $\mathbb{N}^*$. Also, one can show that every clopen set of $\mathbb{N}^*$ is in that form.

Steprāns and Šostak's method

Fix a collection $\mathcal{F}$ consisting of non-empty closed subsets of $\mathbb{N}^*$. Define the space $X(\mathcal{F})$ as the space with underlying set $\mathbb{N} \cup \mathcal{F}$ and the coarsest topology such that

- $\{n\}$ is open for every $n \in \mathbb{N}$,
- $\{K\} \cup A$ is open whenever $K \in \mathcal{F}$, $A \subseteq \mathbb{N}$ and $K \subseteq A^*$.

Proposition (Steprāns and Šostak, 2000)

If $\mathcal{F}$ consists of pairwise disjoint subsets of $\mathbb{N}^*$ then $X(\mathcal{F})$ is a Hausdorff space.

Proof: disjoint closed sets in $\mathbb{N}^*$ can be separated by a clopen set. If $K_1 \subseteq A^*$ and $K_2 \subseteq \mathbb{N}^* \setminus A^* = (\mathbb{N} \setminus A)^*$ then $\{K_1\} \cup A$ and $\{K_2\} \cup (\mathbb{N} \setminus A)$ are the required neighborhoods.

Ensuring CLP-compactness of the factors

Proposition (Steprāns and Šostak, 2000)

Assume that for every clopen set C in $X(\mathcal{F})$, either $A = C \cap \mathbb{N}$ is finite or A is cofinite. Then $X(\mathcal{F})$ is CLP-compact.

To achieve such goal, it is enough to make sure that whenever $A \subseteq \mathbb{N}$ is infinite and coinfinite, there exists $K \in \mathcal{F}$ which is in the closure of both A and $\mathbb{N} \setminus A$. In our case we will set $K = \{\mathcal{U}, \mathcal{V}\}$, where $\mathcal{U} \in A^*$ and $\mathcal{V} \in (\mathbb{N} \setminus A)^*$.
Just enumerate as $\{A_\xi : \xi \in \mathfrak{c}\}$ all such As, then construct $\mathcal{F}$ by transfinite induction in $\mathfrak{c}$ steps: take care of A_ξ at the ξ-th stage.

Making CLP-compactness fail in the product

Proposition (Steprāns and Šostak, 2000)

If $(\bigcup \mathcal{F}_1) \cap (\bigcup \mathcal{F}_2) = \varnothing$ then $X(\mathcal{F}_1) \times X(\mathcal{F}_2)$ can be partitioned into infinitely many non-empty clopen subsets. In particular, it is not CLP-compact.

Proof: clearly, each $\{(n, n)\}$ is open (actually, clopen).
But we also claim that the complement of the diagonal $\Delta = \{(n, n) : n \in \mathbb{N}\}$ is open.
For example, take (K_1, K_2), where each $K_i \in \mathcal{F}_i$. Since $K_1 \cap K_2 = \varnothing$, there exist disjoint $A_1, A_2 \subseteq \mathbb{N}$ such that each $K_i \subseteq A_i^*$. Then $(\{K_1\} \cup A_1) \times (\{K_2\} \cup A_2)$ is an open neighborhood of (K_1, K_2) disjoint from the diagonal.

Constructing our counterexample

1. Construct $\mathcal{F}_i$ for each $i \in \omega$ so that, whenever $n \in \omega$ and $p : n \longrightarrow \omega$, the finite product $X_p = \prod_{i \in n} X(\mathcal{F}_{p(i)})$ is CLP-compact.
2. At the same time, make sure that $\prod_{i \in \omega} X(\mathcal{F}_i)$ can be partitioned into infinitely many disjoint clopen sets. (This uses a trick by Comfort.)
3. Using a machine invented by Frolík for pseudocompact spaces, convert the above example into a single space X such that X^κ is CLP-compact iff κ is finite:

$$X = X(\mathcal{F}_1) \oplus X(\mathcal{F}_2) \oplus \cdots \oplus \{*\}$$

with a natural topology.

CLP-compactness of the finite subproducts

The following definitions isolate the multidimensional versions of 'finiteness' and 'cofiniteness' that we need. Fix $p : n \longrightarrow \omega$.

$$\mathbb{S}_p^N = \bigcup_{i \in n} \Big(\mathbb{N} \times \cdots \times \underbrace{\{0, 1, \ldots, N-1\}}_{i\text{-th coordinate}} \times \cdots \times \mathbb{N} \Big) \subseteq X_p.$$

$$\mathbb{T}_p^N = \mathbb{N}^n \setminus \mathbb{S}_p^N = (\mathbb{N} \setminus \{0, 1, \ldots, N-1\})^n \subseteq X_p.$$

Proposition

Assume that for every clopen set $C \subseteq X_p$, either $C \cap \mathbb{N}^n \subseteq \mathbb{S}_p^N$ or $\mathbb{T}_p^N \subseteq C \cap \mathbb{N}^n$ for some $N \in \omega$. Then X_p is CLP-compact.

