Zero-dimensional σ-homogeneous spaces

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Preliminaries

All spaces are assumed to be separable and metrizable.

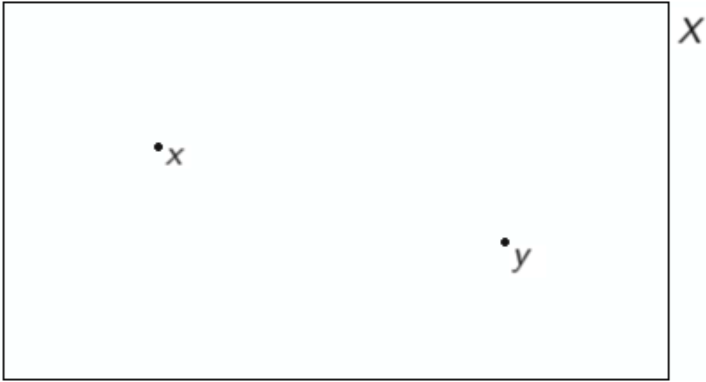
- A space X is *homogeneous* if for every $(x, y) \in X \times X$ there exists a homeomorphism $h : X \longrightarrow X$ such that $h(x) = y$.
- A zero-dimensional space X is *strongly homogeneous* if all its non-empty clopen subspaces are homeomorphic.
- A space X is *rigid* if $|X| \geq 2$ and the only homeomorphism $h : X \longrightarrow X$ is the identity.
- A space is *σ-homogeneous* if it is the union of countably many of its homogeneous subspaces.
- A space is *Borel* if it can be embedded into some Polish space as a Borel set. Similarly define *analytic* and *coanalytic*.
- A space X is *$\mathfrak{c}$-crowded* if it is non-empty and every non-empty open subset of X has size $\mathfrak{c}$.
- A space X is *nowhere $\mathcal{P}$* if it is non-empty and no non-empty open subset of X has $\mathcal{P}$.

Lemma (Terada, 1993)

If X is zero-dimensional and has a π-base consisting of clopen sets that are homeomorphic to X then X is strongly homogeneous.

Lemma (folklore)

If X is strongly homogeneous then X is homogeneous.



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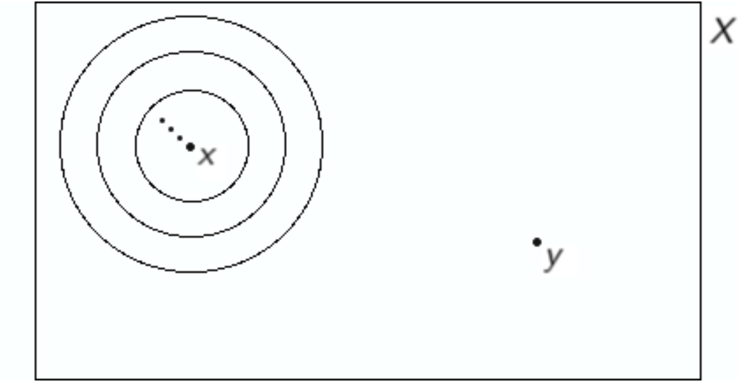

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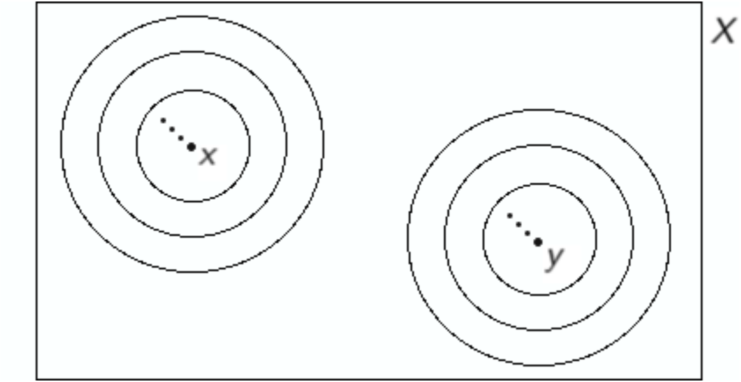

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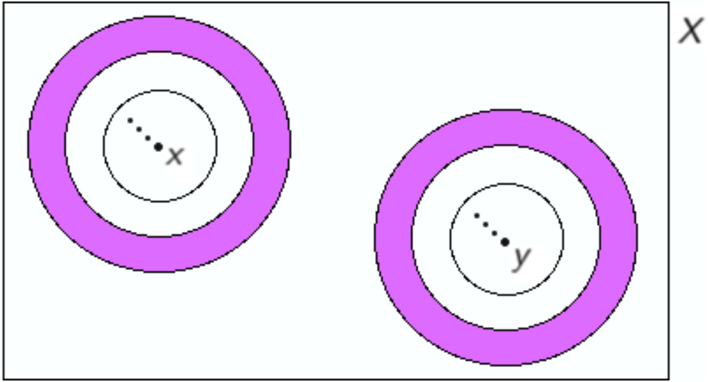

