Seven characterizations of non-meager P-filters

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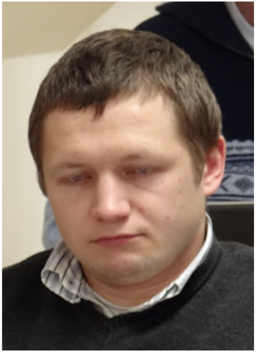
The speaker and his coauthors



One of the above is a serious and dedicated professional...
(Hint: Probably not the idiot in the middle.)

A speed-of-light introduction to P-points

Every filter $\mathcal{F}$ is on ω and $\{\text{cofinite sets}\} \subseteq \mathcal{F} \subsetneq \mathcal{P}(\omega)$.
We identify 2^ω with $\mathcal{P}(\omega)$, so that every filter $\mathcal{F} \subseteq \mathcal{P}(\omega)$ can be viewed as a topological space $\mathcal{F} \subseteq 2^\omega$.
In particular, it makes sense to say that a filter is *non-meager*.
It is easy to see that every ultrafilter is non-meager.

Definition

- A filter $\mathcal{F}$ a P*-filter* if every countable $\mathcal{X} \subseteq \mathcal{F}$ has a pseudointersection in $\mathcal{F}$.
- An ultrafilter $\mathcal{U}$ that is a P-filter is called a P*-point*.

Theorem

- *Under* CH*, there exist* P*-points. (W. Rudin, 1956.)*
- *There exist non-*P*-points. (Folklore.)*
- *It is consistent that there are no* P*-points. (Shelah, 1982.)*

An early characterization

One of the seven characterizations was already known...

Definition

A space X is *completely Baire* (briefly, CB) if every closed subspace of X is Baire.

Theorem (Marciszewski, 1998)

For a filter $\mathcal{F}$, the following are equivalent.

- *$\mathcal{F}$ is a non-meager P-filter.*
- *$\mathcal{F}$ is completely Baire.*

One direction is a very cute tiny little elegant proof... ☺

The other direction is ugly. ☹

(Note: This might just mean that I don't understand it...)

But it doesn't matter because we will give a new, more systematic proof! ☺

Countable dense homogeneity

All spaces are assumed to be separable metrizable.
Given D, E countable dense subsets of X, define $D \sim_X E$ if there exists $h \in \mathcal{H}(X)$ such that $h[D] = E$.

Definition

- We will denote by $\mathrm{TCD}(X)$ the number of equivalence classes of $\sim_X$.
- A space X is *countable dense homogeneous* (briefly, CDH) if $\mathrm{TCD}(X) = 1$.

Notice that $\mathrm{TCD}(X) \leq \mathfrak{c}$ for every space X.
Countable dense homogeneity is a well-studied topic.
Classic examples of CDH spaces are $\mathbb{R}$ (Cantor, 1895), $\mathbb{R}^n$ (Brouwer, 1913), $[0,1]^\omega$ (Fort, 1962).
Non-examples: $\mathbb{Q}$ (trivial), $\mathbb{Q}^\omega$ (Fitzpatrick and Zhou, 1992).

The main results and questions

1. A filter is CDH if and only if it is a non-meager P-filter. (Plus other six characterizations!)
2. Under $\mathfrak{u} < \mathfrak{g}$, every non-meager filter has a non-meager P-subfilter (hence a completely Baire dense subset).
3. Under $\diamondsuit$, there exists a non-meager filter with no non-meager P-subfilters.
4. It is consistent that $\mathsf{TCD}(\mathcal{F}) \in \{1, \mathfrak{c}\}$ for every filter $\mathcal{F}$. (Namely, it holds under $\mathfrak{u} < \mathfrak{g}$. This uses item 2 above.)
5. Is it true in ZFC that *every* non-meager filter has a completely Baire dense subset? Can we get at least *one*?
6. In item 3 above, can we weaken $\diamondsuit$ to CH?
7. Does the dichotomy of 4 hold in ZFC? (This would follow from an affirmative answer to 5.)

The original motivation

It is a common theme that taking powers improves homogeneity-type properties...

Question (Fitzpatrick and Zhou, 1990)

Which subspaces $X \subseteq 2^\omega$ are such that X^ω is CDH?

Theorem (Hrušák and Zamora Avilés, 2005)

For a Borel $X \subseteq 2^\omega$, the following conditions are equivalent.

