The b-adic diaphony of digital $(0, 1)$-sequences: A central limit theorem and a relation to the classical diaphony

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Abstract

The b-adic diaphony is a quantitative measure for the irregularity of distribution of a sequence in the unit interval. In this paper we show that the b-adic diaphony of digital $(0, 1)$-sequences satisfies a central limit theorem. Further we show a relation between the functions $\chi_b^{\delta_j}$ and ψ_b, which appear in the formulas for the classical diaphony and the b-adic diaphony of digital $(0, 1)$-NUT-sequences respectively. This relation implies that the expected value of the squared classical diaphony over all digital $(0, 1)$-NUT-sequences is given up to a constant by the squared b-adic diaphony, which has the same value for any digital $(0, 1)$-sequence over $\mathbb{Z}_b$.

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1 Introduction

The classical diaphony F_N (see [18] or [6, Definition 1.29] or [13, Exercise 5.27, p.162]) of the first N elements of the sequences $\omega = (x_n)_{n \geq 0}$ in $[0, 1)$ is given by

$$
F_N(\omega) := \left(2 \sum_{k=1}^{\infty} \frac{1}{k^2} \left| \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i k x_n} \right|^2 \right)^{1/2}.
$$

It is well known that the classical diaphony is a quantitative measure for the irregularity of distribution of the first N points of a sequence: A sequence ω is uniformly distributed modulo one if and only if $\lim_{N \to \infty} F_N(\omega) = 0$.

In [12] Hellekalek and Leeb introduced the notion of dyadic diaphony which was later generalized by Grozdanov and Stoilova [10] for general integers $b \geq 2$. This b-adic diaphony is similar to the classical diaphony but with the trigonometric functions replaced by Walsh functions in base b. Before we give the exact definition of the b-adic diaphony

we recall the definition of Walsh functions.

Let b be an integer. For a non-negative integer k with base b representation $k = \kappa_{a-1}b^{a-1} + \cdots + \kappa_1 b + \kappa_0$ and a real $x \in [0,1)$ with base b representation $x = \frac{x_1}{b} + \frac{x_2}{b^2} + \cdots$ (unique in the sense that infinitely many of the x_i must be different from $b-1$) the k-th Walsh function in base b is defined as

$$_b\mathrm{wal}_k(x) := \mathrm{e}^{2\pi\mathtt{i}(x_1\kappa_0 + \cdots + x_a\kappa_{a-1})/b}.$$

Now we give the definition of the b-adic diaphony (see [10] or [12]).

Definition 1. Let $b \geq 2$ be an integer. The b-adic diaphony of the first N elements of a sequence $\omega = (x_n)_{n \geq 0}$ in $[0,1)$ is defined by

$$F_{b,N}(\omega) := \left(\frac{1}{b} \sum_{k=1}^{\infty} \frac{1}{b^{2a(k)}} \left| \frac{1}{N} \sum_{n=0}^{N-1} {}_b\mathrm{wal}_k(x_n) \right|^2 \right)^{1/2},$$

where $a(k)$ is the unique determined integer a such that $b^a \leq k < b^{a+1}$. If $b = 2$ we also speak of dyadic diaphony.

The b-adic diaphony is a quantitative measure for the irregularity of distribution of a sequence: a sequence ω is uniformly distributed modulo one if and only if $\lim_{N \to \infty} F_{b,N}(\omega) = 0$. This was shown in [12] for the case $b = 2$ and in [10] for the general case. Further it is shown in [2] that the b-adic diaphony is – up to a factor depending on b and the dimension s – the worst case error for quasi-Monte Carlo integration of functions from a certain Hilbert space $H_{\mathrm{wal},s,\gamma}$, which has been introduced in [4].

Throughout this paper let b be a prime and let $\mathbb{Z}_b$ be the finite field of prime order b. We consider the b-adic diaphony of a special class of sequences in $[0,1)$, namely of so-called digital $(0,1)$-sequences over $\mathbb{Z}_b$. Here 1 is the dimension of the sequence and 0 is the (best possible) quality parameter. This is a special case of digital (t,s)-sequences over $\mathbb{Z}_b$ which were introduced by Niederreiter [14, 15] in a more general setting and provide at the moment the most efficient method to generate sequences with excellent distribution properties.

Before we give the definition of digital $(0,1)$-sequences over $\mathbb{Z}_b$ we introduce some notation: For a vector $\mathbf{c} = (c_1, c_2, \ldots) \in \mathbb{Z}_b^{\infty}$ and $m \in \mathbb{N}$ we denote the vector in $\mathbb{Z}_b^m$ consisting of the first m components of $\mathbf{c}$ by $\mathbf{c}(m)$, i.e. $\mathbf{c}(m) = (c_1, \ldots, c_m)$. Further let $\mathbf{c}_i$ denote the ith row vector of the matrix C.

Definition 2. Let b be a prime and C a $\mathbb{N} \times \mathbb{N}$ matrix over $\mathbb{Z}_b$ with the property that for every $m \in \mathbb{N}$ the vectors

$$\mathbf{c}_1(m), \ldots, \mathbf{c}_m(m)$$

are linearly independent.
For $n \geq 0$ let $n = n_0 + n_1 b + n_2 b^2 + \cdots$ be the base b representation of n. Multiply the vector $\mathbf{n} = (n_0, n_1, \ldots)^{\top} \in \mathbb{Z}_b^{\infty}$ by the matrix C,

$$C \cdot \mathbf{n} =: (x_n(1), x_n(2), \ldots)^{\top} \in \mathbb{Z}_b^{\infty},$$

and set
$$x_n := \frac{x_n(1)}{b} + \frac{x_n(2)}{b^2} + \cdots .$$

Every sequence $(x_n)_{n \geq 0}$ constructed this way is called digital $(0,1)$-sequence over $\mathbb{Z}_b$. The matrix $C = (c_{i,j})_{i,j \geq 1}$ is called generator matrix of the sequence. If the generator matrix C is a non-singular upper triangular matrix the sequence is called a digital $(0,1)$-NUT-sequence.