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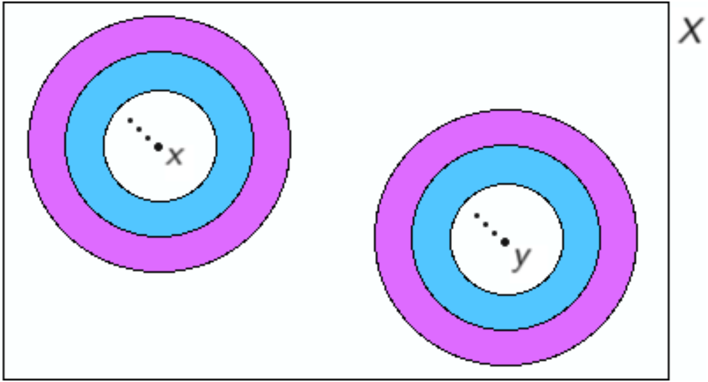


A few words about the axioms

AD denotes the *Axiom of Determinacy*: every game on ω is determined (either Player I or Player II has a winning strategy).
DC denotes the principle of *Dependent Choices*: if R is a binary relation on a non-empty set A such that $\forall a \, \exists b \, (b \, R \, a)$, then there exists $(a_0, a_1, \ldots) \in A^\omega$ such that $a_{n+1} \, R \, a_n$ for each $n \in \omega$.

1. The set-theoretic universe is extremely regular under AD,
2. AD is incompatible with AC,
3. AC $\rightarrow$ DC $\rightarrow$ CC,
4. ZF + DC + AD is consistent (assuming large cardinals),
5. ZF + DC is sufficient to carry out recursions of length ω,
6. DC is equivalent to Baire's Category Theorem for completely metrizable spaces (Blair, 1977),
7. ZF + DC proves Borel Determinacy (Martin, 1975).

Unless we specify otherwise, we will be working in ZF + DC.

An established pattern in set theory

Many properties $\mathcal{P}$ behave as follows:

- Every Borel set of reals satisfies $\mathcal{P}$,
- Under AD, all sets of reals satisfy $\mathcal{P}$,
- Under AC, there exist counterexamples to $\mathcal{P}$,
- Under $\mathrm{V} = \mathrm{L}$, there exist definable (usually coanalytic) counterexamples to $\mathcal{P}$.

The classical regularity properties ($\mathcal{P} =$ "perfect set property", $\mathcal{P} =$ "Lebesgue measurable" and $\mathcal{P} =$ "Baire property") are the most famous instances of this pattern. More entertaining examples include $\mathcal{P} =$ "not a Hamel basis" and $\mathcal{P} =$ "not an ultrafilter". A recent example is $\mathcal{P} =$ "Effros group". This talk is about

$$\mathcal{P} = \text{“}\sigma\text{-homogeneity”},$$

in the context of zero-dimensional spaces.

Wadge theory: basic definitions

Let Z be a set and $\mathbf{\Gamma} \subseteq \mathcal{P}(Z)$. Define $\check{\mathbf{\Gamma}} = \{Z \setminus A : A \in \mathbf{\Gamma}\}$. We say that $\mathbf{\Gamma}$ is *selfdual* if $\mathbf{\Gamma} = \check{\mathbf{\Gamma}}$. Also define $\Delta(\mathbf{\Gamma}) = \mathbf{\Gamma} \cap \check{\mathbf{\Gamma}}$.

Definition (Wadge)

Let Z be a space. Given $A, B \subseteq Z$, we will write $A \leq B$ if there exists a continuous function $f : Z \longrightarrow Z$ such that $A = f^{-1}[B]$. In this case, we will say that A is *Wadge-reducible* to B, and that f *witnesses* the reduction.

Definition (Wadge)

Let Z be a space. Given $A \subseteq Z$, define

$$A{\downarrow} = \{B \subseteq Z : B \leq A\}.$$

We will say that $\mathbf{\Gamma} \subseteq \mathcal{P}(Z)$ is a *Wadge class* if there exists $A \subseteq Z$ such that $\mathbf{\Gamma} = A{\downarrow}$. The set A is *selfdual* if $A{\downarrow}$ is selfdual.

First examples of Wadge classes

From now on, we will assume that Z is an uncountable zero-dimensional Polish space.