- *X^ω is CDH.*
- *X is a G_δ.*

Question (Hrušák and Zamora Avilés, 2005)

Is there a non-G_δ subset X of 2^ω such that X^ω is CDH?

Theorem (Medini and Milovich, 2012)

Assume MA(countable)*.*

- *There exists an ultrafilter* $\mathcal{U}$ *such that* $\mathcal{U}$ *is* CDH*.*
- *There exists an ultrafilter* $\mathcal{U}$ *such that* $\mathcal{U}^\omega$ *is* CDH*.*

The following question is a specific instance of a **very general theme:** do combinatorial constraints on the family $\mathcal{X} \subseteq \mathcal{P}(\omega)$ have topological consequences/equivalents on the space $\mathcal{X} \subseteq 2^\omega$?

Question (Medini and Milovich, 2012)

Is every P*-point necessarily* CDH*?*

Theorem (Hernández-Gutiérrez and Hrušák, 2013)

Let $\mathcal{F}$ *be a non-meager* P*-filter. Then* $\mathcal{F}$ *and* $\mathcal{F}^\omega$ *are* CDH*.*

Great! ☺ Do non-meager P-filters exist?

It's a long-standing open problem... ☹

Question (Just, Mathias, Prikry and Simon, 1990)

Can we construct in ZFC *a non-meager* P*-filter?*

Theorem (Just, Mathias, Prikry and Simon, 1990)

If one of the following assumptions holds, then there exists a non-meager P*-filter.*

- $\mathfrak{t} = \mathfrak{b}$.
- $\mathfrak{b} < \mathfrak{d}$.
- $\mathrm{cof}([\mathfrak{d}]^{\omega}, \subseteq) \leq \mathfrak{d}$.

The last condition is particularly interesting because its negation has large cardinal strength.

Can we at least construct a non-CDH ultrafilter in ZFC?

Theorem (Medini and Milovich, 2012)

Assume MA(countable)*. Then there exists an ultrafilter that is not* CDH*.*

Theorem (Repovš, Zdomskyy and Zhang, to appear)

In ZFC*, there exists a non-meager filter that is not* CDH*.*

The next part of the talk will be about the proof of the following characterization. At the same time we will obtain five more characterizations. (Recall that $5 + 2 = 7$.)

Theorem

For a filter $\mathcal{F}$, the following are equivalent.

- *$\mathcal{F}$ is a non-meager* P*-filter.*
- *$\mathcal{F}$ is countable dense homogeneous.*

Strengthening a result of Miller

Miller showed that P-points are preserved under Miller forcing, then non-chalantly remarked that his proof can be modified to obtain the following result.

Definition

A filter $\mathcal{F}$ has the *strong Miller property* if for every countable crowded $Q \subseteq \mathcal{F}$ there exists a crowded $Q' \subseteq Q$ such that $Q' \subseteq z\uparrow$ for some $z \in \mathcal{F}$.

Theorem (Miller, 1983)

Every P*-point has the strong Miller property.*

Miller's result can be generalized as follows, yielding the **combinatorial core of our proof.**

Theorem

Every non-meager P*-filter has the strong Miller property.*

Actually, the proof gives something stronger.

Lemma (The local Miller lemma)

Let $\mathcal{F}$ be a non-meager (not necessarily P*-)filter. If $Q \subseteq \mathcal{F}$ is countable, crowded and has a pseudointersection in $\mathcal{F}$, then there exists a crowded $Q' \subseteq Q$ such that $Q' \subseteq z\uparrow$ for some $z \in \mathcal{F}$.*

Notice that the strong Miller property implies the following purely topological property.

Definition

A space X has the *Miller property* if for every countable crowded $Q \subseteq X$ there exists a crowded $Q' \subseteq Q$ with compact closure.

The Cantor-Bendixson property

Before stating the main theorem, we need two more definitions.

Definition

A space X has the *Cantor-Bendixson property* if every closed subset of X is either scattered or contains a copy of 2^ω.

Notice that the Cantor-Bendixson property is intermediate between the Miller property and being completely Baire. Just like we did for the Miller property, if $X = \mathcal{F}$ is a filter, we can consider a strong version of this property.