Making CLP-compactness fail in the product

Proposition

Assume that whenever $I \subseteq \omega$ is infinite and $K_i \in \mathcal{F}_i$ for every $i \in I$, there exist $i, j \in I$ such that $K_i \cap K_j = \varnothing$. Then $\prod_{i \in \omega} X(\mathcal{F}_i)$ can be partitioned into infinitely many non-empty clopen subsets. In particular, it is not CLP-compact.

"Proof": Notice that, for each $n \in \omega$, the set

$$C_n = \underbrace{\{n\} \times \cdots \times \{n\}}_{\text{first } n \text{ coordinates}} \times X(\mathcal{F}_n) \times X(\mathcal{F}_{n+1}) \times \cdots$$

is open (actually, clopen). Using the assumption, one can show that the complement of $\bigcup_{n \in \omega} C_n$ is also open.

A scary excerpt: the successor stage

At a successor stage $\xi = \eta + 1$, assume that each $\mathcal{F}_i^\eta$ is given. Let $p = p(\eta) : n \longrightarrow \omega$. First, define $W = \bigcup_{i \in \omega} \bigcup \mathcal{F}_i^\eta$. Set $\tau_i = \pi_i \upharpoonright D_\eta$ for every $i \in n(p)$. Since each τ_i is injective, it makes sense to consider the induced function $\tau_i^* : D_\eta^* \longrightarrow \mathbb{N}^*$.It is possible to choose

$$\mathcal{U}^\eta \in D_\eta^* \setminus ((\tau_0^*)^{-1}[W] \cup \cdots \cup (\tau_{n(p)-1}^*)^{-1}[W]).$$

Let $\mathcal{U}_i^\eta = \tau_i^*(\mathcal{U}^\eta)$ for every $i \in n$.
Similarly choose $\mathcal{V}_i^\eta$ for every $i \in n$.
Conclude the successor stage by setting

$$\mathcal{F}_k^{\eta+1} = \mathcal{F}_k^\eta \cup \{\{\mathcal{U}_i^\eta : i \in p^{-1}(k)\} \cup \{\mathcal{V}_i^\eta : i \in p^{-1}(k)\}\}$$

for every $k \in \mathrm{ran}(p)$ and $\mathcal{F}_k^{\eta+1} = \mathcal{F}_k^\eta$ for every $k \in \omega \setminus \mathrm{ran}(p)$.

h-Homogeneity

Definition (Ostrovskiĭ, 1981; Van Mill, 1981)

A topological space X is *h-homogeneous* (or *strongly homogeneous*) if every non-empty clopen subset of X is homeomorphic to X.

Examples:

- The Cantor set 2^ω, the rationals $\mathbb{Q}$, the irrationals ω^ω. (Use their respective characterizations.)
- Any connected space.
- Any product (h-homogeneous) $\times$ (connected space).
- The Knaster-Kuratowski fan.
- Erdös space $\mathfrak{E} = \{x \in \ell^2 : x_n \in \mathbb{Q} \text{ for all } n \in \omega\}$. (Deep result due to Dijkstra and Van Mill.)

h-Homogeneity has been studied by Terada, Matveev, Medvedev, De La Vega, Motorov, Shelah and Geshcke.

Overview of our results on h-homogeneity

1. In the class of zero-dimensional spaces, h-homogeneity is productive.
2. If the product is pseudocompact, then the zero-dimensionality requirement can be dropped.
3. Clopen sets in pseudocompact products depend on finitely many coordinates.
4. Partial answers to Terada's question: is the infinite power X^ω h-homogeneous for every zero-dimensional first-countable X? Notice that this could be called an 'h-homogeneous Dow-Pearl theorem'.

A useful base for βX

Definition

Given U open in X, define $\text{Ex}(U) = \beta X \setminus \text{cl}_{\beta X}(X \setminus U)$.

Basic facts:

- $\text{Ex}(U)$ is the biggest open set in βX such that its intersection with X is U.
- The collection $\{\text{Ex}(U) : U \text{ is open in } X\}$ is a base for βX.
- If C is clopen in X then $\text{Ex}(C) = \text{cl}_{\beta X}(C)$, hence $\text{Ex}(C)$ is clopen in βX.