For example if we choose as generator matrix the $\mathbb{N} \times \mathbb{N}$ identity matrix, then the resulting digital $(0,1)$-sequence over $\mathbb{Z}_b$ is the well known van der Corput sequence in base b. Hence the concept of digital $(0,1)$-sequences over $\mathbb{Z}_b$ is a generalization of the construction principle of the van der Corput sequence.

To guarantee that the points x_n belong to $[0,1)$ (and not just to $[0,1]$) we need the condition that for each $n \geq 0$ infinitely many of the $x_n(i)$ are different from $b-1$. This condition is always satisfied if we assume that for each $r \geq 1$ we have $c_{i,r} = 0$ for all sufficiently large i. Throughout this article we assume that the generator matrix fulfills this condition (see [15, p.72] where this condition is called (S6)). More information about (t,s)-sequences can be found in the books [15] and [3].

In this article we pursue two main goals:

1. We want to extend the central limit theorem for the dyadic diaphony of digital $(0,1)$-sequences over $\mathbb{Z}_2$ (see [16, Corollary 2.4]) to the b-adic diaphony for primes $b \geq 2$. This will be done in Theorem 4 in Section 2.

2. We want to show a relation between the functions $\chi_b^{\delta_j}$ and ψ_b, which appear in the formulas for the classical diaphony and the b-adic diaphony of digital $(0,1)$-NUT-sequences in base b, see Proposition 5 in Section 2. This relation will imply that the expected value of the squared classical diaphony over all digital $(0,1)$-NUT-sequences is given up to a constant by the squared b-adic diaphony, which has the same value for any digital $(0,1)$-sequence over $\mathbb{Z}_b$, see Theorem 6 in Section 2.

For the classical diaphony it was proven by Faure [7, Theorem 4] that

$$(NF_N(\omega))^2 = \frac{4\pi^2}{b^2} \sum_{j=1}^{\infty} \chi_b^{\delta_j} \left(\frac{N}{b^j} \right), \tag{1}$$

where ω is a digital $(0,1)$-NUT-sequence over $\mathbb{Z}_b$. Here $\chi_b^{\delta_j}$ are certain functions depending on the generator matrix C, which are defined in the following way (see [7]):

Let b be a prime and σ be a permutation of $\mathbb{Z}_b$. Set

$$Z_b^\sigma := \left(\frac{\sigma(0)}{b}, \dots, \frac{\sigma(b-1)}{b} \right).$$

For any integer h, $0 \leq h \leq b-1$, the function $\varphi_{b,h}^\sigma : \mathbb{R} \to \mathbb{R}$ is defined as follows. Let k be an integer with $1 \leq k \leq b$. Then for $x \in \left[\frac{k-1}{b}, \frac{k}{b} \right)$ we set

$$\varphi_{b,h}^\sigma(x) := \begin{cases} A\left(\left[0, \frac{h}{b}\right), k, Z_b^\sigma \right) - hx & \text{if } 0 \leq h \leq \sigma(k-1) \\ (b-h)x - A\left(\left[\frac{h}{b}, 1\right), k, Z_b^\sigma \right) & \text{if } \sigma(k-1) < h < b, \end{cases}$$

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where $A([\alpha, \beta), k, Z_b^\sigma)$ is the number of indices n such that $1 \leq n \leq k$ and $Z_b^\sigma(n) \in [\alpha, \beta) \subseteq [0, 1)$; then the function $\varphi_{b,h}^\sigma$ is extended to the reals by periodicity. Note that $\varphi_{b,0}^\sigma = 0$.

Thus we have associated b functions with the pair (b, σ):

$$\varphi_{b,0}^\sigma = 0, \varphi_{b,1}^\sigma, \ldots, \varphi_{b,b-1}^\sigma.$$

For a given pair (b, σ) we set now

$$\chi_b^\sigma := b \sum_{h=0}^{b-1} (\varphi_{b,h}^\sigma)^2 - \left(\sum_{h=0}^{b-1} \varphi_{b,h}^\sigma \right)^2.$$

For $x \in \mathbb{Z}$ let $[x]_b$ denote the remainder of x, when divided by b, i.e. $[x]_b \in \{0, 1, \ldots, b-1\}$. For simplicity we omit from now on the subscript b, i.e. $[x] = [x]_b$. For $j \geq 1$ the permutation δ_j of $\mathbb{Z}_b$ is defined as

$$\delta_j := \begin{pmatrix} 0 & 1 & 2 & \cdots & b-1 \\ 0 & c_j^j & [2c_j^j] & \cdots & [(b-1)c_j^j] \end{pmatrix},$$

where c_j^j are the diagonal entries of the generator matrix C.

For the b-adic diaphony the author showed in [8, Theorem 6] that

$$(NF_{b,N}(\omega))^2 = \frac{12}{b^2} \sum_{u=1}^\infty \psi_b \left(\frac{N}{b^u} \right) \tag{2}$$

for any digital $(0, 1)$-sequence over $\mathbb{Z}_b$. Here the function ψ_b does not depend on the generator matrix C, so the b-adic diaphony is invariant for all digital $(0, 1)$-sequences over $\mathbb{Z}_b$. The function ψ_b is defined as follows:

Definition 3. Let β be an integer in $\{1, \ldots, b-1\}$. For $x \in \left[\frac{j}{b}, \frac{j+1}{b} \right)$, $j \in \{0, \ldots, b-1\}$ we set

$$\psi_b^\beta(x) := \frac{b^2(b^2-1)}{12} \left| \frac{1}{b} \frac{z_\beta^j - 1}{z_\beta - 1} + z_\beta^j \left(x - \frac{j}{b} \right) \right|^2,$$

where $z_\beta = e^{\frac{2\pi i}{b}\beta} = {}_b\text{wal}_1 \left(\frac{\beta}{b} \right)$; then the function is extended to the reals by periodicity. The function ψ_b is now defined as the mean of the functions ψ_b^β:

$$\psi_b(x) := \frac{1}{b-1} \sum_{\beta=1}^{b-1} \psi_b^\beta(x).$$

The functions $\chi_b^{\delta_j}$ appearing in (1) and ψ_b appearing in (2) have a similar structure (see [1, Propriété 3.3, Propriété 3.5 (ii)] and [8, Lemma 11]):

1. $\chi_b^{\delta_j}(x) = \psi_b(x) = \frac{b^2(b^2-1)}{12} x^2$ for $x \in \left[0, \frac{1}{b} \right)$,

2. $\chi_b^{\delta_j}$ and ψ_b are continuous,

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3. on intervals of the form $\left[\frac{k}{b}, \frac{k+1}{b}\right)$ the functions $\chi_b^{\delta_j}$ and ψ_b are a translation of the parabola $\frac{b^2(b^2-1)}{12}x^2$.

We use the following notation: For $c \in \{1, \ldots, b-1\}$ let χ_b^c denote the function χ_b^σ with the permutation σ of $\mathbb{Z}_b$ given by.

$$\sigma = \begin{pmatrix} 0 & 1 & 2 & \cdots & b-1 \\ 0 & c & [2c] & \cdots & [(b-1)c] \end{pmatrix}.$$

Note that $\chi_b^{\delta_j} \in \{\chi_b^1, \ldots, \chi_b^{b-1}\}$ for all $j \in \mathbb{N}$. For $b = 2$ and $b = 3$ the functions ψ_b and χ_b^c are the same. This can be easily calculated. Hence we have for any digital $(0,1)$-NUT-sequences ω over $\mathbb{Z}_2$ and $\mathbb{Z}_3$

$$F_{2,N}(\omega) = \frac{\sqrt{3}}{\pi} F_N(\omega) \quad \text{and} \quad F_{3,N}(\omega) = \frac{\sqrt{3}}{\pi} F_N(\omega) \tag{3}$$

respectively. I.e., the classical diaphony and the b-adic diaphony of any digital $(0,1)$-NUT-sequences ω over $\mathbb{Z}_b$ are for $b = 2, 3$ up to the constant $\frac{\sqrt{3}}{\pi}$ the same. For greater values of b this is no longer true. The aim of this paper is now to prove a relation between the functions χ_b^c and ψ_b, see Proposition 5. This will imply that the squared (invariant) b-adic diaphony of a digital $(0,1)$-sequence over $\mathbb{Z}_b$ is the expected value over all $(0,1)$-NUT-sequences of the squared classical diaphony, see Theorem 6.

2 Results

We will show a central limit theorem for the b-adic diaphony of digital $(0,1)$-sequences over $\mathbb{Z}_b$.

Theorem 4. *Let b be a prime and ω be a digital $(0,1)$-sequence over $\mathbb{Z}_b$. Then for any real y we have*

$$\frac{1}{b^m} \left| \left\{ N < b^m : (NF_{b,N}(\omega))^2 \leq \frac{b^2-1}{6b} \log_b N + y \frac{12}{b^2} \sqrt{\frac{\log_b N(b^4-1)b^2}{25920}} \right\} \right|$$
$$= \Phi(y) + o(1) \qquad (as \ m \to \infty),$$

where

$$\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-\frac{t^2}{2}} dt$$

denotes the normal distribution function and $\log_b$ denotes the logarithm to the base b. I.e., the squared b-adic diaphony of a digital $(0,1)$-sequence over $\mathbb{Z}_b$ satisfies a central limit theorem.

The proof of this theorem will be given in Section 3.

For $b = 2$ and $b = 3$ the functions ψ_b and χ_b^c are the same. For greater values of b this is no longer true, i.e. the functions χ_b^c depend on the parameter c, but we have the following nice relation between ψ_b and χ_b^c:

Proposition 5. *Let b be a prime and ψ_b and χ_b^c as above, then we have*

$$\psi_b(x) = \frac{1}{b-1} \sum_{c=1}^{b-1} \chi_b^c(x). \tag{4}$$

The proof of this theorem will be given in Section 4.

For $b = 2$ and $b = 3$ the classical and the b-adic diaphony are up to a constant the same, see (3). For $b \geq 5$ such a simple relation is no longer true, but we still have a relation between the classical diaphony and the b-adic diaphony of digital $(0, 1)$-NUT-sequences over $\mathbb{Z}_b$.

Theorem 6. *The expected value of the squared classical diaphony over all $(0, 1)$-NUT-sequences over $\mathbb{Z}_b$ is given by*

$$\mathbb{E}((NF_N(\omega))^2) = \frac{\pi^2}{3}(NF_{b,N})^2,$$

where $(NF_{b,N})^2$ is the squared b-adic diaphony of any digital $(0, 1)$-sequence over $\mathbb{Z}_b$. (Note that the b-adic diaphony is invariant for all digital $(0, 1)$-sequences over $\mathbb{Z}_b$.)

Proof: For $(0, 1)$-NUT-sequences the diagonal entries must be in $\{1, \dots, b-1\}$. If we consider the expectation of the diaphony over all $(0, 1)$-NUT-sequences, each diagonal entry appears with the same probability $1/(b-1)$. The functions $\chi_b^{\delta_j}$ can therefore be replaced by $\frac{1}{b-1} \sum_{c=1}^{b-1} \chi_b^c(x) = \psi_b(x)$ and we have

$$\mathbb{E}((NF_N(\omega))^2) = \frac{4\pi^2}{b^2} \sum_{j=1}^{\infty} \frac{1}{b-1} \sum_{c=1}^{b-1} \chi_b^c\left(\frac{N}{b^j}\right) = \frac{4\pi^2}{b^2} \sum_{j=1}^{\infty} \psi_b\left(\frac{N}{b^j}\right) = \frac{\pi^2}{3}(NF_{b,N})^2.$$

$\square$

3 Proof of Theorem 4

For the proof of this theorem we will need the following central limit theorem [17, Theorem 2.3.1].