Definition

A filter $\mathcal{F}$ has the *strong Cantor-Bendixson property* if every closed subset of $\mathcal{F}$ is either scattered or contains a copy K of 2^ω such that $K \subseteq z\uparrow$ for some $z \in \mathcal{F}$.

The seven characterizations (finally!)

Theorem

For a filter $\mathcal{F}$, the following are equivalent.

1. *$\mathcal{F}$ is a non-meager P-filter.*
2. *$\mathcal{F}$ has the strong Miller property.*
3. *$\mathcal{F}$ has the Miller property.*
4. *$\mathcal{F}$ has the strong Cantor-Bendixson property.*
5. *$\mathcal{F}$ has the Cantor-Bendixson property.*
6. *$\mathcal{F}$ is completely Baire.*
7. *$\mathcal{F}$ is relatively countable dense homogeneous in 2^ω.*
8. *$\mathcal{F}$ is countable dense homogeneous.*

First, prove that (1), . . . , (6) are equivalent, using the proof of Marciszewski for (6) $\rightarrow$ (1). Then, (1) $\rightarrow$ (7) follows from the proof of Hernández-Gutiérrez and Hrušák, (7) $\rightarrow$ (8) is trivial, and the proof of (8) $\rightarrow$ (3) is in the next slide.

CDH filters have the Miller property

Assume that $\mathcal{F}$ is countable dense homogeneous. In particular, by a future slide (that you will probably never see), $\mathcal{F}$ must be non-meager.

Let $E = \{\text{cofinite sets}\}$. Notice that *every* subset of E has a pseudointersection in $\mathcal{F}$. (Just take ω!)

Now let $Q \subseteq \mathcal{F}$ be countable and crowded. Extend Q to a countable dense subset D of $\mathcal{F}$.

Let $h : \mathcal{F} \longrightarrow \mathcal{F}$ be a homeomorphism such that $h[D] = E$.

Since $\mathcal{F}$ is non-meager, the local Miller lemma shows that $R = h[Q]$ has a crowded subset R' with compact closure.

Since h is a homeomorphism, it follows that $Q' = h^{-1}[R']$ is a crowded subset of Q with compact closure. Therefore $\mathcal{F}$ has the Miller property.

How to get $\mathfrak{c}$ types of countable dense sets

The following is essentially due to Hrušák and Van Mill (2013).

Theorem

Assume that one of the following conditions holds.

- *X is not completely Baire but has a completely Baire dense subset.*
- *X is not Baire but every uncountable open subset of X contains a copy of 2^ω.*

Then $\mathrm{TCD}(X) = \mathfrak{c}$.

Using the first condition, we will show that it is consistent that $\mathrm{TCD}(\mathcal{F}) \in \{1, \mathfrak{c}\}$ for every filter $\mathcal{F}$. On the other hand, the second condition can be used to prove the following.

Corollary

Let $\mathcal{F}$ be a meager filter. Then $\mathrm{TCD}(\mathcal{F}) = \mathfrak{c}$.

Two consistent characterizations

Given a space $X \subseteq 2^\omega$, let RTCD(X) be the number of *relative* types of countable dense subsets of X in 2^ω.

Theorem

Assume $\mathfrak{u} < \mathfrak{g}$. For a filter $\mathcal{F}$, the following are equivalent.

1. *$\mathcal{F}$ is a non-meager* P*-filter.*
2. TCD($\mathcal{F}$) $< \mathfrak{c}$.
3. RTCD($\mathcal{F}$) $< \mathfrak{c}$.

We already know that (1) $\to$ (3), and (3) $\to$ (2) is obvious.
To show that (2) $\to$ (1), assume that that $\mathcal{F}$ is meager or a non-P-filter. We will show that TCD($\mathcal{F}$) $= \mathfrak{c}$.
If $\mathcal{F}$ is meager, we already known that TCD($\mathcal{F}$) $= \mathfrak{c}$. So we can assume that $\mathcal{F}$ is a non-meager non-P-filter.
In particular, $\mathcal{F}$ is not CB. So it would be enough to show that $\mathcal{F}$ has a CB dense subset...

How do we get a CB dense subset?

Question

Does every non-meager filter have a CB *dense subset?*

Consistently, yes. Actually, something stronger is true.