☢ It is not true that βX is zero-dimensional whenever X is zero-dimensional. (Dowker, 1957.) ☢

When does β commute with $\prod$?

Theorem (Glicksberg, 1959)

The product $\prod_{i\in I} X_i$ is C^*-embedded in $\prod_{i\in I}\beta X_i$ if and only if $\prod_{i\in I} X_i$ is pseudocompact.

In that case,

$$\prod_{i\in I}\beta X_i \cong \beta\left(\prod_{i\in I} X_i\right).$$

More precisely, there exists a homeomorphism

$$h : \prod_{i\in I}\beta X_i \longrightarrow \beta\left(\prod_{i\in I} X_i\right)$$

such that $h \restriction \prod_{i\in I} X_i = \mathrm{id}$.

The productivity of h-homogeneity

Theorem (Terada, 1993)

If X_i is h-homogeneous and zero-dimensional for every $i \in I$ and $P = \prod_{i \in I} X_i$ is compact or non-pseudocompact, then P is h-homogeneous.

Proof of the compact case, for $P = X \times Y$:
Observe that $n \times X \cong X$ whenever $1 \leq n < \omega$.
So $n \times X \times Y \cong X \times Y$ whenever $1 \leq n < \omega$.
Let C be non-empty and clopen in $X \times Y$. By compactness, zero-dimensionality and ✂, find clopen rectangles C_i such that

$$C = C_1 \oplus \cdots \oplus C_n.$$

By h-homogeneity, $C \cong n \times X \times Y \cong X \times Y$.

Theorem

If $X \times Y$ is pseudocompact, then every clopen set C can be written as a finite union of open rectangles.

Proof: By Glicksberg's theorem, there exists a homeomorphism

$$h : \beta X \times \beta Y \longrightarrow \beta(X \times Y)$$

such that $h(x, y) = (x, y)$ whenever $(x, y) \in X \times Y$.
Since $\{\mathrm{Ex}(U) : U \text{ is open in } X\}$ is a base for βX and $\{\mathrm{Ex}(V) : V \text{ is open in } Y\}$ is a base for βY, the collection

$$\mathcal{B} = \{\mathrm{Ex}(U) \times \mathrm{Ex}(V) : U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$$

is a base for $\beta X \times \beta Y$.

Therefore $\{h[B] : B \in \mathcal{B}\}$ is a base for $\beta(X \times Y)$.
Hence we can write $\mathrm{Ex}(C) = h[B_1] \cup \cdots \cup h[B_n]$ for some $B_1, \ldots, B_n \in \mathcal{B}$ by compactness.
Finally, if we let $B_i = \mathrm{Ex}(U_i) \times \mathrm{Ex}(V_i)$ for each i, we get

$$\begin{aligned} C &= \mathrm{Ex}(C) \cap X \times Y \\ &= (h[B_1] \cup \cdots \cup h[B_n]) \cap h[X \times Y] \\ &= h[B_1 \cap X \times Y] \cup \cdots \cup h[B_n \cap X \times Y] \\ &= (B_1 \cap X \times Y) \cup \cdots \cup (B_n \cap X \times Y) \\ &= (U_1 \times V_1) \cup \cdots \cup (U_n \times V_n). \end{aligned}$$

But we would like *clopen* rectangles... ☹
Why? Because then we could prove the following.
(Notice that zero-dimensionality is not needed.)

Theorem

Assume that $X \times Y$ is pseudocompact. If X and Y are h-homogeneous then $X \times Y$ is h-homogeneous.

Proof: If X and Y are both connected then $X \times Y$ is connected, so assume without loss of generality that X is not connected. It follows that $X \cong n \times X$ whenever $1 \leq n < \omega$.

...then finish the proof as in the compact case.

Lemma

Let $C \subseteq X \times Y$ be a clopen set that can be written as the union of finitely many rectangles. Then C can be written as the union of finitely many pairwise disjoint clopen rectangles. ☺

[✎ *Draws an enlightening picture on the board.*]

Proof: For every $x \in X$, let $C_x = \{y \in Y : (x, y) \in C\}$ be the corresponding vertical cross-section. For every $y \in Y$, let $C^y = \{x \in X : (x, y) \in C\}$ be the corresponding horizontal cross-section. Since C is clopen, each cross-section is clopen.