Theorem 7. *Suppose S_m is given by*

$$S_m = \sum_{j=1}^{m} f_j(X_j),$$

where $X_j = \{b^j U\}$, U a random variable that is uniformly distributed modulo 1, and f_j are uniformly bounded and uniformly Lipschitz continuous. If the variance of S_m satisfies $\mathbb{V}(S_m) \geq m^\alpha$ for some $\alpha > 2/3$, then S_m satisfies a central limit theorem:

$$\frac{S_m - \mathbb{E}(S_m)}{\sqrt{\mathbb{V}(S_m)}} \longrightarrow \mathcal{N}(0, 1).$$

Remark 8. One can easily check, that the above theorem also holds with the slightly weaker condition $\mathbb{V}(S_m) \geq m^\alpha$ for all $m \geq m_0$, for some $\alpha > 2/3$ and $m_0 \in \mathbb{N}$.

Lemma 9. *The function ψ_b is Lipschitz-continuous.*

Proof: Let $x, y \in \left[\frac{j}{b}, \frac{j+1}{b}\right]$, $j \in \{0, \ldots, b-1\}$. On this interval ψ_b is a translation of the parabola $\frac{b^2(b^2-1)}{12} x^2$ and therefore Lipschitz-continuous with constant C_j.
Let $x \in \left[\frac{j}{b}, \frac{j+1}{b}\right)$, $y \in \left[\frac{l}{b}, \frac{l+1}{b}\right)$, $l, j \in \{0, \ldots, b-1\}$, $l > j$, and $C := \max_{j \in \{0,\ldots,b-1\}} C_j$, then we have

$$
\begin{aligned}
&|\psi_b(y) - \psi_b(x)| \\
&\leq \left|\psi_b(y) - \psi_b\left(\frac{l}{b}\right)\right| + \left|\psi_b\left(\frac{l}{b}\right) - \psi_b\left(\frac{l-1}{b}\right)\right| + \cdots + \left|\psi_b\left(\frac{j+1}{b}\right) - \psi_b^\beta(x)\right| \\
&\leq C \left|y - \frac{l}{b}\right| + C \left|\frac{l}{b} - \frac{l-1}{b}\right| + \cdots + C \left|\frac{j+1}{b} - x\right| \\
&= C|y - x|.
\end{aligned}
$$

So we have shown that ψ_b is Lipschitz-continuous. $\qquad\square$

We will apply Theorem 7 to

$$
S_m = \sum_{j=1}^m \psi_b(Xb^j), \tag{5}
$$

where X is uniformly distributed on $[0, 1)$. The function ψ_b is bounded (see [8, Lemma 12] and Lipschitz continuous (see Lemma 9). Now we have to compute $\mathbb{E}(S_m)$ and $\mathbb{V}(S_m)$.

Lemma 10. *With S_m as in (5), we have*

$$
\begin{aligned}
\mathbb{E}(S_m) &= \frac{mb(b^2 - 1)}{72}, \\
\mathbb{V}(S_m) &= \frac{mb^2(b^4 - 1)}{25920} + \frac{b^3(1 + b + b^2)(1 - b^{-2m})}{12960}.
\end{aligned}
$$

Proof: In the following we will use the two formulas

$$
\sum_{k=1}^{b-1} \frac{\mathrm{e}^{\frac{2\pi i}{b} kl}}{\sin^2\left(\frac{k\pi}{b}\right)} = \frac{b^2 - 1}{3} + 2|l|(|l| - b), \quad \text{for } l \in \{-b, \ldots, b\}, \tag{6}
$$

(see [3, Corollary A.23]) and

$$
\sum_{k=1}^{b-1} \frac{\mathrm{e}^{\frac{2\pi i}{b} kl} - 1}{\mathrm{e}^{\frac{2\pi i}{b} k} - 1} = b - l, \quad \text{for } l \in \{1, \ldots, b\}, \tag{7}
$$

which can be easily deduced from (6) in the following way.

$$
\sum_{k=1}^{b-1} \frac{\mathrm{e}^{\frac{2\pi i}{b} kl} - 1}{\mathrm{e}^{\frac{2\pi i}{b} k} - 1} = \sum_{k=1}^{b-1} \frac{(\mathrm{e}^{\frac{2\pi i}{b} kl} - 1)(\mathrm{e}^{-\frac{2\pi i}{b} k} - 1)}{(\mathrm{e}^{\frac{2\pi i}{b} k} - 1)(\mathrm{e}^{-\frac{2\pi i}{b} k} - 1)}
$$

$$= \sum_{k=1}^{b-1} \frac{e^{\frac{2\pi i}{b}k(l-1)} - e^{\frac{2\pi i}{b}kl} - e^{\frac{2\pi i}{b}k} + 1}{4\sin^2\left(\frac{k\pi}{b}\right)}$$

$$\stackrel{(6)}{=} b - l.$$

Let $\beta \in \{1, \ldots, b-1\}$ and $j \in \{0, \ldots, b-1\}$, with the abbreviations

$$A_{\beta,j} := \mathrm{Re}\left(\frac{1}{b}\frac{e^{\frac{2\pi i}{b}\beta j} - 1}{e^{\frac{2\pi i}{b}\beta j}(e^{\frac{2\pi i}{b}\beta} - 1)}\right) \quad \text{and} \quad B_{\beta,j} := \mathrm{Im}\left(\frac{1}{b}\frac{e^{\frac{2\pi i}{b}\beta j} - 1}{e^{\frac{2\pi i}{b}\beta j}(e^{\frac{2\pi i}{b}\beta} - 1)}\right)$$

the function ψ_b^β on the interval $\left[\frac{j}{b}, \frac{j+1}{b}\right)$ equals

$$\psi_b^\beta(x) = \frac{b^2(b^2-1)}{12}\left|A_{\beta,j} + iB_{\beta,j} + x - \frac{j}{b}\right|^2.$$

