Topological applications of Wadge theory

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Everybody loves

homogeneous stuff!

Topological homogeneity

A space is homogeneous if all points “look alike” from a global point of view:

Definition

A space X is *homogeneous* if for every $x, y \in X$ there exists a homeomorphism $h : X \longrightarrow X$ such that $h(x) = y$.

Non-examples:

- $\omega + 1$ (Because of the limit point)
- $[0, 1]^n$ whenever $1 \leq n < \omega$
  (Points on the boundary are different from points in the interior)
- The Stone-Čech remainder $\omega^* = \beta\omega \setminus \omega$
  (W. Rudin, 1956, under CH, because of P-points)
  (Frolík, 1967, using a cardinality argument)
  (Kunen, 1978, by proving the existence of weak P-points)

Examples:

- Any topological group
- Any product of homogeneous spaces
- Any open subspace of a zero-dimensional homogeneous space
- The Hilbert cube $[0,1]^\omega$ (Keller, 1931)
- X^ω for every zero-dimensional first-countable X
  (Dow and Pearl, 1997, based on work of Lawrence)

Homogeneous spaces are decently understood.
Compact homogeneous spaces are shrouded in mystery:

Question (Van Douwen, 1970s)

Is there a compact homogeneous space with more than $\mathfrak{c}$ pairwise disjoint non-empty open sets?

Question (W. Rudin, 1958)

Is there an infinite compact homogeneous space with no non-trivial convergent ω-sequences?

Strong homogeneity

Definition

A space X is *strongly homogeneous* (or *h-homogeneous*) if every non-empty clopen subspace of X is homeomorphic to X.

Examples:

- Any connected space
- $\mathbb{Q}$, 2^ω, ω^ω (Use their characterizations)
- Any product of zero-dimensional strongly homogeneous spaces (Medini, 2011, building on work of Terada, 1993)
- Erdős space $\mathfrak{E} = \{x \in \ell^2 : x_n \in \mathbb{Q} \text{ for all } n \in \omega\}$ (Dijkstra and van Mill, 2010)

Non-examples:

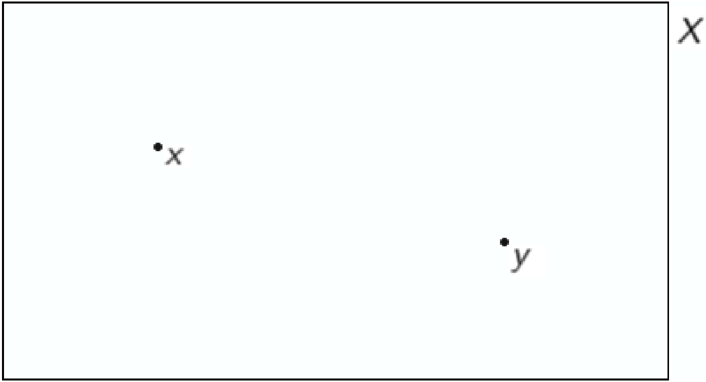
- Discrete spaces with at least two elements
- $\omega \times 2^\omega$

Is "strong" a good choice of word?

Not particularly. For example, ω^* is strongly homogeneous but not homogeneous. Things get better under additional assumptions:

Theorem (folklore)

Let X be a first-countable zero-dimensional space. If X is strongly homogeneous then X is homogeneous.



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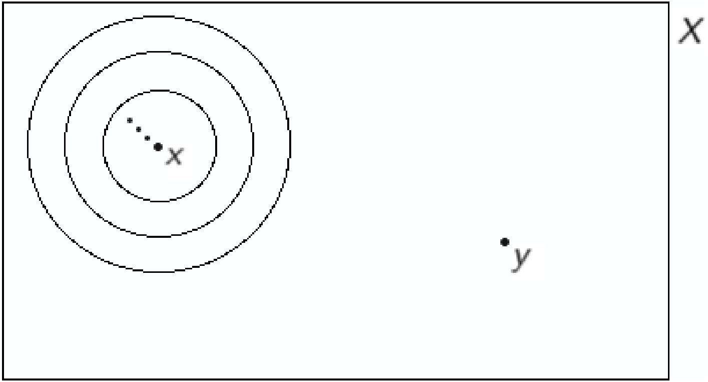

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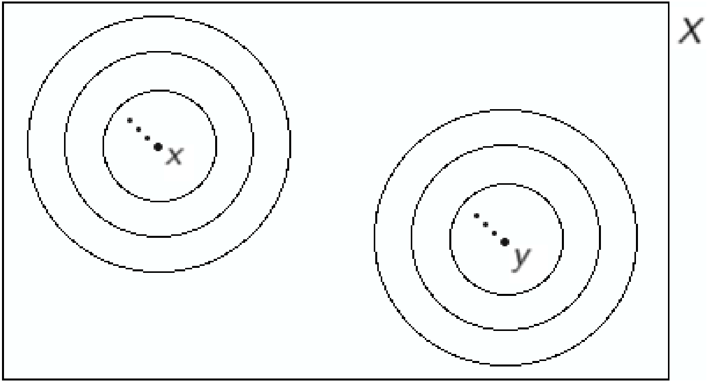

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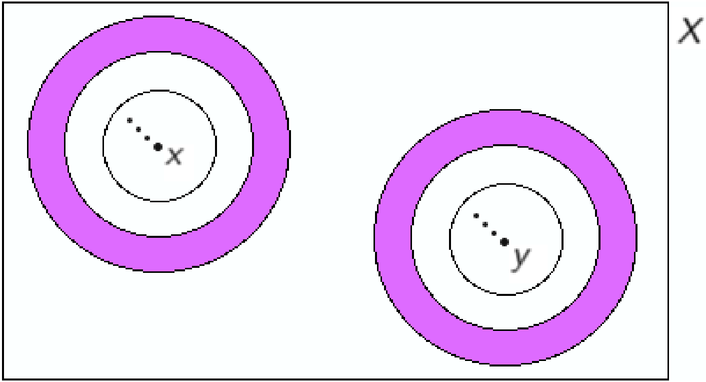

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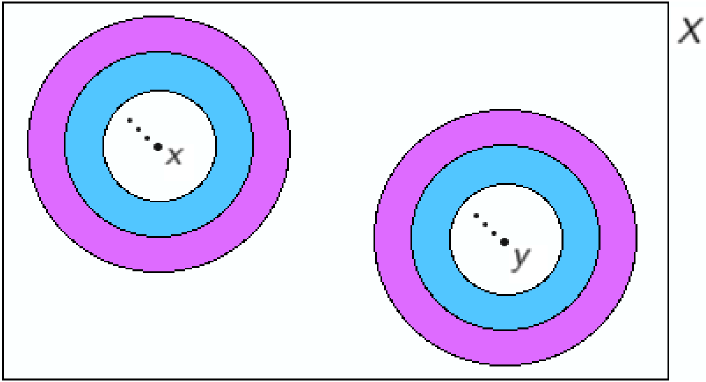


The locally compact case (the trivial case)

From now on, all spaces are separable and metrizable.

Proposition

Let X be a locally compact zero-dimensional space. Then the following conditions are equivalent:

- *X is homogeneous*
- *X is discrete, $X \approx \omega \times 2^\omega$, or $X \approx 2^\omega$*

Two open questions

Question (Terada, 1993)

Is X^ω strongly homogeneous for every zero-dimensional space X?

Question (Medvedev, 2012)

Is X strongly homogeneous for every meager zero-dimensional homogeneous space X?

An example of van Douwen

Theorem (van Douwen, 1984)

There exists a subspace X of $\mathbb{R}$ with the following properties:

- *X is a Bernstein subset of $\mathbb{R}$*
- *X is a subgroup of $(\mathbb{R}, +)$*
- *There exists a measure μ on the Borel subsets of X such that $A \approx B$ implies $\mu(A) = \mu(B)$ whenever $A, B \subseteq X$ are Borel*

Given a Borel subset A of X, the measure of A is defined by:

$$\mu(A) = \text{Lebesgue measure of } \widetilde{A}$$

where $\widetilde{A}$ is a Borel subset of $\mathbb{R}$ such that $\widetilde{A} \cap X = A$.

Corollary

There exists a zero-dimensional homogeneous space that is not locally compact space and not strongly homogeneous.

The main result

In his remarkable Ph.D. thesis, van Engelen obtained a complete classification of the zero-dimensional homogeneous Borel spaces. As a corollary, he proved the following:

Theorem (van Engelen, 1986)

Let X be a zero-dimensional Borel space that is not locally compact. If X is homogeneous then X is strongly homogeneous.