- $\{\varnothing\}$ and $\{Z\}$. (These are the minimal ones.)
- $\boldsymbol{\Delta}^0_1(Z)$ is their immediate successor.
  (Generated by an arbitrary proper clopen set.)

Let $1 \leq \xi < \omega_1$. Recall that $\boldsymbol{\Sigma}^0_\xi(Z)$ has a 2^ω-universal set U. This means that $U \in \boldsymbol{\Sigma}^0_\xi(2^\omega \times Z)$ and

$$\boldsymbol{\Sigma}^0_\xi(Z) = \{U_x : x \in 2^\omega\}$$

where $U_x = \{y \in Z : (x, y) \in U\}$ denotes the vertical section.

- $\boldsymbol{\Sigma}^0_\xi(Z)$ and $\boldsymbol{\Pi}^0_\xi(Z)$. (Generated by a universal set.)
- $\boldsymbol{\Sigma}^1_n(Z)$ and $\boldsymbol{\Pi}^1_n(Z)$ for $n \geq 1$. (As above.)

The Wadge hierachy: a first glimpse

$$
\begin{tikzcd}
 & \vdots & \\
\boldsymbol{\Sigma}^0_2(Z) \arrow[ur, dash] & & \boldsymbol{\Pi}^0_2(Z) \arrow[ul, dash] \\
 & ??? \arrow[ul, dash] \arrow[ur, dash] & \\
\boldsymbol{\Sigma}^0_1(Z) \arrow[ur, dash] & & \boldsymbol{\Pi}^0_1(Z) \arrow[ul, dash] \\
 & \boldsymbol{\Delta}^0_1(Z) \arrow[ul, dash] \arrow[ur, dash] & \\
\{\varnothing\} \arrow[ur, dash] & & \{Z\} \arrow[ul, dash]
\end{tikzcd}
$$

The difference hierarchy

Given $1 \leq \eta < \omega_1$, define D_η as follows:

- $\mathrm{D}_1(A_0) = A_0$,
- $\mathrm{D}_2(A_0, A_1) = A_1 \setminus A_0$,
- $\mathrm{D}_3(A_0, A_1, A_2) = A_0 \cup (A_2 \setminus A_1)$,

$$\vdots$$

- $\mathrm{D}_\omega(A_0, A_1, \ldots) = (A_1 \setminus A_0) \cup (A_3 \setminus A_2) \cup \cdots$,
- $\mathrm{D}_{\omega+1}(A_0, A_1, \ldots, A_\omega) = A_0 \cup (A_2 \setminus A_1) \cup \cdots \cup (A_\omega \setminus \bigcup_{n<\omega} A_n)$,

$$\vdots$$

The following are non-selfdual Wadge classes for $1 \leq \eta, \xi < \omega_1$:

$$\mathrm{D}_\eta(\mathbf{\Sigma}^0_\xi(Z)) = \{\mathrm{D}_\eta(A_\mu : \mu < \eta) : \text{the } A_\mu \in \mathbf{\Sigma}^0_\xi(Z) \text{ are increasing}\}.$$

Theorem (Hausdorff, Kuratowski)

$\mathbf{\Delta}^0_{\xi+1}(Z) = \bigcup_{1 \leq \eta < \omega_1} \mathrm{D}_\eta(\mathbf{\Sigma}^0_\xi(Z))$.

Why do we need determinacy?

Lemma (Wadge)

Assume AD*. Let $A, B \subseteq Z$. Then either $A \leq B$ or $B \leq Z \setminus A$.*

Here are two simple applications:

- In the poset $\mathbb{W}(Z)$ of all Wadge classes in Z ordered by $\subseteq$, antichains have size at most 2,
- If $\boldsymbol{\Gamma}$ is a Wadge class and $A \in \boldsymbol{\Gamma} \setminus \check{\boldsymbol{\Gamma}}$ then $A{\downarrow} = \boldsymbol{\Gamma}$.

Theorem (Martin, Monk)

Assume AD*. The poset $\mathbb{W}(Z)$ is well-founded.*

This yields the definition of Wadge rank.

- By the two results above, $\mathbb{W}(Z)$ becomes a well-order if we identify every Wadge class $\boldsymbol{\Gamma}$ with its dual class $\check{\boldsymbol{\Gamma}}$.
- The length of this well-order is $\Theta = \sup\{\alpha \in \mathrm{On} : \text{there exists a surjection } \omega^\omega \longrightarrow \alpha\}$.

The analysis of selfdual Wadge classes

Definition

Given $\xi < \omega_1$, define $\mathrm{PU}_\xi(\boldsymbol{\Gamma})$ as the collection of all sets of the form

$$\bigcup_{n\in\omega}(A_n \cap V_n),$$

where $A_n \in \boldsymbol{\Gamma}$ for $n \in \omega$ and $\{V_n : n \in \omega\} \subseteq \boldsymbol{\Delta}^0_{1+\xi}(Z)$ is a partition of Z. A set in this form is called a *partitioned union* of sets in $\boldsymbol{\Gamma}$.

