Elementary submodels of the universe

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Reasons for giving this talk.

- Practice with Beamer.
- Prerequisite to Erik's talk on PFA.
- Elementary submodels are cool.
- Arkhangelski's theorem is cool.
- The proof Arkhangelski's theorem using elementary submodels is *really* cool.
- I get to use `itemize` a lot.

Definition

Let M, N be $\mathcal{L}$-structures. We say that M is an *elementary submodel* of N (written $M \preccurlyeq N$) if for every formula $\varphi(x_1, \ldots x_n)$ and every $a_1, \ldots, a_n \in M$

$$M \vDash \varphi[a_1, \ldots, a_n] \text{ if and only if } N \vDash \varphi[a_1, \ldots, a_n]$$

- When $n = 0$, the definition says that $M \equiv N$ (i.e. M and N are *elementarily equivalent*: they satisfy the same sentences).
- When $n > 0$, the definition says that things in M have exactly the same properties in M that they have in N.

Definition

Let x be a set. The *transitive closure* of x is

$$\mathrm{trcl}(x) = \bigcup_{n\in\omega}\{y : y \text{ is an element of } \underbrace{\cdots \text{ an element of}}_{n \text{ times}} x\}$$

Definition

Let θ be a cardinal. $H(\theta) = \{x : |\mathrm{trcl}(x)| < \theta\}$.

Theorem (Reflection)

Given any finite set of formulas Δ, there are arbitrarily large cardinals θ such that every $\varphi \in \Delta$ is absolute for $H(\theta), V$.

Our language is just $\mathcal{L} = \{\in\}$.

Theorem (Löwenheim-Skolem-Tarski)

Let N be an $\mathcal{L}$-structure and $S \subseteq N$. Fix any cardinal λ with $\max(|S|, \omega) \leq \lambda \leq |N|$. Then there exists M such that

- $M \preccurlyeq N$
- $S \subseteq M$
- $|M| = \lambda$

- Idea: use the theorem with $N = V$ to get an elementary submodel of the universe.
- Problem: it doesn't make sense (V is a proper class).
- Solution: take $M \preccurlyeq H(\theta)$ for a sufficiently large cardinal θ.

Theorem

Let M be any elementary submodel. Then $\omega \subseteq M$ and $\omega \in M$.

Since M models enough of ZFC, there is a unique element 0^M that M thinks of as the empty set:

$$M \vDash 0^M \text{ is the empty set.}$$

Hence, by elementarity

$$H(\theta) \vDash 0^M \text{ is the empty set.}$$

But $H(\theta)$ is a *transitive* model of enough of ZFC, hence $0^M = 0$ is the *real* empty set by absoluteness.

Similarly, there is a unique element 1^M of M such that:

$$M \models 1^M \text{ is the successor of } 0.$$

By elementarity and absoluteness, we see that 1^M is the real 1.
If we continue this way we see that $\omega \subseteq M$.
Finally, to see that $\omega \in M$, notice that (by the Axioms of Infinity and Comprehension) there is a unique element ω^M of M such that

$$M \models \omega^M \text{ is the set of all natural numbers,}$$

Again, by elementarity and absoluteness, ω^M must be the real set of natural numbers.

Theorem

Let M be a countable elementary submodel. Then $\delta = \omega_1 \cap M$ is a (countable) ordinal.

Since obviously $\delta \subseteq \omega_1$, it is well-ordered by $\in$.
To show that δ is transitive, take $\alpha \in \delta$. We want to show $\alpha \subseteq \delta$.
Since $\alpha \in M$ and

$$H(\theta) \vDash \exists f : \omega \to \alpha \text{ which is surjective,}$$

by elementarity and absoluteness there must be such a function $f \in M$.
Since $\omega \in M$, we see that $f(n) \in M$ for every $n \in \omega$.
Hence $\alpha = \{f(n) : n \in \omega\} \subseteq M$.
Since $\alpha \subseteq \omega_1$ is obvious, we get $\alpha \subseteq \omega_1 \cap M = \delta$.

Notice that in general neither of the implications $x \in M \Rightarrow x \subseteq M$ or $x \subseteq M \Rightarrow x \in M$ is true.
For example $\omega_1 \in M$ by elementarity (for any $M \preccurlyeq H(\aleph_2)$), but $\omega_1 \subseteq M$ is impossible if M is countable.
For any $f : \omega \to M$, it's easy to see that $f \subseteq M$ (since $n \in M$ and $f(n) \in M$, we must have $\langle n, f(n) \rangle \in M$ by elementarity).
On the other hand, if $|M| < 2^{\aleph_0}$, then there must be some $f : \omega \to M$ such that $f \notin M$.

Definition

An elementary submodel M is *countably closed* if every function $f : \omega \to M$ actually belongs to M (more concisely ${}^{\omega}M \subseteq M$).

Theorem

Given any S of size at most $2^{\aleph_0}$, there is a countably closed elementary submodel M of size $2^{\aleph_0}$ such that $S \subseteq M$.

Idea of the proof: take an elementary chain of lenght ω_1, making sure $|M_\alpha| \leq 2^{\aleph_0}$ at every stage.
Start with any $M_0 \supseteq S$.
At a limit stage γ, take the union $M_\gamma = \bigcup_{\alpha<\gamma} M_\alpha$.
Successor stages: given M_α, observe that

$$|{}^\omega M_\alpha| = |M_\alpha|^{\aleph_0} \leq (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}.$$

So just take any $M_{\alpha+1} \supseteq M_\alpha \cup {}^\omega M_\alpha$.
Finally, put $M = M_{\omega_1}$.

Theorem (Arkhangelski)

If (X, τ) is a first countable compact Hausdorff topological space then $|X| \leq 2^{\aleph_0}$.

From now on we will fix θ big enough and a countably closed $M \preccurlyeq H(\theta)$ of size $2^{\aleph_0}$ such that $(X, \tau) \in M$.
The structure of the proof is the following.

- Prove that $X \cap M$ is a compact subspace of X.
- Use the compactness of $X \cap M$ to show that actually $X \subseteq M$.

Lemma

$X \cap M$ is closed in X (hence compact).

Take $x \in \overline{X \cap M}$. We will show that $x \in M$.
In $H(\theta)$: by first countability, there exists a sequence $s = \langle x_n : n \in \omega \rangle$ in $X \cap M$ converging to x.
So, since X is Hausdorff,

$$H(\theta) \vDash s \text{ has a unique limit}$$

Since $s \in M$ by ω-closedness,

$$M \vDash s \text{ has a unique limit}$$

by elementarity. Such limit must be x by elementarity.

Now assume, in order to get a contradiction, that $z \in X \setminus M$.
For every $x \in X \cap M$, let $\mathcal{B}_x \in M$ be a countable local base at x.
For every $x \in X \cap M$, choose $U_x \in \mathcal{B}_x$ such that $z \notin U_x$.
By compactness, there are $x_1, \ldots, x_n \in X \cap M$ such that

$$X \cap M \subseteq U_{x_1} \cup \cdots \cup U_{x_n}.$$

But every $U_{x_i} \in M$ (because every $\mathcal{B}_x \subseteq M$).
So we can talk about them:

$$M \models X \subseteq U_{x_1} \cup \cdots \cup U_{x_n}.$$

On the other hand, z witnesses that

$$H(\theta) \models X \not\subseteq \bigcup U_{x_1} \cup \cdots \cup U_{x_n},$$

contradicting $M \preccurlyeq H(\theta)$.