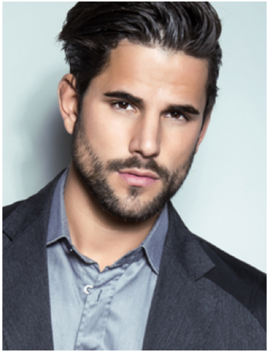
Can the "Borel" assumption be dropped? Certainly not in ZFC, by van Douwen's example. However:

Theorem (Carroy, Medini, Müller)

Work in $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}$. *Let X be a zero-dimensional space that is not locally compact. If X is homogeneous then X is strongly homogeneous.*

This gives consistent "yes" answers to Terada's and Medvedev's questions. (It is still open whether AD is really needed for those.) From now on, we will work in $\mathsf{ZF} + \mathsf{DC}$.

The Wadge Brigade



Raphaël Carroy **Andrea Medini** **Sandra Müller**

The Wadge Brigade (for real)

Raphaël Carroy

Andrea Medini

Sandra Müller

Wadge theory: basic definitions

Let Z be a set and $\boldsymbol{\Gamma} \subseteq \mathcal{P}(Z)$. Define $\check{\boldsymbol{\Gamma}} = \{Z \setminus A : A \in \boldsymbol{\Gamma}\}$. We say that $\boldsymbol{\Gamma}$ is *selfdual* if $\boldsymbol{\Gamma} = \check{\boldsymbol{\Gamma}}$. Also define $\Delta(\boldsymbol{\Gamma}) = \boldsymbol{\Gamma} \cap \check{\boldsymbol{\Gamma}}$.

Definition (Wadge, 1984)

Let Z be a space. Given $A, B \subseteq Z$, we will write $A \leq B$ if there exists a continuous function $f : Z \longrightarrow Z$ such that $A = f^{-1}[B]$. In this case, we will say that A is *Wadge-reducible* to B, and that f *witnesses* the reduction.

Definition (Wadge, 1984)

Let Z be a space. Given $A \subseteq Z$, define

$$[A] = \{B \subseteq Z : B \leq A\}$$

We will say that $\boldsymbol{\Gamma} \subseteq \mathcal{P}(Z)$ is a *Wadge class* if there exists $A \subseteq Z$ such that $\boldsymbol{\Gamma} = [A]$. The set A is *selfdual* if $[A]$ is selfdual.

First examples of Wadge classes

From now on, unless we specify otherwise, we will always assume that Z is an uncountable zero-dimensional Polish space.

- $\{\varnothing\}$ and $\{Z\}$ (These are the minimal ones)
- $\mathbf{\Delta}^0_1(Z)$ is their immediate successor
  (Generated by an arbitrary proper clopen set)

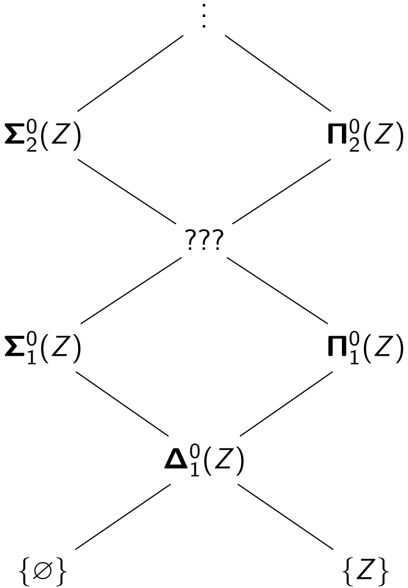
Let $1 \leq \xi < \omega_1$. Recall that $\mathbf{\Sigma}^0_\xi(Z)$ has a 2^ω-universal set U. This means that $U \in \mathbf{\Sigma}^0_\xi(2^\omega \times Z)$ and

$$\mathbf{\Sigma}^0_\xi(Z) = \{U_x : x \in 2^\omega\}$$

where $U_x = \{y \in Z : (x, y) \in U\}$ denotes the vertical section.

- $\mathbf{\Sigma}^0_\xi(Z)$ and $\mathbf{\Pi}^0_\xi(Z)$ (Generated by a universal set)
- $\mathbf{\Sigma}^1_n(Z)$ and $\mathbf{\Pi}^1_n(Z)$ for $n \geq 1$ (As above)

The Wadge hierachy: a first glimpse



Why do we need determinacy?

Lemma (Wadge, 1984)

Assume AD*. Let* $A, B \subseteq Z$*. Then either* $A \leq B$ *or* $B \leq Z \setminus A$*.*

Here are two simple (but very useful) applications:

- In the poset $\mathbb{W}(Z)$ of all Wadge classes in Z ordered by $\subseteq$, antichains have size at most 2
- If $\boldsymbol{\Gamma}$ is a Wadge class and $A \in \boldsymbol{\Gamma} \setminus \check{\boldsymbol{\Gamma}}$ then $[A] = \boldsymbol{\Gamma}$

Theorem (Martin, Monk)

Assume AD*. The poset* $\mathbb{W}(Z)$ *is well-founded.*

This yields the definition of Wadge rank.

- By the two results above, $\mathbb{W}(Z)$ becomes a well-order if we identify every Wadge class $\boldsymbol{\Gamma}$ with its dual class $\check{\boldsymbol{\Gamma}}$
- The length of this well-order is Θ

From now on, we will always assume that AD holds.

The analysis of selfdual Wadge classes

Definition

Given $\xi < \omega_1$, define $\mathrm{PU}_\xi(\boldsymbol{\Gamma})$ as the collection of all sets of the form

$$\bigcup_{n\in\omega}(A_n \cap V_n)$$

where $A_n \in \boldsymbol{\Gamma}$ for $n \in \omega$ and $\{V_n : n \in \omega\} \subseteq \boldsymbol{\Delta}^0_{1+\xi}(Z)$ is a partition of Z. A set in this form is called a *partitioned union* of sets in $\boldsymbol{\Gamma}$.

The following fundamental result reduces the study of selfdual Wadge classes to the study of non-selfdual Wadge classes:

Theorem (see Motto Ros, 2009)

Let $\boldsymbol{\Delta}$ be a selfdual Wadge class. Then there exist non-selfdual Wadge classes $\boldsymbol{\Gamma}_n$ for $n \in \omega$ such that

$$\boldsymbol{\Delta} = \mathrm{PU}_0\left(\bigcup_{n\in\omega}(\boldsymbol{\Gamma}_n \cup \check{\boldsymbol{\Gamma}}_n)\right)$$

Hausdorff operations

Definition (Hausdorff, 1927)

Given $D \subseteq \mathcal{P}(\omega)$, define

$$\mathcal{H}_D(A_0, A_1, \ldots) = \{x \in Z : \{n \in \omega : x \in A_n\} \in D\}$$

whenever $A_0, A_1, \ldots \subseteq Z$. We will call functions of this form *Hausdorff operations* (or ω-*ary Boolean operations*).

Given $n \in \omega$, set $\mathcal{U}_n = \{A \subseteq \omega : n \in A\}$. Then:

- $\mathcal{H}_{\mathcal{U}_n}(A_0, A_1, \ldots) = A_n$
- $\bigcap_{i \in I} \mathcal{H}_{D_i}(A_0, A_1, \ldots) = \mathcal{H}_D(A_0, A_1, \ldots)$, where $D = \bigcap_{i \in I} D_i$
- $\bigcup_{i \in I} \mathcal{H}_{D_i}(A_0, A_1, \ldots) = \mathcal{H}_D(A_0, A_1, \ldots)$, where $D = \bigcup_{i \in I} D_i$
- $Z \setminus \mathcal{H}_D(A_0, A_1, \ldots) = \mathcal{H}_{\mathcal{P}(\omega) \setminus D}(A_0, A_1, \ldots)$ for all $D \subseteq \mathcal{P}(\omega)$

Hence, any operation obtained by combining unions, intersections and complements can be expressed as a Hausdorff operation.