We have

$$\sum_{\beta=1}^{b-1} A_{\beta,j} = \sum_{\beta=1}^{b-1} \mathrm{Re}\left(\frac{1}{b}\frac{e^{\frac{2\pi i}{b}\beta j} - 1}{e^{\frac{2\pi i}{b}\beta j}(e^{\frac{2\pi i}{b}\beta} - 1)}\right)$$

$$= -\frac{1}{b}\mathrm{Re}\left(\sum_{\beta=1}^{b-1}\frac{e^{\frac{2\pi i}{b}\beta(b-j)} - 1}{e^{\frac{2\pi i}{b}\beta} - 1}\right) \stackrel{(7)}{=} -\frac{j}{b} \qquad (8)$$

and

$$\sum_{\beta=1}^{b-1}(A_{\beta,j}^2 + B_{\beta,j}^2) = \sum_{\beta=1}^{b-1}\left|\frac{1}{b}\frac{e^{\frac{2\pi i}{b}\beta j} - 1}{e^{\frac{2\pi i}{b}\beta j}(e^{\frac{2\pi i}{b}\beta} - 1)}\right|^2$$

$$= \frac{1}{b^2}\sum_{\beta=1}^{b-1}\frac{\left(e^{\frac{2\pi i}{b}\beta j} - 1\right)\left(e^{-\frac{2\pi i}{b}\beta j} - 1\right)}{\left(e^{\frac{2\pi i}{b}\beta} - 1\right)\left(e^{-\frac{2\pi i}{b}\beta} - 1\right)}$$

$$= \frac{1}{2b^2}\mathrm{Re}\left(\sum_{\beta=1}^{b-1}\frac{1 - e^{\frac{2\pi i}{b}\beta j}}{\sin^2\left(\frac{\beta\pi}{b}\right)}\right)$$

$$\stackrel{(6)}{=} \frac{j(b-j)}{b^2}. \qquad (9)$$

Now we are ready to compute $\mathbb{E}(S_m)$ and $\mathbb{V}(S_m)$.

$$\mathbb{E}(S_m) = \int_0^1 \sum_{w=1}^m \psi_b(xb^w)dx = \frac{1}{b-1}\sum_{w=1}^m\sum_{\beta=1}^{b-1}\int_0^{b^w}\psi_b^\beta(x)\frac{1}{b^w}dx$$

$$= \frac{m}{b-1}\sum_{\beta=1}^{b-1}\sum_{j=0}^{b-1}\int_{\frac{j}{b}}^{\frac{j+1}{b}}\frac{b^2(b^2-1)}{12}\left|A_{\beta,j} + iB_{\beta,j} + x - \frac{j}{b}\right|^2dx$$

$$= \frac{mb^2(b+1)}{12}\sum_{\beta=1}^{b-1}\sum_{j=0}^{b-1}\int_0^{\frac{1}{b}}\left|A_{\beta,j} + iB_{\beta,j} + x\right|^2dx$$

$$= \frac{mb^2(b+1)}{12} \sum_{j=0}^{b-1} \sum_{\beta=1}^{b-1} \left(\frac{1}{3b^3} + \frac{A_{\beta,j}}{b^2} + \frac{A_{\beta,j}^2 + B_{\beta,j}^2}{b} \right)$$

$$\overset{(8),(9)}{=} \frac{mb(b^2-1)}{72}.$$

Now we compute $\mathbb{V}(S_m)$.

$$\mathbb{E}(S_m^2) = \int_0^1 \sum_{w=1}^m \sum_{v=1}^m \psi_b(xb^w)\psi_b(xb^v)\mathrm{d}x$$

$$= \sum_{w=1}^m \sum_{v=1}^{w-1} \frac{1}{b^w} \int_0^{b^w} \psi_b(x)\psi_b\left(\frac{x}{b^{w-v}}\right) \mathrm{d}x + \sum_{w=1}^m \sum_{v=w+1}^m \frac{1}{b^v} \int_0^{b^v} \psi_b(x)\psi_b\left(\frac{x}{b^{v-w}}\right) \mathrm{d}x$$

$$+ \sum_{w=1}^m \frac{1}{b^w} \int_0^{b^w} \psi_b(x)^2\mathrm{d}x$$

$$= \sum_{w=1}^m \sum_{v=1}^{w-1} \frac{1}{b^{w-v}} \int_0^{b^{w-v}} \psi_b(x)\psi_b\left(\frac{x}{b^{w-v}}\right) \mathrm{d}x$$

$$+ \sum_{w=1}^m \sum_{v=w+1}^m \frac{1}{b^{v-w}} \int_0^{b^{v-w}} \psi_b(x)\psi_b\left(\frac{x}{b^{v-w}}\right) \mathrm{d}x$$

$$+ m \int_0^1 \psi_b(x)^2\mathrm{d}x.$$

For $k \geq 0$ we consider now the term

$$J_k := \frac{1}{b^k} \int_0^{b^k} \psi_b(x)\psi_b\left(\frac{x}{b^k}\right) \mathrm{d}x$$

$$= \frac{1}{b^k} \sum_{j=0}^{b-1} \sum_{l=jb^k}^{(j+1)b^k-1} \frac{1}{(b-1)^2} \sum_{\beta=1}^{b-1} \sum_{\eta=1}^{b-1} \int_{\frac{l}{b}}^{\frac{l+1}{b}} \psi_b^\beta(x)\psi_b^\eta\left(\frac{x}{b^k}\right) \mathrm{d}x$$