Let $\mathcal{A}$ be the Boolean subalgebra of the clopen algebra of X generated by $\{C^y : y \in Y\}$. Since $\mathcal{A}$ is finite, it must be atomic. Let $P_1, \ldots, P_m$ be the atoms of $\mathcal{A}$. Similarly, let $\mathcal{B}$ be the Boolean subalgebra of the clopen algebra of Y generated by $\{C_x : x \in X\}$, and let $Q_1, \ldots, Q_n$ be the atoms of $\mathcal{B}$.

Observe that the rectangles $P_i \times Q_j$ are clopen and pairwise disjoint. Furthermore, given any i, j, either $P_i \times Q_j \subseteq C$ or $P_i \times Q_j \cap C = \varnothing$. Hence C is the union of a (finite) collection of such rectangles.

☕

Corollary

Assume that $X = X_1 \times \cdots \times X_n$ is pseudocompact. If each X_i is h-homogeneous then X is h-homogeneous.

An obvious modification of the proof of the theorem yields:

Theorem

Assume that $X = \prod_{i \in I} X_i$ is pseudocompact. If $C \subseteq X$ is clopen then C can be written as the union of finitely many open rectangles.

Corollary

Assume that $X = \prod_{i \in I} X_i$ is pseudocompact. If $C \subseteq X$ is clopen then C depends on finitely many coordinates.

[*The speaker takes a walk down memory lane...*]

Theorem

Assume that $X = \prod_{i \in I} X_i$ is pseudocompact. If X_i is h-homogeneous for every $i \in I$ then X is h-homogeneous.

Proof: let $C \subseteq X$ be clopen and non-empty.
Then there exists a finite subset F of I such that C is homeomorphic to $C' \times \prod_{i \in I \setminus F} X_i$, where C' is a clopen subset of $\prod_{i \in F} X_i$.
But $\prod_{i \in F} X_i$ is h-homogeneous, so

$$C \cong C' \times \prod_{i \in I \setminus F} X_i \cong \prod_{i \in F} X'_i \times \prod_{i \in I \setminus F} X_i \cong X.$$

Conclusions on products of h-homogeneous spaces

Putting together our results with Terada's theorem, we obtain the following.

Theorem (Medini, 2011)

If X_i is h-homogeneous and zero-dimensional for every $i \in I$ and $X = \prod_{i \in I} X_i$ then X is h-homogeneous.

After all this work...

Question

Is h-homogeneity productive?

Notice that any counterexample product space would have to be non-zero-dimensional and non-pseudocompact.

CLP-compactness can help

In some cases, CLP-compactness can help in showing that a product is h-homogeneous.

Theorem

Let $X = X_1 \times \cdots \times X_n$. If each X_i is h-homogeneous, sequential and CLP-compact, then X is h-homogeneous.

Proof: By Steprāns' theorem, X is CLP-compact. So every clopen set can be written as the disjoint union of finitely many clopen rectangles.

Corollary

Any finite power of the Knaster-Kuratowski fan is h-homogeneous.

Homogeneity vs h-homogeneity

All spaces are assumed to be zero-dimensional and first-countable from now on.

Definition

A space X is *homogeneous* if for every $x, y \in X$ there exists a homeomorphism $f : X \longrightarrow X$ such that $f(x) = y$.

By a picture-proof, h-homogeneity implies homogeneity. ✎
Erik van Douwen constructed a compact homogeneous space that is not h-homogeneous.

Theorem (Motorov, 1989)

If X is a compact homogeneous space of uncountable cellularity then X is h-homogeneous.

Infinite powers

Question (Terada, 1993)

Is X^ω always h-homogeneous?

The following remarkable theorem answers a question of Gruenhage and Zhou, and is based on work by Lawrence. Partial results were obtained by Medvedev, Motorov and Van Engelen.

Theorem (Dow and Pearl, 1997)

X^ω is always homogeneous.

However, Terada's question remains open.

Motorov's main result

Theorem (Motorov, 1989)

If X has a π-base consisting of clopen sets that are homeomorphic to X then X is h-homogeneous.

Proof: Let C be a non-empty clopen set in X.
By first-countability, write

$$X = \{x\} \cup \bigcup_{n\in\omega} X_n \qquad \text{and} \qquad C = \{y\} \cup \bigcup_{n\in\omega} C_n$$

where the X_n are disjoint, clopen, they converge to x but do not contain x, and the C_n are disjoint, clopen, they converge to y but do not contain y.
[✎ *Finishes the proof by juggling with clopen sets.*]

Divisibility

Definition

A space F is a *factor* of X (or X is *divisible* by F) if there exists Y such that $F \times Y \cong X$. If $F \times X \cong X$ then F is a *strong factor* of X (or X is *strongly divisible* by F).