The difference hierarchy

Given $1 \leq \eta < \omega_1$, define the Hausdorff operation D_η as follows:

- $\mathrm{D}_1(A_0) = A_0$
- $\mathrm{D}_2(A_0, A_1) = A_1 \setminus A_0$
- $\mathrm{D}_3(A_0, A_1, A_2) = A_0 \cup (A_2 \setminus A_1)$

$$\vdots$$

- $\mathrm{D}_\omega(A_0, A_1, \ldots) = (A_1 \setminus A_0) \cup (A_3 \setminus A_2) \cup \cdots$
- $\mathrm{D}_{\omega+1}(A_0, A_1, \ldots, A_\omega) = A_0 \cup (A_2 \setminus A_1) \cup \cdots \cup (A_\omega \setminus \bigcup_{n<\omega} A_n)$

$$\vdots$$

Given $1 \leq \xi < \omega_1$, define:

$$\mathrm{D}_\eta(\boldsymbol{\Sigma}^0_\xi) = \{\mathrm{D}_\eta(A_\mu : \mu < \eta) : \text{each } A_\mu \in \boldsymbol{\Sigma}^0_\xi \text{ and } (A_\mu : \mu < \eta) \text{ is increasing}\}$$

It can be shown that $\mathrm{D}_\eta(\boldsymbol{\Sigma}^0_\xi) \subsetneq \mathrm{D}_\mu(\boldsymbol{\Sigma}^0_\xi)$ whenever $\eta < \mu$.

Wadge classes from Hausdorff operations

Definition

Given $D \subseteq \mathcal{P}(\omega)$, define

$$\boldsymbol{\Gamma}_D(Z) = \{\mathcal{H}_D(A_0, A_1, \ldots) : A_0, A_1, \ldots \in \boldsymbol{\Sigma}^0_1(Z)\}$$

By fixing a 2^ω-universal set for $\boldsymbol{\Sigma}^0_1(Z)$ and "applying $\mathcal{H}_D$ to it", one obtains the following:

Theorem (Addison for $Z = \omega^\omega$)

Let $D \subseteq \mathcal{P}(\omega)$. Then $\boldsymbol{\Gamma}_D(Z)$ is a non-selfdual Wadge class.

In particular, each $\mathrm{D}_\eta(\boldsymbol{\Sigma}^0_1(Z))$ is a non-selfdual Wadge class.
In fact, it can be shown that they and their duals exhaust the non-selfdual Wadge classes contained in $\boldsymbol{\Delta}^0_2(Z)$.
The analog statement for $\boldsymbol{\Delta}^0_3(Z)$ is false! However:

Theorem (Hausdorff and Kuratowski)

$\boldsymbol{\Delta}^0_{\xi+1}(Z) = \bigcup_{1 \leq \eta < \omega_1} \mathrm{D}_\eta(\boldsymbol{\Sigma}^0_\xi(Z))$

Relativization: yet another reason to love Hausdorff operations

When one tries to give a systematic exposition of Wadge theory, it soon becomes apparent that it would be very useful to be able to say when A and B belong to "the same" Wadge class $\mathbf{\Gamma}$, even when $A \subseteq Z$ and $B \subseteq W$ for distinct ambient spaces Z and W. (This is clear in some particular cases, like $\mathbf{\Gamma} = \mathbf{\Pi}^0_2$ or $\mathbf{\Gamma} = \mathrm{D}_5(\mathbf{\Sigma}^0_1)$, but what about arbitrary, possibly more "exotic" Wadge classes?)

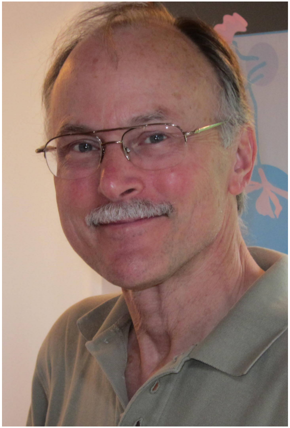
It turns out that Hausdorff operations allow us to do exactly that in a rather elegant way. The first ingredient is the following result, proved by Van Wesep in his Ph.D. thesis:

Theorem (Van Wesep, 1977, for $Z = \omega^\omega$)

The following are equivalent:

- $\mathbf{\Gamma}$ *is a non-selfdual Wadge class in* Z
- *There exists* $D \subseteq \mathcal{P}(\omega)$ *such that* $\mathbf{\Gamma} = \mathbf{\Gamma}_D(Z)$

Robert Van Wesep: medical scientist, mathematician, poet



Plus Ultra

The whole world having been
into its ultrapower injected
The latter being founded well,
if all goes as expected
The sets whose images contain
the point of criticality
Return an ultrafilter with
a dividend: normality!

Relativization: the crucial lemma

The second ingredient is the following “Relativization Lemma”. (Similar result have appeared in work of van Engelen, and even earlier in work of Louveau and Saint-Raymond.)

Lemma

Let Z and W be arbitrary topological spaces, and let $D \subseteq \mathcal{P}(\omega)$.

- *Assume that $W \subseteq Z$. Then $A \in \mathbf{\Gamma}_D(W)$ iff there exists $\widetilde{A} \in \mathbf{\Gamma}_D(Z)$ such that $A = \widetilde{A} \cap W$*
- *If $f : Z \longrightarrow W$ is continuous and $B \in \mathbf{\Gamma}_D(W)$ then $f^{-1}[B] \in \mathbf{\Gamma}_D(Z)$*
- *If $h : Z \longrightarrow W$ is a homeomorphism then $A \in \mathbf{\Gamma}_D(Z)$ iff $h[A] \in \mathbf{\Gamma}_D(W)$*

It is hard to understate how much confusion and ugliness was cleared up by this lemma...

Reasonably closed Wadge classes

Given $i \in 2$, set:

$$Q_i = \{x \in 2^\omega : x(n) = i \text{ for all but finitely many } n \in \omega\}$$

Notice that every element of $2^\omega \setminus (Q_0 \cup Q_1)$ is obtained by alternating finite blocks of zeros and finite blocks of ones.

Define the function $\phi : 2^\omega \setminus (Q_0 \cup Q_1) \longrightarrow 2^\omega$ by setting

$$\phi(x)(n) = \begin{cases} 0 & \text{if the } n^{\text{th}} \text{ block of zeros of } x \text{ has even length} \\ 1 & \text{otherwise} \end{cases}$$

where we start counting with the 0^{th} block of zeros. It is easy to check that ϕ is continuous.

Definition (Steel, 1980)

Let $\mathbf{\Gamma}$ be a Wadge class in 2^ω. We will say that $\mathbf{\Gamma}$ is *reasonably closed* if $\phi^{-1}[A] \cup Q_0 \in \mathbf{\Gamma}$ for every $A \in \mathbf{\Gamma}$.

Why would anybody need that?

Lemma (Harrington)

Let $\mathbf{\Gamma} = [B]$ *be a reasonably closed Wadge class in* 2^ω. *If* $A \leq B$ *then this is witnessed by an injective function.*

The above lemma will be useful to us because every injective continuous function $f : 2^\omega \longrightarrow 2^\omega$ is an embedding.

Proof.

Let $A^* = \phi^{-1}[A] \cup Q_0$. Since $\mathbf{\Gamma}$ is reasonably closed, we can fix $\sigma : 2^{<\omega} \longrightarrow 2^{<\omega}$ such that $f_\sigma : 2^\omega \longrightarrow 2^\omega$ witnesses $A^* \leq B$. We will construct $\tau : 2^{<\omega} \longrightarrow 2^{<\omega}$ such that $f_\tau : 2^\omega \longrightarrow 2^\omega$ witnesses $A \leq A^*$ and $f_\sigma \circ f_\tau$ is injective.
Make sure that

1. $\tau(s)$ always ends with a 1
2. There are exactly $|s|$ blocks of zeros in $\tau(s)$
3. $s(n)$ is the parity of the n^{th} block of zeros in $\tau(s)$

Begin by setting $\tau(\varnothing) = \langle 1 \rangle$.
Given $s \in 2^{<\omega}$, notice that $\tau(s)^\frown\vec{0} \in A^*$ and $\tau(s)^\frown\vec{1} \notin A^*$.
Since f_σ witnesses that $A^* \leq B$, we must have $f_\sigma(\tau(s)^\frown\vec{0}) \in B$ and $f_\sigma(\tau(s)^\frown\vec{1}) \notin B$. Therefore, we can find $k \in \omega$ such that

$$\sigma(\tau(s)^\frown 0^k) \neq \sigma(\tau(s)^\frown 1^k)$$

Now simply pick $\tau(s^\frown i) \supseteq \tau(s)^\frown i^k$ for $i = 0, 1$ satisfying conditions (1), (2) and (3).
To check that f_τ has the desired properties, observe that

- $\mathrm{ran}(f_\tau) \subseteq 2^\omega \setminus (Q_0 \cup Q_1)$ (By conditions 1 and 2)
- $\phi(f_\tau(x)) = x$ for every $x \in 2^\omega$ (By conditions 1 and 3)

□

Our main tool: Steel's theorem

Given a Wadge class $\boldsymbol{\Gamma}$ in 2^ω and $X \subseteq 2^\omega$, we will say that X is *everywhere properly* $\boldsymbol{\Gamma}$ if $X \cap [s] \in \boldsymbol{\Gamma} \setminus \check{\boldsymbol{\Gamma}}$ for every $s \in 2^{<\omega}$.