The following fundamental result reduces the study of selfdual Wadge classes to the study of non-selfdual Wadge classes:

Theorem

Assume AD. *Let* $\boldsymbol{\Delta}$ *be a selfdual Wadge class in* Z. *Then there exist non-selfdual Wadge classes* $\boldsymbol{\Gamma}_n$ *in* Z *for* $n \in \omega$ *such that*

$$\boldsymbol{\Delta} = \mathrm{PU}_0\left(\bigcup_{n\in\omega}(\boldsymbol{\Gamma}_n \cup \check{\boldsymbol{\Gamma}}_n)\right).$$

The "pair of socks" problem

Observe that non-selfdual Wadge classes seem to suffer from the "pair of socks" problem. In other (less ridiculous) words, given a non-selfdual Wadge class $\boldsymbol{\Gamma}$, there is no obvious way to distinguish $\boldsymbol{\Gamma}$ from $\check{\boldsymbol{\Gamma}}$.

The following result gives an elegant solution to this problem. Recall that $\boldsymbol{\Gamma} \subseteq \mathcal{P}(Z)$ has the *separation property* if whenever $A, B \in \boldsymbol{\Gamma}$ are disjoint then there exists $C \in \Delta(\boldsymbol{\Gamma})$ such that $A \subseteq C \subseteq Z \setminus B$.

Theorem (Steel, 1981; Van Wesep, 1978)

Assume AD*. Let* $\boldsymbol{\Gamma}$ *be a non-selfdual Wadge class. Then exactly one of* $\boldsymbol{\Gamma}$ *and* $\check{\boldsymbol{\Gamma}}$ *has the separation property.*

Quite amusingly, we will not care at all about what the separation property says exactly: we will only need a way to choose without using AC. (Recall that our ambient theory is $\mathsf{ZF} + \mathsf{DC}$.)

A(nother) theorem of Steel

The following result allows us to make the considerable jump from "having the same complexity" to "being homeomorphic".

It is at the heart of *all* topological applications of Wadge theory.

We will say that $\mathbf{\Gamma}$ is *reasonably closed* if [redacted] for every [redacted].

Theorem (Steel, 1980)

Assume AD*. Let* $\mathbf{\Gamma}$ *be a reasonably closed Wadge class in* 2^ω*, and let* $X, Y \subseteq 2^\omega$ *be such that the following conditions hold:*

- X *and* Y *are either both comeager or both meager,*
- *For every basic clopen subset* U *of* 2^ω*, both* $X \cap U$ *and* $Y \cap U$ *have complexity exactly* $\mathbf{\Gamma}$ *(i.e. they belong to* $\mathbf{\Gamma} \setminus \check{\mathbf{\Gamma}}$*).*

Then there exists a homeomorphism $h : 2^\omega \longrightarrow 2^\omega$ *with* $h[X] = Y$*.*

Exercise: show that $\mathbb{Q}^\omega \approx \{x \in \omega^\omega : \lim_{n\to\infty} x_n = \infty\}$.

Homogeneous spaces, the case above $\boldsymbol{\Delta}$

In his remarkable Ph.D. thesis, van Engelen classified all the zero-dimensional homogeneous Borel spaces. In particular, he discovered that $\boldsymbol{\Delta} = \Delta(\mathrm{D}_\omega(\boldsymbol{\Sigma}^0_2))$ is the crucial dividing line.

It turns out that Steel's theorem can be converted (after some significant Wadge-theoretic struggle) into the following uniqueness theorem. Its Borel version is due to van Engelen.

Theorem (Carroy, Medini, Müller, 2020)

Assume AD. *Let X, Y be zero-dimensional homogeneous spaces of complexity higher than $\boldsymbol{\Delta}$ such that the following conditions hold:*

- *X and Y are either both Baire or both meager,*
- *X and Y have exactly the same complexity.*

Then X is homeomorphic to Y.

For the purposes of this talk, we will not need the above theorem. (We will need to use Steel's theorem.) But hey, it seemed fitting!

Homogeneous spaces, the case below $\mathbf{\Delta}$

Let us begin by stating some classical characterizations, where X is assumed to be zero-dimensional:

- $X \approx \mathbb{Q}$ iff X is countable and nowhere singleton,
- $X \approx 2^\omega$ iff X is compact and nowhere countable,
- $X \approx \mathbb{Q} \times 2^\omega$ iff X is σ-compact, nowhere compact and nowhere countable,
- $X \approx \omega^\omega$ iff X is complete and nowhere compact,
- $X \approx \mathbb{Q} \times \omega^\omega$ iff X is strongly σ-complete, nowhere complete and nowhere σ-compact.