$$= \frac{b^4(b^2-1)^2}{144(b-1)^2 b^k} \sum_{j=0}^{b-1} \sum_{l=jb^k}^{(j+1)b^k-1} \sum_{\beta=1}^{b-1} \sum_{\eta=1}^{b-1}$$

$$\int_{\frac{l}{b}}^{\frac{l+1}{b}} \left| A_{\beta,l} + iB_{\beta,l} + x - \frac{l}{b} \right|^2 \left| A_{\eta,j} + iB_{\eta,j} + \frac{x}{b^k} - \frac{j}{b} \right|^2 \mathrm{d}x$$

$$= \frac{b^4(b+1)^2}{144b^k} \sum_{j=0}^{b-1} \sum_{l=jb^k}^{(j+1)b^k-1} \sum_{\beta=1}^{b-1} \sum_{\eta=1}^{b-1}$$

$$\int_0^{\frac{1}{b}} \left| A_{\beta,l} + iB_{\beta,l} + x \right|^2 \left| A_{\eta,j} + iB_{\eta,j} + \frac{x}{b^k} + \frac{l - jb^k}{b^{k+1}} \right|^2 \mathrm{d}x$$

$$=: \frac{b^4(b+1)^2}{144b^k} \sum_{j=0}^{b-1} \sum_{l=jb^k}^{(j+1)b^k-1} I(j,k,l).$$

Let now $C_{j,k,l} := \frac{l-jb^k}{b^{k+1}}$ and $a \in \{0, \ldots, b-1\}$ such that $l = rb + a$. Note that in the case

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$k = 0$ we have $l = j = a$ and $C_{j,0,l} = 0$. With (8) and (9) we obtain

$$
\begin{aligned}
I(j,k,l) := \sum_{\beta=1}^{b-1} \sum_{\eta=1}^{b-1} \Bigg[& (A_{\beta,l}^2 + B_{\beta,l}^2)(A_{\eta,j}^2 + B_{\eta,j}^2)\frac{1}{b} \\
& + (A_{\beta,l}^2 + B_{\beta,l}^2)\left(\frac{1}{3b^{2k+3}} + C_{j,k,l}^2\frac{1}{b} + C_{j,k,l}\frac{1}{b^{k+2}}\right) \\
& + (A_{\beta,l}^2 + B_{\beta,l}^2)A_{\eta,j}\left(\frac{1}{b^{k+2}} + 2C_{j,k,l}\frac{1}{b}\right) + A_{\beta,l}(A_{\eta,j}^2 + B_{\eta,j}^2)\frac{1}{b^2} \\
& + A_{\beta,l}\left(\frac{1}{2b^{2k+4}} + C_{j,k,l}^2\frac{1}{b^2} + C_{j,k,l}\frac{4}{3b^{k+3}}\right) + A_{\beta,l}A_{\eta,j}\left(\frac{4}{3b^{k+3}} + C_{j,k,l}\frac{2}{b^2}\right) \\
& + (A_{\eta,j}^2 + B_{\eta,j}^2)\frac{1}{3b^3} + A_{\eta,j}\left(\frac{1}{2b^{k+4}} + C_{j,k,l}\frac{2}{3b^3}\right) \\
& + \left(\frac{1}{5b^{2k+5}} + C_{j,k,l}^2\frac{1}{3b^3} + C_{j,k,l}\frac{1}{2b^{k+4}}\right)\Bigg] \\
= & \frac{a(b-a)j(b-j)}{b^4}\frac{1}{b} + \frac{a(b-a)(b-1)}{b^2}\left(\frac{1}{3b^{2k+3}} + C_{j,k,l}^2\frac{1}{b} + C_{j,k,l}\frac{1}{b^{k+2}}\right) \\
& - \frac{a(b-a)j}{b^3}\left(\frac{1}{b^{k+2}} + 2C_{j,k,l}\frac{1}{b}\right) - \frac{aj(b-j)}{b^3}\frac{1}{b^2} \\
& - \frac{a(b-1)}{b}\left(\frac{1}{2b^{2k+4}} + C_{j,k,l}^2\frac{1}{b^2} + C_{j,k,l}\frac{4}{3b^{k+3}}\right) + \frac{aj}{b^2}\left(\frac{4}{3b^{k+3}} + C_{j,k,l}\frac{2}{b^2}\right) \\
& + \frac{j(b-j)(b-1)}{b^2}\frac{1}{3b^3} - \frac{j(b-1)}{b}\left(\frac{1}{2b^{k+4}} + C_{j,k,l}\frac{2}{3b^3}\right) \\
& + (b-1)^2\left(\frac{1}{5b^{2k+5}} + C_{j,k,l}^2\frac{1}{3b^3} + C_{j,k,l}\frac{1}{2b^{k+4}}\right).
\end{aligned}
$$

For $k > 0$ we obtain now

$$
\begin{aligned}
J_k &= \frac{b^4(b+1)^2}{144b^k} \sum_{j=0}^{b-1} \sum_{r=jb^{k-1}}^{(j+1)b^{k-1}-1} \sum_{a=0}^{b-1} I(j,k,rb+a) \\
&= \frac{b^{1-2k}(b^2-1)^2\left(-1 - b\left(1 + b - 5b^{2k}\right)\right)}{25920}.
\end{aligned}
$$

For $k = 0$ we obtain

$$
\begin{aligned}
J_0 &= \frac{b^4(b+1)^2}{144b^k} \sum_{j=0}^{b-1} I(j,0,j) \\
&= \frac{(b-1)\,b(b+1)^2\left(1 + b\left(3b-1\right)\right)}{12960}.
\end{aligned}
$$

Now we have

$$
\begin{aligned}
\mathbb{V}(S_m) &= \mathbb{E}(S_m^2) - \mathbb{E}(S_m)^2 \\
&= \sum_{w=1}^{m}\sum_{v=1}^{w-1} J_{w-v} + \sum_{w=1}^{m}\sum_{v=w+1}^{m} J_{v-w} + mJ_0 - m^2\frac{(b^2-1)^2 b^2}{72^2}
\end{aligned}
$$

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$$= \frac{mb^2(b^4 - 1)}{25920} + \frac{b^3(1 + b + b^2)(1 - b^{-2m})}{12960}.$$

$\square$

Remark 11. For $b = 2$ we obtain

$$\mathbb{E}(S_m) = \frac{m}{12} \quad \text{and} \quad \mathbb{V}(S_m) = \frac{m}{432} + \frac{7(1 - 2^{-2m})}{1620},$$

as in [16, Proof of Corollary 2.4].