Theorem (Steel, 1980)

Let $\boldsymbol{\Gamma}$ be a reasonably closed Wadge class in 2^ω. Assume that X and Y are subsets of 2^ω that satisfy the following:

- *X and Y are everywhere properly $\boldsymbol{\Gamma}$*
- *X and Y are either both meager or both comeager*

Then there exists a homeomorphism $h : 2^\omega \longrightarrow 2^\omega$ such that $h[X] = Y$.

Proof.

Without loss of generality, fix closed nowhere dense subsets X_n and Y_n of 2^ω for $n \in \omega$ such that $X \subset \bigcup_{n\in\omega} X_n$ and $Y \subset \bigcup_{n\in\omega} Y_n$. We will combine Harrington's Lemma with Knaster-Reichbach systems. (To be continued...)

Knaster-Reichbach covers

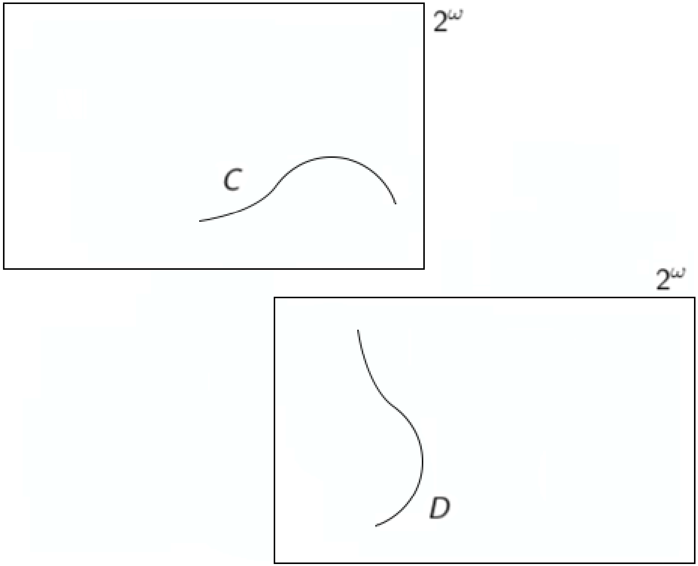
Fix a homeomorphism $h : C \longrightarrow D$ between closed nowhere dense subsets of 2^ω. We will say that $\langle \mathcal{U}, \mathcal{V}, \psi \rangle$ is a *Knaster-Reichbach cover* (briefly, a KR-cover) for $\langle 2^\omega \setminus C, 2^\omega \setminus D, h \rangle$ if the following conditions hold:

- $\mathcal{U}$ is a cover of $2^\omega \setminus C$ consisting of pairwise disjoint non-empty clopen subsets of 2^ω
- $\mathcal{V}$ is a cover of $2^\omega \setminus D$ consisting of pairwise disjoint non-empty clopen subsets of 2^ω
- $\psi : \mathcal{U} \longrightarrow \mathcal{V}$ is a bijection
- If $f : 2^\omega \longrightarrow 2^\omega$ is a bijection such that $h \subseteq f$ and $f[U] = \psi(U)$ for every $U \in \mathcal{U}$ (we say that f *respects* ψ), then f is continuous on C and f^{-1} is continuous on D

Lemma (van Engelen, 1986; see also Medini, 2015)

Let $h : C \longrightarrow D$ be a homeomorphism between closed nowhere dense subsets of 2^ω. Then there exists a KR-cover *for* $\langle 2^\omega \setminus C, 2^\omega \setminus D, h \rangle$.

Knaster-Reichbach covers



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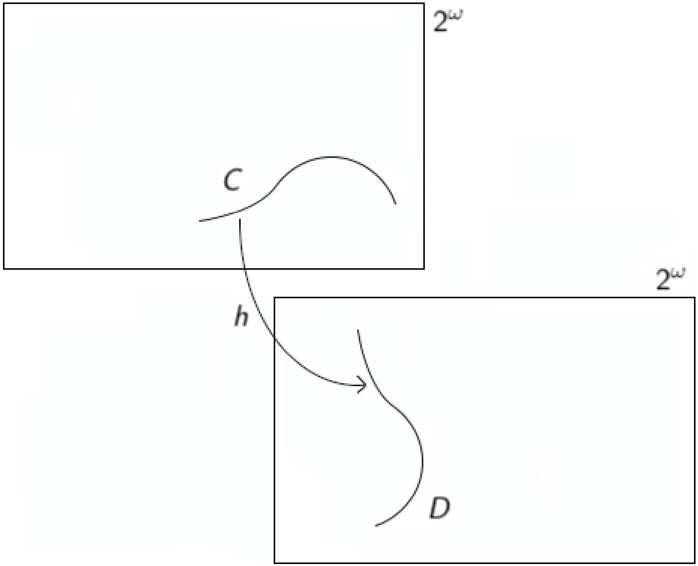

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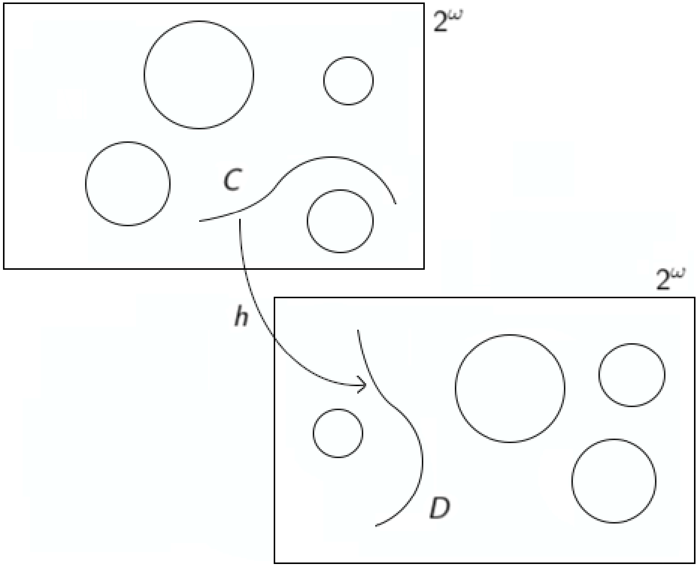

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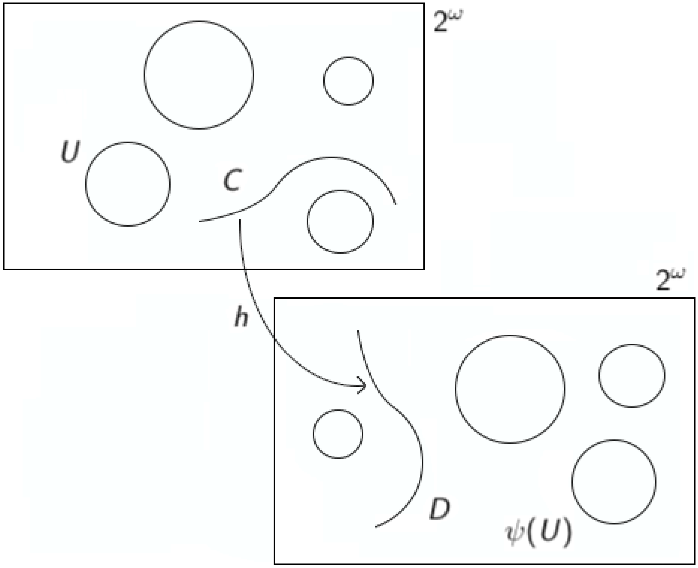
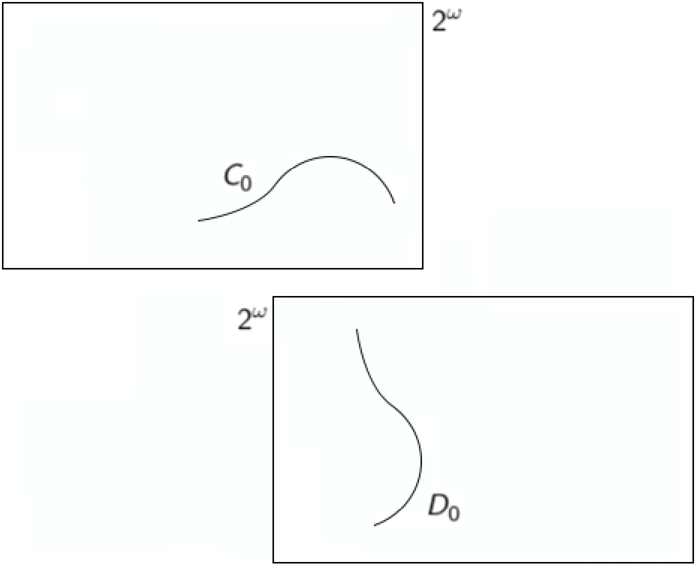