Van Douwen and van Mill made one more step in this direction by defining the spaces $\mathbb{T}$ and $\mathbb{S}$. Finally, van Engelen completed this line of research by describing the remaining ω steps.

Modulo trivialities, these are precisely the zero-dimensional homogeneous spaces of complexity below $\mathbf{\Delta}$.

He defined properties $\mathcal{P}_n^{(i)}$ and ordered them in type ω as follows:

$$\mathcal{P}^1_{-2} \prec \mathcal{P}^1_{-1} \prec \mathcal{P}^2_{-2} \prec \mathcal{P}^2_{-1} \prec \cdots$$
$$\cdots \prec \mathcal{P}_{4k} \prec \mathcal{P}_{4k+1} \prec \mathcal{P}^1_{4k+2} \prec \mathcal{P}^1_{4k+3} \prec \mathcal{P}^2_{4k+2} \prec \mathcal{P}^2_{4k+3} \prec \cdots .$$

For every property $\mathcal{P}_n^{(i)}$, van Engelen defined a class of spaces $\mathcal{X}_n^{(i)}$ with the following properties:

Theorem (van Engelen, 1985)

For a zero-dimensional space X, the following conditions are equivalent:

- $X \in \mathcal{X}_n^{(i)}$,
- X *is* $\mathcal{P}_n^{(i)}$ *and nowhere* $\mathcal{P}_m^{(j)}$ *whenever* $\mathcal{P}_m^{(j)} \prec \mathcal{P}_n^{(i)}$.

Theorem (van Engelen, 1985)

Up to homeomorphism, each $\mathcal{X}_n^{(i)}$ contains exactly one element, which is strongly homogeneous.

$$
\begin{tikzcd}
1 \in \mathcal{X}^1_{-2} \arrow[d, no head] & & 2^\omega \in \mathcal{X}^2_{-2} \arrow[d, no head] \\
\mathbb{Q} \in \mathcal{X}^1_{-1} \arrow[dr, no head] & & \mathbb{Q} \times 2^\omega \in \mathcal{X}^2_{-1} \arrow[dl, no head] \\
& \omega^\omega \in \mathcal{X}_0 \arrow[d, no head] & \\
& \mathbb{Q} \times \omega^\omega \in \mathcal{X}_1 \arrow[dl, no head] \arrow[dr, no head] & \\
\mathbb{T} \in \mathcal{X}^1_2 \arrow[d, no head] & & \mathbb{S} \in \mathcal{X}^2_2 \arrow[d, no head] \\
\vdots & & \vdots
\end{tikzcd}
$$

The positive result

Theorem (Ostrovsky, 2011)

Every zero-dimensional Borel space is σ-homogeneous.

Ostrovsky used the techniques of van Engelen's thesis.
The foundation of these techniques lies in Louveau's (somewhat impenetrable) 1983 article that classifies the Borel Wadge classes.
Using instead material from Louveau's unpublished book, it is possible to extend these techniques beyond the Borel realm.

Theorem

Assume AD. *Then every zero-dimensional space is σ-homogeneous.*

The idea is to associate to every $X \subseteq 2^\omega$ a strongly homogeneous clopen subspace $\mathrm{HC}(X)$ of X. If $X \neq \varnothing$, we will have $\mathrm{HC}(X) \neq \varnothing$.

Corollary (van Engelen, Miller, Steel, 1987)

Assume AD. *Then there are no zero-dimensional rigid spaces.*

Proof of the theorem, assuming that HC has already been defined

Given $X \subseteq 2^\omega$, define X_α for every ordinal α as follows:

- $X_0 = X$,
- $X_{\alpha+1} = X_\alpha \setminus \mathsf{HC}(X_\alpha)$,
- $X_\gamma = \bigcap_{\alpha<\gamma} X_\alpha$ if γ is a limit ordinal.

Since $X_0 \supseteq X_1 \supseteq \cdots$ are closed in X, the sequence must stabilize at some countable ordinal δ. Notice that $X_\delta = \varnothing$, otherwise we would have $X_{\delta+1} \subsetneq X_\delta$ by the definition of HC.
It follows that

$$X = \bigcup_{\alpha<\delta} \mathsf{HC}(X_\alpha),$$

which shows that X is the union of countably many pairwise disjoint, strongly homogeneous, closed subspaces.

Construction of HC

The idea is simply to take a non-empty clopen subspace U of X of "minimal complexity". Start by fixing a well-order of $\boldsymbol{\Delta}^0_1(2^\omega)$.