Problem (Motorov, 1989)

Is X^ω always divisible by 2?

As we observed already, h-homogeneity implies divisibility by 2. We will show that Terada's question is equivalent to Motorov's question. Actually, even weaker conditions suffice.

Lemma

The following are equivalent.

1. F *is a factor of* X^ω.
2. $F \times X^\omega \cong X^\omega$.
3. $F^\omega \times X^\omega \cong X^\omega$.

Proof: The implications $2 \to 1$ and $3 \to 1$ are clear.
Assume 1. Then there exists Y such that $F \times Y \cong X^\omega$, hence

$$X^\omega \cong (X^\omega)^\omega \cong (F \times Y)^\omega \cong F^\omega \times Y^\omega.$$

Since multiplication by F or by F^ω does not change the right hand side, it follows that 2 and 3 hold.

The key lemma

Lemma

$X = (Y \oplus 1)^\omega$ is h-homogeneous.

Proof: Recall that $1 = \{0\}$. For each $n \in \omega$, define

$$U_n = \underbrace{\{0\} \times \{0\} \times \cdots \times \{0\}}_{n \text{ times}} \times (Y \oplus 1) \times (Y \oplus 1) \times \cdots$$

Observe that $\{U_n : n \in \omega\}$ is a local base for X at $(0, 0, \ldots)$ consisting of clopen sets that are homeomorphic to X.
But X is homogeneous by the Dow-Pearl theorem, therefore it has a base (hence a π-base) consisting of clopen sets that are homeomorphic to X.
It follows from Motorov's result that X is h-homogeneous.

Lemma

Let $X = (Y \oplus 1)^\omega$. Then

$$X \cong Y^\omega \times (Y \oplus 1)^\omega \cong 2^\omega \times Y^\omega.$$

Proof: Observe that

$$X \cong (Y \oplus 1) \times X \cong (Y \times X) \oplus X,$$

hence $X \cong Y \times X$ by h-homogeneity. It follows that $X \cong Y^\omega \times (Y \oplus 1)^\omega$. Finally,

$$Y^\omega \times (Y \oplus 1)^\omega \cong (Y^\omega \times (Y \oplus 1))^\omega \cong (Y^\omega \oplus Y^\omega)^\omega \cong 2^\omega \times Y^\omega,$$

that concludes the proof.

Theorem (Medini, 2011)

The following are equivalent.

1. $X^\omega \cong (X \oplus 1)^\omega$.
2. $X^\omega \cong Y^\omega$ *for some* Y *with at least one isolated point.*
3. X^ω *is h-homogeneous.*
4. X^ω *has a clopen subset that is strongly divisible by* 2.
5. X^ω *has a proper clopen subspace homeomorphic to* X^ω.
6. X^ω *has a proper clopen subspace as a factor.*

Proof: The implication $1 \to 2$ is trivial; the implication $2 \to 3$ follows from the lemma; the implications $3 \to 4 \to 5 \to 6$ are trivial.

Assume that 6 holds. Let C be a proper clopen subset of X^ω that is also a factor of X^ω and let $D = X^\omega \setminus C$. Then

$$\begin{aligned} X^\omega &\cong (C \oplus D) \times X^\omega \\ &\cong (C \times X^\omega) \oplus (D \times X^\omega) \\ &\cong X^\omega \oplus (D \times X^\omega) \\ &\cong (1 \oplus D) \times X^\omega, \end{aligned}$$

hence $X^\omega \cong (1 \oplus D)^\omega \times X^\omega$. Since $(1 \oplus D)^\omega \cong 2^\omega \times D^\omega$ by the lemma, it follows that $X^\omega \cong 2^\omega \times X^\omega$.
Therefore 1 holds by the lemma.

The pseudocompact case

The next two theorems show that in the pseudocompact case we can say something more.

Theorem

Assume that X^ω is pseudocompact. Then $C^\omega \cong (X \oplus 1)^\omega$ for every non-empty proper clopen subset C of X^ω.

Theorem

Assume that X^ω is pseudocompact. Then the following are equivalent.

1. *X^ω is h-homogeneous.*
2. *X^ω has a proper clopen subspace C such that $C \cong Y^\omega$ for some Y.*

Ultraparacompactness

The following notion allows us to give us a positive answer to Terada's question for a certain class of spaces.

Definition

A space X is *ultraparacompact* if every open cover of X has a refinement consisting of pairwise disjoint clopen sets.

A metric space X is ultraparacompact if and only if $\dim X = 0$.

Theorem

If X^ω is ultraparacompact and non-Lindelöf then X^ω is h-homogeneous.