Knaster-Reichbach systems

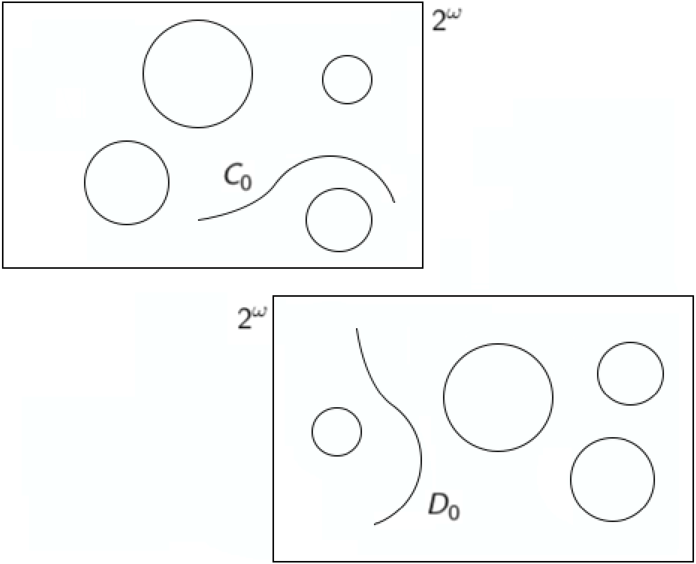
Fix an admissible metric on 2^ω. We will say that a sequence $\langle\langle h_n, \mathcal{K}_n\rangle : n \in \omega\rangle$ is a *Knaster-Reichbach system* (briefly, a KR-system) if the following conditions are satisfied:

- Each $h_n : C_n \longrightarrow D_n$ is a homeomorphism between closed nowhere dense subsets of 2^ω
- $h_m \subseteq h_n$ whenever $m \leq n$
- Each $\mathcal{K}_n = \langle \mathcal{U}_n, \mathcal{V}_n, \psi_n\rangle$ is a KR-cover for $\langle 2^\omega \setminus C_n, 2^\omega \setminus D_n, h_n\rangle$
- $\mathrm{mesh}(\mathcal{U}_n) \leq 2^{-n}$ and $\mathrm{mesh}(\mathcal{V}_n) \leq 2^{-n}$ for each n
- $\mathcal{U}_m$ refines $\mathcal{U}_n$ and $\mathcal{V}_m$ refines $\mathcal{V}_n$ whenever $m \geq n$
- Given $U \in \mathcal{U}_m$ and $V \in \mathcal{U}_n$ with $m \geq n$, then $U \subseteq V$ if and only if $\psi_m(U) \subseteq \psi_n(V)$

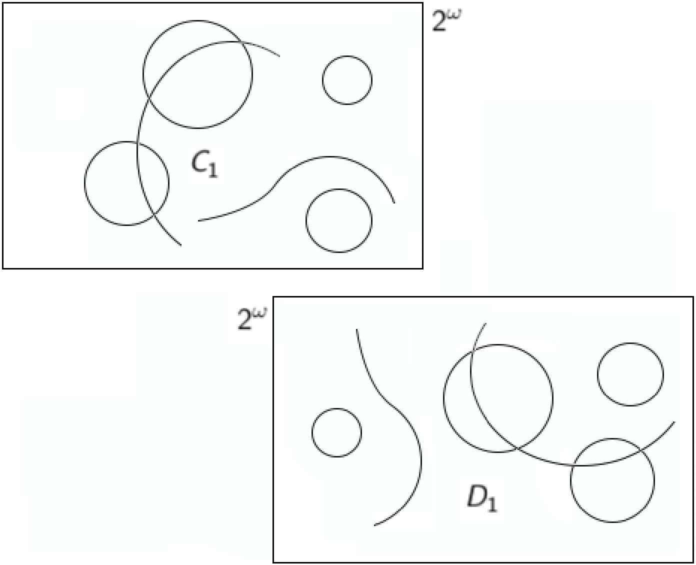
Knaster-Reichbach systems



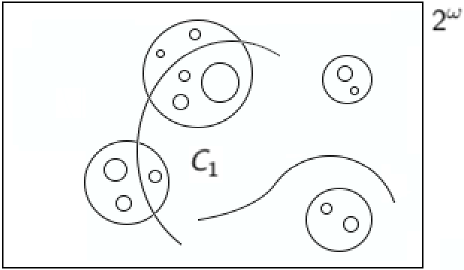
Knaster-Reichbach systems

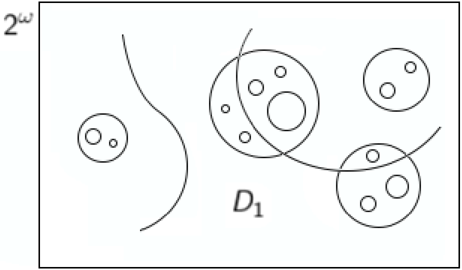


Knaster-Reichbach systems



Knaster-Reichbach systems





Why do we care about Knaster-Reichbach systems?

Because they give us homeomorphisms!

Theorem (see Medini, 2015)

Assume that $\langle\langle h_n, \mathcal{K}_n\rangle : n \in \omega\rangle$ *is a* KR*-system. Then there exists a homeomorphism* $h : 2^\omega \longrightarrow 2^\omega$ *such that* $h \supseteq \bigcup_{n\in\omega} h_n$*.*

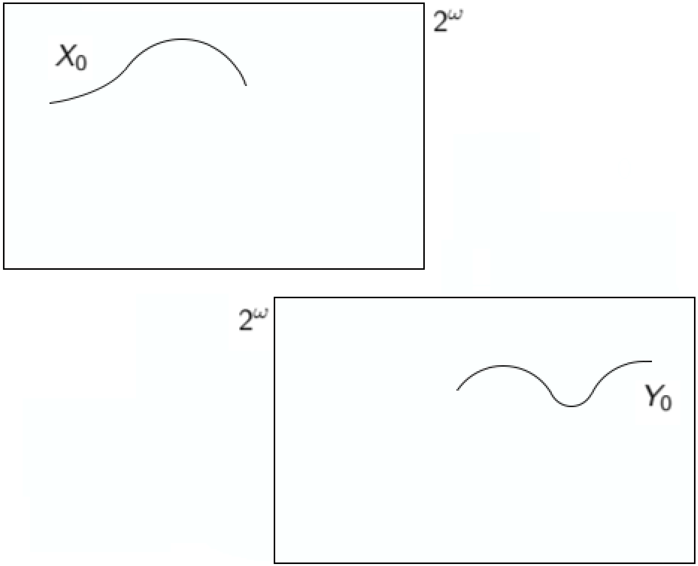
Corollary

Let X *and* Y *be subspaces of* 2^ω*. Assume that* $\langle\langle h_n, \mathcal{K}_n\rangle : n \in \omega\rangle$ *is a* KR*-system satisfying the following additional conditions:*

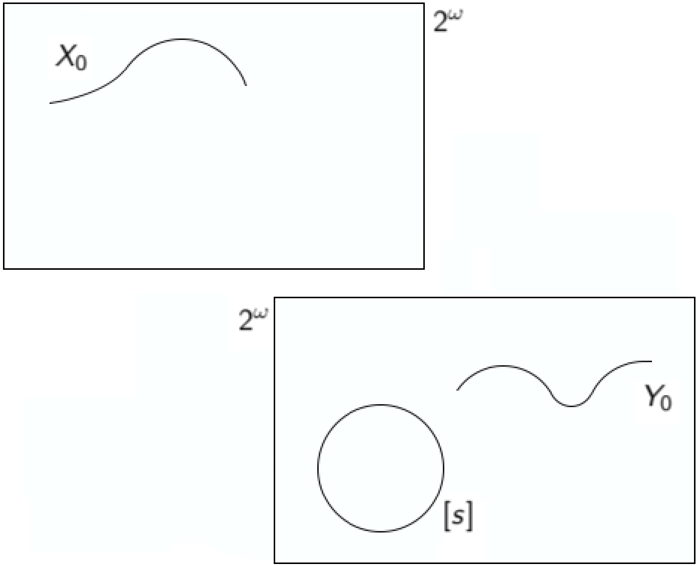
- $X \subseteq \bigcup_{n\in\omega} C_n$
- $Y \subseteq \bigcup_{n\in\omega} D_n$
- $h_n[X \cap C_n] = Y \cap D_n$ *for each* n