Case 1: $X = \varnothing$.
In this case, simply set $\mathsf{HC}(X) = \varnothing$.

Case 2: X has a non-empty open subspace of complexity below $\boldsymbol{\Delta}$.
Pick the minimal property $\mathcal{P}$ of the form $\mathcal{P}^{(i)}_n$ such that, for some clopen subset C of 2^ω, the following conditions hold:

- $C \cap X \neq \varnothing$,
- $C \cap X$ has $\mathcal{P}$.

Then set $\mathsf{HC}(X) = C \cap X$ for the minimal C as above.

It is clear that $\mathsf{HC}(X)$ is clopen in X and non-empty.
Using van Engelen's theorems, one sees that $\mathsf{HC}(X) \in \mathcal{X}^{(i)}_n$, hence $\mathsf{HC}(X)$ is strongly homogeneous.

Case 3: X is nowhere $\mathbf{\Delta}$.
By the Martin-Monk theorem, we can pick the minimal Wadge class $\mathbf{\Gamma}$ in 2^ω such that, for some clopen subset C of 2^ω, the following conditions hold:

- $C \cap X \neq \varnothing$,
- $C \cap X \in \mathbf{\Gamma}$.

If $\mathbf{\Gamma}$ and $\check{\mathbf{\Gamma}}$ are both acceptable choices, pick the one that has the separation property. Then set $\mathsf{HC}(X) = C \cap X$ for the minimal C that satisfies the above conditions, plus the following:

- $C \cap X$ is either meager or Baire.

It is clear that $\mathsf{HC}(X)$ is clopen in X and non-empty.
Using Steel's Theorem and Terada's Lemma, it is not hard to see that $\mathsf{HC}(X)$ is strongly homogeneous. (The technically demanding part is to show that $\mathbf{\Gamma}$ is reasonably closed.)

A counterexample in ZFC

The naive definition of "hereditarily rigid" would be silly.
But a small tweak yields an interesting notion:

Definition

A space X is $\mathfrak{c}$-*hereditarily rigid* if X is $\mathfrak{c}$-crowded and every $\mathfrak{c}$-crowded subspace of X is rigid.

Theorem

There exists a ZFC *example of a zero-dimensional* $\mathfrak{c}$*-hereditarily rigid space.*

Corollary

There exists a ZFC *example of a zero-dimensional space that is not* σ*-homogeneous.*

To prove the corollary, use the fact that $\mathrm{cof}(\mathfrak{c}) > \aleph_0$ plus the fact that every homogeneous space of size $\mathfrak{c}$ is $\mathfrak{c}$-crowded.

Proof of the theorem

It will be enough to construct a subspace X of 2^ω such that $|X| = \mathfrak{c}$ and every $\mathfrak{c}$-crowded subspace of X is rigid.

Begin by enumerating as $\{h_\alpha : \alpha \in \mathfrak{c}\}$ all homeomorphisms between G_δ subsets of 2^ω, making sure that $h_0 : 2^\omega \longrightarrow 2^\omega$ is the identity.

Pick x_α for $\alpha \in \mathfrak{c}$ such that the following conditions hold:

- $x_\alpha \neq h_\beta^{-1}(x_\gamma)$ whenever $\beta, \gamma < \alpha$,
- $x_\alpha \neq h_\beta(x_\gamma)$ whenever $\beta, \gamma < \alpha$.

Set $X = \{x_\alpha : \alpha \in \mathfrak{c}\}$, and notice that $|X| = \mathfrak{c}$ by the choice of h_0.

Now assume that $S \subseteq X$ is $\mathfrak{c}$-crowded and $h : S \longrightarrow S$ is a homeomorphism. By Lavrentieff's Lemma, it is possible to fix $\delta \in \mathfrak{c}$ such that $h \subseteq h_\delta$.

It will be enough to show that $h(x_\alpha) = x_\alpha$ for every $\alpha > \delta$. In fact, since S is $\mathfrak{c}$-crowded, it will follow that h is the identity.

So fix $\alpha > \delta$. Let β be such that $h(x_\alpha) = x_\beta$. We will prove that $\beta = \alpha$ by showing that the other cases are impossible.
Case 1: $\beta > \alpha$.
The fact that $h_\delta(x_\alpha) = x_\beta$ contradicts the construction.
Case 2: $\beta < \alpha$.
The fact that $h_\delta^{-1}(x_\beta) = x_\alpha$ contradicts the construction.