Theorem

Assume $\mathfrak{u} < \mathfrak{g}$. *Then every non-meager filter has a non-meager* P*-subfilter.*

To prove the above theorem, we will make use of some 'coherence of filters' voodoo:

Theorem (see the article of Blass in the Handbook)

Assume $\mathfrak{u} < \mathfrak{g}$. *The there exists a* P*-point* $\mathcal{U}$ *such that for every non-meager filter* $\mathcal{F}$ *there exists a finite-to-one function* $f : \omega \longrightarrow \omega$ *such that* $f(\mathcal{U}) = f(\mathcal{F})$.

Recall that, given $f : \omega \longrightarrow \omega$ and $\mathcal{X} \subseteq \mathcal{P}(\omega)$,

$$f(\mathcal{X}) = \{x \subseteq \omega : f^{-1}[x] \in \mathcal{X}\}.$$

If f is finite-to-one, it is easy to see that $f(\mathcal{F})$ is a filter (resp. ultrafilter) whenever $\mathcal{F}$ is a filter (resp. ultrafilter).
Fix a non-meager filter $\mathcal{F}$. Get a finite-to-one $f : \omega \longrightarrow \omega$ such that $f(\mathcal{U}) = f(\mathcal{F})$, where $\mathcal{U}$ is the P-point given by the theorem.
It is clear that $\langle\{f^{-1}[x] : x \in f(\mathcal{F})\}\rangle = \langle\{f^{-1}[x] : x \in f(\mathcal{U})\}\rangle$ is a subfilter of $\mathcal{F}$.
By the following proposition, it is a non-meager P-filter.

Proposition

Let $\mathcal{F}$ be a filter and let $f : \omega \longrightarrow \omega$ be finite-to-one.

1. *If $\mathcal{F}$ is a* P*-filter, then $f(\mathcal{F})$ is a* P*-filter.*
2. *If $\mathcal{F}$ is a* P*-filter, then $\langle\{f^{-1}[x] : x \in \mathcal{F}\}\rangle$ is a* P*-filter.*
3. *If $\mathcal{F}$ is non-meager, then $\langle\{f^{-1}[x] : x \in \mathcal{F}\}\rangle$ is non-meager.*

Does that hold in ZFC?

It depends on what you mean by ‘that’...
If ‘that’ $=$ ‘every non-meager filter has a non-meager P-subfilter’, then no. It is actually independent of ZFC.

Theorem

Assume $\diamond$. Then there exists an ultrafilter $\mathcal{U}$ such that whenever $\mathcal{X} \subseteq \mathcal{U}$, one of the following holds.

1. *$\mathcal{X}$ has a countable subset with no pseudointersection in $\mathcal{U}$. (In particular, $\mathcal{X}$ is not a P-filter.)*
2. *$\mathcal{X}$ has a pseudointersection. (In particular, $\mathcal{X}$ is meager.)*

Whenever you prove something using $\diamond$, there's a natural question to ask:

Question

Can we weaken to CH the assumption of $\diamond$?

If ‘that’ $=$ ‘every non-meager filter has a CB dense subset’, then we don’t know. Actually we don’t even know the answer to the following question.

Question

Can we construct in ZFC *a (non-meager) filter that is not* CB *but has a* CB *dense subset?*

The following question goes in the ‘opposite direction’.

Question

For a filter $\mathcal{F}$, is having a CB *dense subset equivalent to having a non-meager* P*-subfilter?*

Question

Let D be a CB *dense subset of a filter $\mathcal{F}$. Is $\langle D \rangle$ necessarily a (non-meager)* P*-filter?*

If ‘that’ $=$ ‘the characterization involving $\mathrm{TCD}(\mathcal{F})$’, then we also don’t know. It is equivalent to the following question.

Question

Is it consistent that there exists a filter $\mathcal{F}$ such that $1 < \mathrm{TCD}(\mathcal{F}) < \mathfrak{c}$?

The following is also open.

Question

Can we construct in ZFC a non-meager filter $\mathcal{F}$ such that $\mathrm{TCD}(\mathcal{F}) = \mathfrak{c}$?

We know that such a filter exists under $\mathfrak{u} < \mathfrak{g}$. Furthermore:

Theorem

Assume $\mathrm{MA}(\text{countable})$. Then there exists an ultrafilter $\mathcal{U}$ such that $\mathrm{TCD}(\mathcal{U}) = \mathfrak{c}$.