Then there exists a homeomorphism $h : 2^\omega \longrightarrow 2^\omega$ *such that* $h \supseteq \bigcup_{n\in\omega} h_n$ *and* $h[X] = Y$*.*

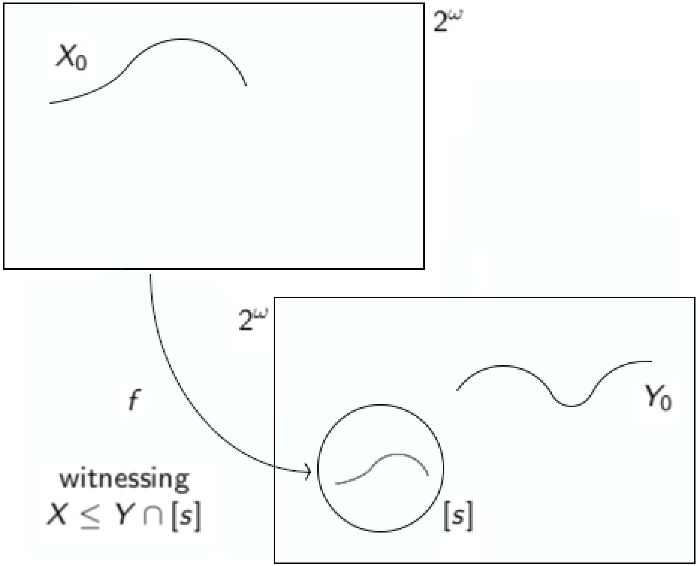
Proof of Steel's theorem



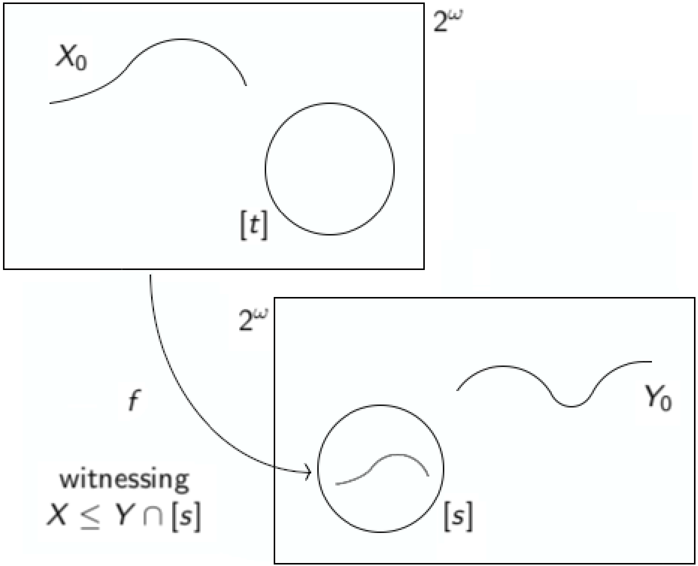
Proof of Steel's theorem



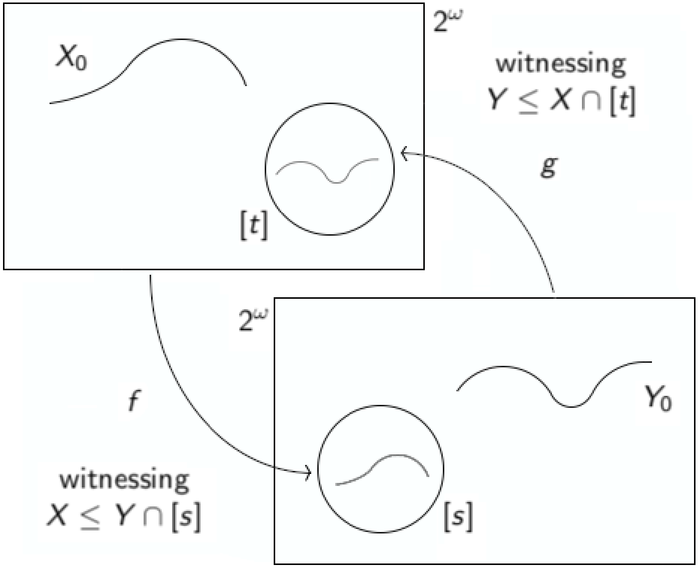
Proof of Steel's theorem



Proof of Steel's theorem



Proof of Steel's theorem



Proof of Steel's theorem

Remember that our strategy is to construct a KR-system $\langle\langle h_n, \mathcal{K}_n\rangle : n \in \omega\rangle$. We have seen how to begin:

- $C_0 = X_0 \cup g[Y_0]$
- $D_0 = Y_0 \cup f[X_0]$
- $h_0 = (f \restriction X_0) \cup (g^{-1} \restriction g[Y_0])$

Then obtain a KR-cover $\langle \mathcal{U}_0, \mathcal{V}_0, \psi_0\rangle$ for $\langle 2^\omega \setminus C_0, 2^\omega \setminus D_0, h_0\rangle$.

The next step is like the first one, but with the following changes:

- Instead of working between 2^ω and 2^ω, work between U and $\psi_0(U)$, where $U \in \mathcal{U}_0$
- Instead of looking at X_0 and Y_0, look at $X_1 \cap U$ and $Y_1 \cap \psi_0(U)$
- Repeat for every $U \in \mathcal{U}_0$, then union up the partial homeomorphisms to get h_1

Keep going like this for ω more steps...

□

A sufficient condition for reasonability

Lemma (Step 3)

Let $\mathbf{\Gamma}$ *be a Wadge class in* 2^ω *that is closed under intersections with* $\mathbf{\Pi}^0_2$ *sets and unions with* $\mathbf{\Sigma}^0_2$ *sets. Then* $\mathbf{\Gamma}$ *is reasonably closed.*

Proof.

Pick $A \in \mathbf{\Gamma}$. We need to show that $\phi^{-1}[A] \cup Q_0 \in \mathbf{\Gamma}$.
By Van Wesep's Theorem, fix $D \subseteq \mathcal{P}(\omega)$ such that $\mathbf{\Gamma} = \mathbf{\Gamma}_D(2^\omega)$.

Set $Z = 2^\omega \setminus (Q_0 \cup Q_1)$, and notice that $\phi^{-1}[A] \in \mathbf{\Gamma}_D(Z)$ by the Relativization Lemma. Therefore, again by the Relativization Lemma, there exists $B \in \mathbf{\Gamma}_D(2^\omega) = \mathbf{\Gamma}$ such that $B \cap Z = \phi^{-1}[A]$.

Since $Z \in \mathbf{\Pi}^0_2(2^\omega)$, it follows from our assumptions that $\phi^{-1}[A] \in \mathbf{\Gamma}$, hence $\phi^{-1}[A] \cup Q_0 \in \mathbf{\Gamma}$. □

In particular, it follows that both $\mathbf{\Sigma}^0_\xi(2^\omega)$ and $\mathbf{\Pi}^0_\xi(2^\omega)$ are reasonably closed whenever $3 \leq \xi < \omega_1$. (But we will need to be much more sophisticated than that!)

Three steps to reasonability

$\mathbf{\Gamma} = [X]$ for some homogeneous $X \subseteq 2^\omega$ such that $X \notin \Delta(\mathrm{D}_\omega(\mathbf{\Sigma}^0_2))$

$\downarrow$

$\mathbf{\Gamma}$ is a good Wadge class

$\downarrow$

$\mathbf{\Gamma}$ is closed under $\cap\, \mathbf{\Pi}^0_2$ and $\cup\, \mathbf{\Sigma}^0_2$

$\downarrow$

$\mathbf{\Gamma}$ is reasonably closed

The notion of level

From now on, $\xi < \omega_1$ and $\mathbf{\Gamma}$, $\mathbf{\Lambda}$ are Wadge classes in Z.