The idea of the following proof is the same as in [5, Proof of Theorem 2], see also [17].
Proof of Theorem 4: Let $M = b^m$. We will not prove the limit relation of Theorem 4 but

$$\frac{1}{M} \left| \left\{ N < M : (NF_{b,N}(\omega))^2 \leq \frac{b^2 - 1}{6b} \log_b M + y\frac{12}{b^2} \sqrt{\frac{\log_b M(b^4 - 1)b^2}{25920}} \right\} \right|$$
$$= \Phi(y) + o(1) \qquad (\text{as } m \to \infty). \tag{10}$$

It is easy to see that both limit relations are equivalent. This follows directly by restricting N with $M/(\log M) \leq N \leq M$. Of course, N with $N < M/(\log M)$ do not matter in the limit. For the remaining ones we have $|\log M - \log N| \ll \log\log M$ and this difference does not matter in the limit either.

The idea of the proof of (10) is to use (2) and approximate $(NF_{b,N}(\omega))^2$ as a sum of weakly dependent random variables.

We use now the same argument as Drmota et al. in [5, Proof of Theorem 2] (for the case $b = 2$) for general primes b, see also [17]. Since $M = b^m$, the digits N_j of the b-adic expansion $N = N_0 + bN_1 + \cdots + b^{m-1}N_{m-1}$ can be considered as independent random variables that are uniformly distributed on $\{0, 1, \ldots, b-1\}$ (by assuming that all numbers $N < b^m$ have equal probability b^{-m}). Then

$$U_r = \left\{ \frac{N}{b^r} \right\} = \frac{N_{r-1}}{b} + \frac{N_{r-2}}{b^2} + \cdots + \frac{N_0}{b^r}$$

is very close to a random variable that is uniformly distributed on $[0, 1)$. In fact we can add missing digits $N_{-1}, N_{-2}, \ldots$ that are independent and uniformly distributed on $\{0, 1, \ldots, b-1\}$ and get

$$U_r' = U_r + \sum_{j=1}^{\infty} \frac{N_{-j}}{b^{r+j}}.$$

Then U_r' is exactly uniformly distributed on $[0, 1)$ and $|U_r - U_r'| \leq b^{-r}$.
Furthermore, U_r and U_{r+k} get more and more independent as k gets large. More precisely, $U_{r+k} - b^{-k}U_r$ and U_r are independent. This means, that a slight modification of order b^{-k} makes U_r and U_{r+k} independent.
Note that

$$\frac{b^2}{12}(NF_{b,N}(\omega))^2 = \sum_{r=1}^{m} \psi_b(U_r) + \mathcal{O}(1)$$

for all $N < b^m$ (see [8, Proof of Corollary 9]). Thus, $\frac{b^2}{12}(NF_{b,N}(\omega))^2$ can be represented (up to a small error term) by a sum of weakly dependent random variables.

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We further note that the above smoothing by introducing missing digits can be also obtained by considering the following slight variation of the above probability model. Let $\tilde{N}$ be a random variable that is uniformly distributed on $[0, b^m)$, then $\tilde{U}_r = \{\tilde{N}/b^{r-1}\}$ is uniformly distributed on $[0, 1)$ and the common distribution of $(\tilde{U}_r)_{1 \leq r \leq m}$ is the same as that of $(U'_r)_{1 \leq r \leq m}$. Furthermore, if we set $\xi = \tilde{N}/b^m$ and $V_r = \tilde{U}_{m-r+1}$ then ξ is uniformly distributed on $[0, 1)$ and

$$V_r = \{\xi b^r\}.$$

Since $|V_r - U_{m-r+1}| \leq b^{-(m-r+1)}$ and ψ_b is Lipschitz continuous we also have

$$
\begin{aligned}
\frac{b^2}{12}(NF_{b,N}(\omega))^2 &= \sum_{r=1}^m \psi_b\left(\frac{N}{b^r}\right) + \mathcal{O}(1) \\
&= \sum_{r=1}^m \psi_b\left(\frac{N}{b^{m-r+1}}\right) + \mathcal{O}(1) \\
&= \sum_{r=1}^m \psi_b(\xi b^r) + \sum_{r=1}^m \left(\psi_b\left(\frac{N}{b^{m-r+1}}\right) - \psi_b(\xi b^r)\right) + \mathcal{O}(1) \\
&= \sum_{r=1}^m \psi_b(\xi b^r) + \mathcal{O}(1).
\end{aligned}
$$

Now by Theorem 7 it follows that

$$S_m = \sum_{r=1}^m \psi_b(\xi b^r)$$

satisfies a central limit theorem. With Lemma 10 we thus have

$$\frac{S_m - m(b^2-1)b/72}{\sqrt{\frac{m(b^4-1)b^2}{25920} + \frac{b^3(1+b+b^2)(1-b^{-2m})}{12960}}} \longrightarrow \mathcal{N}(0,1) \qquad (\text{as } m \to \infty)$$

and also all moments converge. Since $(NF_{b,N}(\omega))^2 = \frac{12}{b^2}S_m + \mathcal{O}(1)$ we get up to the factor $12/b^2$ the same limit relation for $(NF_{b,N}(\omega))^2$ if N is uniformly distributed on $\{0, 1, \ldots, b^m - 1\}$. In the limit the second term of the variance can be neglected and the result follows. $\qquad \square$

4 Proof of Proposition 5

Since ψ_b and χ_b^c are on intervals of the form $\left[\frac{k}{b}, \frac{k+1}{b}\right], k \in \{0, \ldots, b-1\}$, translations of the same parabola, it is enough to show that equation (4) holds for $x = \frac{k}{b}, k = 0, 1, \ldots, b$. In the case where $b = 2$, the functions ψ_b and χ_b^c are the same, so we consider henceforward only odd primes b. Since $\psi_b(0) = \chi_b^c(0) = 0$ we assume in the following $k \geq 1$.