The proof actually shows that X has the following property:

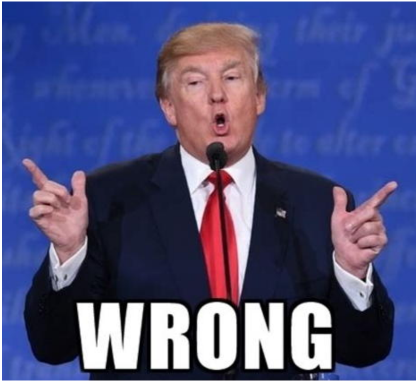
Definition

A space X is *strongly $\mathfrak{c}$-hereditarily rigid* if X is $\mathfrak{c}$-crowded and whenever S and T are $\mathfrak{c}$-crowded subspaces of X and $h : S \longrightarrow T$ is a homeomorphism, then $S = T$ and h is the identity.

Question

Is there a ZFC *example of a zero-dimensional $\mathfrak{c}$-hereditarily rigid space that is not strongly $\mathfrak{c}$-hereditarily rigid?*

Obviously, if you're a topologist, studying computability theory is a complete waste of time...



Definable counterexamples under $\mathsf{V} = \mathsf{L}$

In his 1989 paper, Miller sketched a method for constructing coanalytic versions of certain pathological sets of reals (in the spirit of Gödel's coanalytic set without the perfect set property).

In 2014, Vidnyánszky gave a "black box" version of Miller's method. Using this, it's not hard to prove the following:

Lemma

Assume $\mathsf{V} = \mathsf{L}$. Then there exists $X \subseteq \omega^\omega$ such that:

- *X is coanalytic,*
- *X is dense in ω^ω and $\mathfrak{c}$-crowded,*
- *Every element of X is self-constructible,*
- *If $x, y \in X$ and $x \neq y$ then $\omega_1^x \neq \omega_1^y$.*

Given $x \in \omega^\omega$, we denote by ω_1^x the smallest ordinal not computable from x. We say that x is *self-constructible* if $x \in \mathsf{L}_{\omega_1^x}$.

Lemma

Assume $\mathrm{V} = \mathrm{L}$. *Let* $X \subseteq \omega^\omega$ *be as in the previous lemma and set* $Y = \omega^\omega \setminus X$. *Then:*

- X *and* Y *are* $\mathfrak{c}$*-crowded,*
- X *is strongly* $\mathfrak{c}$*-hereditarily rigid,*
- X *is not* σ*-homogeneous,*
- Y *is rigid but not* $\mathfrak{c}$*-hereditarily rigid,*
- Y *is not* σ*-homogeneous with Borel witnesses.*

Theorem

Assume $\mathrm{V} = \mathrm{L}$. *Then there exists a zero-dimensional coanalytic space that is not* σ*-homogeneous.*

Theorem (van Engelen, Miller, Steel, 1987)

Assume $\mathrm{V} = \mathrm{L}$. *Then there exist both analytic and coanalytic examples of zero-dimensional rigid spaces.*

Proof of the lemma: first a useful claim

We claim that the following holds:

⊛ If $S, T \subseteq \omega^\omega$ and $h : S \longrightarrow T$ is a homeomorphism without fixed points, then $\{x \in S \cap X : h(x) \in X\}$ is countable.

To prove the claim, assume that $h : S \longrightarrow T$ is a homeomorphism such that $\{x \in S \cap X : h(x) \in X\}$ is uncountable. By Lavrentieff's Lemma, we can extend h to a homeomorphism $\widetilde{h}$ between G_δ sets. Pick a countable ordinal δ such that $\widetilde{h}$ is coded in L_δ.

By the injectivity condition, we can fix $x \in S \cap X$ such that $h(x) \in X$ and $\omega_1^x \geq \delta$. Observe that $x \in \mathrm{L}_{\omega_1^x}$ by self-constructibility. Set $y = h(x)$, and observe that $y \in \mathrm{L}_{\omega_1^x}$. Since $\omega_1^x \notin \mathrm{L}_{\omega_1^x}$, it follows that ω_1^x is not computable from y. So we must have $\omega_1^y \leq \omega_1^x$ by minimality.

A similar argument, applied to h^{-1}, shows that $\omega_1^x \leq \omega_1^y$. Therefore $\omega_1^x = \omega_1^y$, hence $x = y$ by the injectivity condition.

Proof that X is strongly $\mathfrak{c}$-hereditarily rigid

Let $h : S \longrightarrow T$ a homeomorphism between $\mathfrak{c}$-crowded subspaces of X. If $h(x) \neq x$ for some $x \in S$, then the Hausdorff property plus $\mathfrak{c}$-crowdedness would contradict $\circledast$.

Proof that Y is $\mathfrak{c}$-crowded

Observe that X does not contain copies of 2^ω. Since X has the property of Baire, it follows that X is meager. Therefore Y is comeager, which implies $\mathfrak{c}$-crowded.