Definition (Louveau, Saint-Raymond, 1988)

- $\ell(\mathbf{\Gamma}) \geq \xi$ if $\mathrm{PU}_\xi(\mathbf{\Gamma}) = \mathbf{\Gamma}$
- $\ell(\mathbf{\Gamma}) = \xi$ if $\ell(\mathbf{\Gamma}) \geq \xi$ and $\ell(\mathbf{\Gamma}) \not\geq \xi + 1$
- $\ell(\mathbf{\Gamma}) = \omega_1$ if $\ell(\mathbf{\Gamma}) \geq \xi$ for every $\xi < \omega_1$

We refer to $\ell(\mathbf{\Gamma})$ as the *level* of $\mathbf{\Gamma}$.

Examples:

- $\ell(\mathbf{\Gamma}) \geq 0$ for every $\mathbf{\Gamma}$
- $\ell(\{\varnothing\}) = \ell(\{Z\}) = \omega_1$
- $\ell(\mathbf{\Sigma}^1_n) = \ell(\mathbf{\Pi}^1_n) = \omega_1$ for every $n \in \omega$
- $\ell(\mathbf{\Sigma}^0_{1+\xi}) = \ell(\mathbf{\Pi}^0_{1+\xi}) = \xi$

It is true (but not easy to prove) that for every non-selfdual Wadge class $\mathbf{\Gamma}$ there exists $\xi \leq \omega_1$ such that $\ell(\mathbf{\Gamma}) = \xi$.

The expansion theorem

Definition (Wadge, 1984)

$$\mathbf{\Gamma}^{(\xi)} = \{f^{-1}[A] : A \in \mathbf{\Gamma} \text{ and } f : Z \longrightarrow Z \text{ is } \mathbf{\Sigma}^0_{1+\xi}\text{-measurable}\}$$

We will refer to $\mathbf{\Gamma}^{(\xi)}$ as an *expansion* of $\mathbf{\Gamma}$. To see what happens with regard to Hausdorff operations, it can be shown that

$$\mathbf{\Gamma}_D(Z)^{(\xi)} = \{\mathcal{H}_D(A_0, A_1, \ldots) : A_0, A_1, \ldots \in \mathbf{\Sigma}^0_{1+\xi}(Z)\}$$

Theorem (Louveau)

Assume that $\mathbf{\Gamma}$ is non-selfdual. Then the following conditions are equivalent:

- $\ell(\mathbf{\Gamma}) \geq \xi$
- *There exists a non-selfdual $\mathbf{\Lambda}$ such that $\mathbf{\Lambda}^{(\xi)} = \mathbf{\Gamma}$*

Good Wadge classes

Definition

We will say that $\mathbf{\Gamma}$ is *good* if the following are satisfied:

- $\mathbf{\Gamma}$ is non-selfdual
- $\Delta(\mathrm{D}_\omega(\mathbf{\Sigma}^0_2)) \subseteq \mathbf{\Gamma}$
- $\ell(\mathbf{\Gamma}) \geq 1$

Lemma (Step 2)

If $\mathbf{\Gamma}$ is good then $\mathbf{\Gamma}$ is closed under $\cap\, \mathbf{\Pi}^0_2$ and $\cup\, \mathbf{\Sigma}^0_2$.

Proof.

Andretta, Hjorth and Neeman proved that if $\Delta(\mathrm{D}_\omega(\mathbf{\Sigma}^0_1)) \subseteq \mathbf{\Lambda}$ then $\mathbf{\Lambda}$ is closed under $\cap\, \mathbf{\Pi}^0_1$ and $\cup\, \mathbf{\Sigma}^0_1$. Since $\ell(\mathbf{\Gamma}) \geq 1$ there exists $\mathbf{\Lambda}$ such that $\mathbf{\Lambda}^{(1)} = \mathbf{\Gamma}$. Apply the above mentioned result to $\mathbf{\Lambda}$, then transfer it to $\mathbf{\Gamma}$ using expansions. □

The proof of Step 1

Let $X \subseteq 2^\omega$ be dense and homogeneous, with $X \notin \Delta(\mathrm{D}_\omega(\mathbf{\Sigma}^0_2))$. We need to show that $[X]$ is a good Wadge class.

Fix a minimal non-selfdual $\mathbf{\Gamma}$ such that there exists a non-empty $U \in \mathbf{\Sigma}^0_1(2^\omega)$ such that $X \cap U \in \mathbf{\Gamma}$ or $X \cap U \in \check{\mathbf{\Gamma}}$. Assume that $X \cap U \in \mathbf{\Gamma}$. First we will show that $\mathbf{\Gamma}$ is good, then that $[X] = \mathbf{\Gamma}$.

Assume, in order to get a contradiction, that $X \cap U \in \Delta(\mathrm{D}_\omega(\mathbf{\Sigma}^0_2))$. Notice that

$$\mathcal{U} = \{h[X \cap U] : h \text{ is a homeomorphism of } X\}$$

is a cover of X because X is homogeneous and dense in 2^ω.

Furthermore, since $\mathrm{D}_\omega(\mathbf{\Sigma}^0_2)$ is a good Wadge class, the following lemma shows that each element of $\mathcal{U}$ belongs to it:

Lemma (Good Wadge classes are "topological")

Let $\mathbf{\Gamma}$ be a good Wadge class in Z. If $A \in \mathbf{\Gamma}$ and $B \approx A$ then $B \in \mathbf{\Gamma}$.

Using a countable subcover of $\mathcal{U}$, write X as a partitioned union of sets in $\mathrm{D}_\omega(\mathbf{\Sigma}^0_2)$, where the elements of the partition are $\mathbf{\Delta}^0_2$. More specifically, let $\mathcal{V} = \{h[U_n \cap X] : n \in \omega\}$ be a countable subcover of $\mathcal{U}$, and set $A_n = h[U_n \cap X]$. Let $\widetilde{V_n} \in \mathbf{\Sigma}^0_1(2^\omega)$ be such that $\widetilde{V_n} \cap X = A_n$. Let V_n disjointify the $\widetilde{V_n}$, then set $V_{-1} = 2^\omega \setminus (\bigcup_{n \in \omega} \widetilde{V_n})$ and $A_{-1} = \varnothing$. It is clear that each V_n is $\mathbf{\Delta}^0_2$, while

$$X = \bigcup_{-1 \leq n < \omega} (A_n \cap V_n)$$

Since $\ell(\mathrm{D}_\omega(\mathbf{\Sigma}^0_2)) \geq 1$, it follows that $X \in \mathrm{D}_\omega(\mathbf{\Sigma}^0_2)$. A similar argument shows that $X \in \check{\mathrm{D}}_\omega(\mathbf{\Sigma}^0_2)$. This contradicts our assumptions, so $X \cap U \notin \Delta(\mathrm{D}_\omega(\mathbf{\Sigma}^0_2))$.

Notice that the above reasoning actually proves the following:

Lemma

Let $\mathbf{\Gamma}$ be a good Wadge class, and let $X \subseteq Z$ be homogeneous. If $X \cap U \in \mathbf{\Gamma}$ for some non-empty $U \in \mathbf{\Sigma}^0_1(Z)$ then $X \in \mathbf{\Gamma}$.

It remains to show that $\ell(\boldsymbol{\Gamma}) \geq 1$. Assume, in order to get a contradiction, that $\ell(\boldsymbol{\Gamma}) = 0$. Then, applying the following with $Z = U$ will contradict the minimality of $\boldsymbol{\Gamma}$:

Lemma

Assume that $\boldsymbol{\Gamma}$ is non-selfdual and that $\ell(\boldsymbol{\Gamma}) = 0$. Let $X \in \boldsymbol{\Gamma}$ be codense in 2^ω. Then there exist a non-empty $V \in \boldsymbol{\Delta}^0_1(Z)$ and a non-selfdual $\boldsymbol{\Lambda}$ such that $\boldsymbol{\Lambda} \subsetneq \boldsymbol{\Gamma}$ and $X \cap V \in \boldsymbol{\Lambda}$.

Now that we know that $\boldsymbol{\Gamma}$ is a good Wadge class, since $X \cap U \in \boldsymbol{\Gamma}$, we can apply the previous lemma to see that $X \in \boldsymbol{\Gamma}$, so $[X] \subseteq \boldsymbol{\Gamma}$. It remains to show that $[X] \subsetneq \boldsymbol{\Gamma}$ is impossible.

If X is non-selfdual, this would directly contradict minimality of $\boldsymbol{\Gamma}$. Otherwise, minimality would be contradicted after applying the analysis of the selfdual sets.