Proof that Y is rigid

Let $h : Y \longrightarrow Y$ be a homeomorphism. Let $\widetilde{h} : G \longrightarrow G$ be a homeomorphism that extends h, where G is G_δ in ω^ω. Notice that $\omega^\omega \setminus G$ is countable, hence $G \cap X$ is $\mathfrak{c}$-crowded (and dense in ω^ω). Therefore $\widetilde{h} \restriction (G \cap X)$ is the identity, and so is h.

Proof that Y is not σ-homogeneous with Borel witnesses

Assume that $Y = \bigcup_{n\in\omega} Y_n$, where each Y_n is homogeneous and Borel in Y. Fix a countable clopen base $\mathcal{B}$ for ω^ω. Whenever there exists $U \in \mathcal{B}$ and a homeomorphism $h : G \longrightarrow H$ between G_δ subsets of ω^ω with no fixed point such that $G \cap Y_n = U \cap Y_n$ and $h[G \cap Y_n] = H \cap Y_n$, declare $(U, n) \in I$, and fix $h_i : G_i \longrightarrow H_i$ with these properties. Then define the following set, where $i = (U_i, n_i)$:

$$X' = \bigcap_{i\in I} \{x \in \omega^\omega : x \notin G_i \text{ or } (x \in G_i \text{ and } h_i(x) \in Y \setminus Y_{n_i})\},$$

and observe that X' is analytic because the witnesses are Borel.

If $x \in X' \cap Y_n$ then $Y_n = \{x\}$, so $X' \setminus X$ is countable.

If $i = (U, n) \in I$ then $\circledast$ shows that $h_i(x) \in Y \setminus Y_n$ for all but countably many $x \in G_i \cap X$. So $X \setminus X'$ is also countable.

In conclusion, $X \Delta X'$ is countable, which is a contradiction.

Open (and closed) questions

Answering the following question would give us a complete picture of σ-homogeneity in the zero-dimensional realm:

Question

Is every analytic zero-dimensional space σ-homogeneous?

There seems to be a parallel between rigidity and the lack of σ-homogeneity. (For example, under AD, they are vacuously equivalent.) So the following question seemed natural:

Question (Not anymore!)

In ZFC, *is there a zero-dimensional σ-homogeneous rigid space?*

The following result gives a resounding "yes":

Theorem (van Engelen, van Mill, 1983)

In ZFC, *there exist homogeneous subspaces X_1 and X_2 of $\mathbb{R}$ such that $X = X_1 \cup X_2$ is rigid. Furthermore, $X_1 \cap X_2 = \varnothing$, $X_1 \approx X_2$ and X is a Baire space.*

More than mere σ-homogeneity

Our positive result gives a very strong form of σ-homogeneity. Can the stronger versions be distinguished from the standard one?

Question (Not anymore!)

In ZFC, *is there a zero-dimensional σ-homogeneous space that is not σ-homogeneous with closed witnesses?*

By a Baire category argument, the example of van Engelen and van Mill shows that the answer to the above question is also "yes".

But the following "covering vs. partition" question is still open:

Question

In ZFC, *is there a zero-dimensional σ-homogeneous space that is not σ-homogeneous with pairwise disjoint witnesses?*

More on hereditary rigidity

Theorem (Medini, van Mill, Zdomskyy, 2016)

There exists a ZFC *example of a subspace X of 2^ω with the following properties, where $Y = 2^\omega \setminus X$:*

- *X is Bernstein,*
- *X is homogeneous,*
- *Y is rigid.*

It turns out that Y is not $\mathfrak{c}$-hereditarily rigid. But:

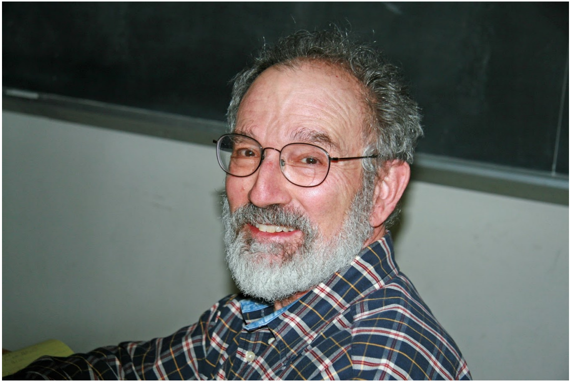
Question

Under $\mathrm{V} = \mathrm{L}$, *is there a coanalytic zero-dimensional rigid space that is not $\mathfrak{c}$-hereditarily rigid?*

Question

In ZFC, *is there a zero-dimensional $\aleph_1$-hereditarily rigid space?*

Kenneth Kunen (1943-2020)



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