□

Finishing the proof

Let X be a zero-dimensional homogeneous space that is not locally compact. Without loss of generality, assume that X is a dense subspace of 2^ω. If $X \in \Delta(\mathrm{D}_\omega(\mathbf{\Sigma}^0_2))$, then X is strongly homogeneous by van Engelen's results. Therefore, we can also assume without loss of generality that $X \notin \Delta(\mathrm{D}_\omega(\mathbf{\Sigma}^0_2))$.

Fix $s \in 2^{<\omega}$, and let $Y = X \cap [s]$. As in the proof of Step 1, using also the Relativization Lemma, one can show that X and Y are everywhere properly $\mathbf{\Gamma} = [X]$ (in 2^ω and $[s] \approx 2^\omega$ respectively).

Since X is homogeneous, either X is meager or it is Baire (hence comeager in 2^ω by AD). The same will be true of Y.

Hence $Y \approx X$ by Steel's theorem. The following result concludes the proof that X is strongly homogeneous:

Theorem (Terada, 1993)

Let X be a space. Assume that X has a base $\mathcal{B} \subseteq \mathbf{\Delta}^0_1(X)$ such that $U \approx X$ for every $U \in \mathcal{B}$. Then X is strongly homogeneous.

Open questions

As we have seen, for spaces of complexity higher than $\Delta(\mathrm{D}_\omega(\boldsymbol{\Sigma}^0_2))$, Baire category and Wadge class are sufficient to uniquely identify a homogeneous zero-dimensional space. This is the "uniqueness" part of the classification. But the "existence" part is still open:

Question

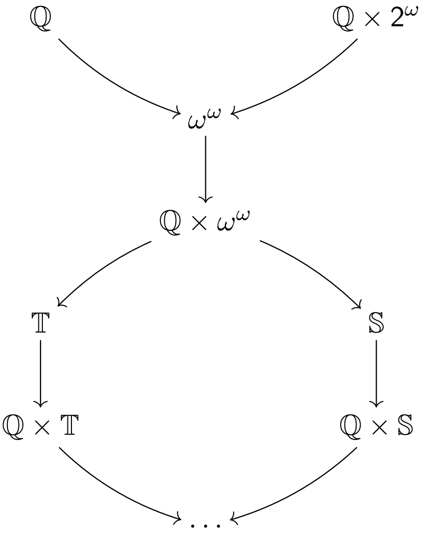
For exactly which good Wadge classes $\boldsymbol{\Gamma}$ is there a homogeneous X such that $\boldsymbol{\Gamma} = [X]$? For which ones is there a meager such X? For which ones is there a Baire such X?

Does the usual pattern of results under AD hold?

Question

Assuming $\mathrm{V} = \mathrm{L}$, is it possible to construct a zero-dimensional $\boldsymbol{\Pi}^1_1$ or $\boldsymbol{\Sigma}^1_1$ space that is homogeneous, not locally compact, and not strongly homogeneous?

What happens below $\Delta(\mathsf{D}_\omega(\mathbf{\Sigma}^0_2))$?



Filters and semifilters

Whenever $\mathcal{X} \subseteq \mathcal{P}(\omega)$, we will identify $\mathcal{X}$ with the subspace of 2^ω consisting of the characteristic functions of elements of $\mathcal{X}$.

Definition

A *semifilter* is a collection $\mathcal{S} \subseteq \mathcal{P}(\omega)$ that satisfies the following conditions:

1. $\varnothing \notin \mathcal{S}$ and $\omega \in \mathcal{S}$
2. If $X \in \mathcal{S}$ and $X =^* Y \subseteq \omega$ then $Y \in \mathcal{S}$
3. If $X \in \mathcal{S}$ and $X \subseteq Y \subseteq \omega$ then $Y \in \mathcal{S}$

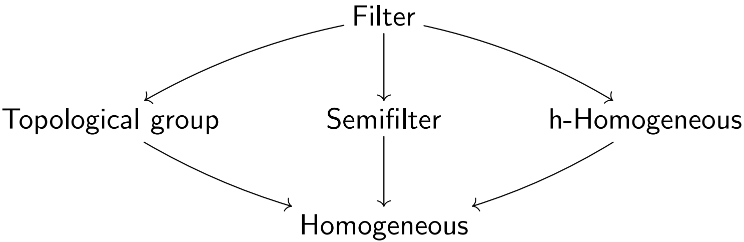
Notice that $\mathrm{Fin} \cap \mathcal{S} = \varnothing$ and $\mathrm{Cof} \subseteq \mathcal{S}$ for every semifilter $\mathcal{S}$. In particular, no semifilter is locally compact.

Definition

A *filter* is a semifilter $\mathcal{F}$ such that the following holds:

4. If $X, Y \in \mathcal{F}$ then $X \cap Y \in \mathcal{F}$

Filters are deliciously homogeneous!



A characterization of Borel filters

As we have seen, the combinatorial structure of filters imposes strong constraints on their topology. But is it possible to go in the other direction as well?
In other words, given a space, is it possible to recognize whether it is homeomorphic to a filter?
This problem has a very elegant solution in the Borel realm:

Theorem (van Engelen, 1994)

Let X be a zero-dimensional Borel space that is not locally compact. Then the following conditions are equivalent:

- *X is homeomorphic to a filter*
- *X is homogeneous, meager, and homeomorphic to its square*

The above characterization inspired the following ZF + DC result:

Theorem (Medini and Zdomskyy, 2016)

Every filter is homeomorphic to its square.

What about semifilters?

Theorem (Medini, 2019)

Let X be a zero-dimensional Borel space that is not locally compact. Then the following conditions are equivalent:

- *X is homeomorphic to a semifilter*
- *X is homogeneous*

Easy counterexamples show that the "Borel" assumption cannot be altogether dropped in ZFC, but the following two natural questions are open (hopefully, not for long):

Question

Under AD, *can the "Borel" assumption be dropped in the above characterization of semifilters?*

Question

Under AD, *can the "Borel" assumption be dropped in van Engelen's characterization of filters?*

Two concrete non-trivial examples: $\mathbb{S}$ and $\mathbb{T}$

Theorem (van Mill, 1983; van Douwen)

Let X be a zero-dimensional space.

- *$X \approx \mathbb{S}$ if and only if X is the union of a complete subspace and a σ-compact subspace, X is nowhere σ-compact, and X is nowhere the union of a complete and a countable subspace*
- *$X \approx \mathbb{T}$ if and only if X is the union of a complete subspace and a countable subspace, X is nowhere σ-compact, and X is nowhere complete*

Fix infinite sets Ω_1 and Ω_2 such that $\Omega_1 \cup \Omega_2 = \omega$ and $\Omega_1 \cap \Omega_2 = \varnothing$. Define

$$\mathcal{T} = \{x_1 \cup x_2 : x_1 \subseteq \Omega_1,\ x_2 \subseteq \Omega_2, \text{ and}$$

$$(x_1 \notin \mathrm{Fin}(\Omega_1) \text{ or } x_2 \in \mathrm{Cof}(\Omega_2))\}$$

It is clear that $\mathcal{T}$ is a semifilter. Furthermore, $\mathcal{T}$ is the union of the following spaces:

- $\{x \subseteq \omega : x \cap \Omega_1 \notin \mathrm{Fin}(\Omega_1)\} \approx \omega^\omega \times 2^\omega \approx \omega^\omega$
- $\{x_1 \cup x_2 : x_1 \in \mathrm{Fin}(\Omega_1) \text{ and } x_2 \in \mathrm{Cof}(\Omega_2)\} \approx \mathbb{Q}$

Using the fact that $\mathcal{T}$ is homogeneous, one can easily see that $\mathcal{T}$ is nowhere σ-compact and nowhere complete. Hence $\mathcal{T} \approx \mathbb{T}$.

To describe $\mathbb{S}$, also fix an infinite $\Omega \subseteq \Omega_2$ such that $\Omega_2 \setminus \Omega$ is infinite. Define

$$\mathcal{S} = \{x_1 \cup x_2 : x_1 \subseteq \Omega_1,\ x_2 \subseteq \Omega_2, \text{ and}$$

$$(x_1 \notin \mathrm{Fin}(\Omega_1) \text{ or } \Omega \subseteq^* x_2)\}$$

Using an argument similar to the one that works for $\mathcal{T}$, one can show that $\mathcal{S} \approx \mathbb{S}$.

Thank you for your attention

and have a good evening!