From the definition of the functions $\varphi_{b,h}^\sigma$ we get

$$\varphi_{b,h}^c\left(\frac{k}{b}\right) = \begin{cases} 0 & h = 0, \\ A\left([0, \frac{h}{b}), k, \left(0, \frac{c}{b}, \ldots, \frac{[(b-1)c]}{b}\right)\right) - h\frac{k}{b} & h \leq [(k-1)c], \\ (b-h)\frac{k}{b} - A\left([\frac{h}{b}, 1), k, \left(0, \frac{c}{b}, \ldots, \frac{[(b-1)c]}{b}\right)\right) & [(k-1)c] < h < b, \end{cases}$$

$$= \begin{cases} 0 & h = 0, \\ A\left(\left[0, \frac{h}{b}\right), k, \left(0, \frac{c}{b}, \ldots, \frac{[(b-1)c]}{b}\right)\right) - h\frac{k}{b} & h \neq 0, \end{cases}$$

$$=: \begin{cases} 0 & h = 0, \\ A_{h,b,k,c} - \frac{kh}{b} & h \neq 0. \end{cases}$$

With $\chi_b^c = b \sum_{h=0}^{b-1} (\varphi_{b,h}^c)^2 - \left(\sum_{h=0}^{b-1} \varphi_{b,h}^c\right)^2$ we get

$$\frac{1}{b-1} \sum_{c=1}^{b-1} \chi_b^c\left(\frac{k}{b}\right) = \frac{1}{b-1} \sum_{c=1}^{b-1} b \sum_{h=1}^{b-1} \left(A_{h,b,k,c} - \frac{kh}{b}\right)^2 - \frac{1}{b-1} \sum_{c=1}^{b-1} \left(\sum_{h=1}^{b-1} \left(A_{h,b,k,c} - \frac{kh}{b}\right)\right)^2$$

$$= \frac{b}{b-1} \sum_{c=1}^{b-1} \sum_{h=1}^{b-1} A_{h,b,k,c}^2 - \frac{2k}{b-1} \sum_{h=1}^{b-1} h \sum_{c=1}^{b-1} A_{h,b,k,c} + \frac{k^2}{b} \sum_{h=1}^{b-1} h^2$$

$$- \frac{1}{b-1} \sum_{c=1}^{b-1} \left(\sum_{h=1}^{b-1} A_{h,b,k,c}\right)^2 + k \sum_{h=1}^{b-1} \sum_{c=1}^{b-1} A_{h,b,k,c} - \frac{k^2(b-1)^2}{4}.$$

We will need the following three lemmas to simplify the above expression.

Lemma 12. *For $h \neq 0$, $k \geq 1$, b a prime, we have*

$$\sum_{c=1}^{b-1} A_{h,b,k,c} = b + hk - h - k.$$

Proof: We have

$$\sum_{c=1}^{b-1} A_{h,b,k,c} = \sum_{j=0}^{h-1} \sum_{c=1}^{b-1} |\{l \in \{0, \ldots, k-1\} : lc \equiv j \bmod b\}|$$

$$= \sum_{c=1}^{b-1} |\{l \in \{0, \ldots, k-1\} : lc \equiv 0 \bmod b\}| + \sum_{j=1}^{h-1} \sum_{c=1}^{b-1} \sum_{\substack{l=1 \\ lc \equiv j \bmod b}}^{k-1} 1$$

$$= b - 1 + \sum_{j=1}^{h-1} \sum_{l=1}^{k-1} \sum_{\substack{c=1 \\ lc \equiv j \bmod b}}^{b-1} 1 = b - 1 + \sum_{j=1}^{h-1} \sum_{l=1}^{k-1} 1$$

$$= b + hk - h - k.$$

$\square$

Lemma 13. *For $k \geq 1$, b a prime, we have*

$$\sum_{c=1}^{b-1} \sum_{h=1}^{b-1} A_{h,b,k,c}^2 = (b-1)^2 k^2 - \sum_{c=1}^{b-1} \sum_{l=0}^{k-1} \sum_{\tilde{l}=0}^{k-1} \max([lc], [\tilde{l}c]).$$

Proof: Since $A_{b,b,k,c} = k$ we have

$$\sum_{c=1}^{b-1} \sum_{h=1}^{b-1} A_{h,b,k,c}^2 = -(b-1)k^2 + \sum_{c=1}^{b-1} \sum_{h=1}^{b} A_{h,b,k,c}^2$$

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$$= -(b-1)k^2 + \sum_{c=1}^{b-1}\sum_{h=1}^{b}\left(\sum_{j=0}^{h-1}\sum_{\substack{l=0 \\ lc\equiv j \bmod b}}^{k-1} 1\right)^2$$

$$= -(b-1)k^2 + \sum_{c=1}^{b-1}\sum_{h=1}^{b}\sum_{j=0}^{h-1}\sum_{i=0}^{h-1}\sum_{\substack{l=0 \\ lc\equiv j \bmod b}}^{k-1}\sum_{\substack{\tilde{l}=0 \\ \tilde{l}c\equiv i \bmod b}}^{k-1} 1$$

$$= -(b-1)k^2 + \sum_{c=1}^{b-1}\sum_{l=0}^{k-1}\sum_{\tilde{l}=0}^{k-1}\sum_{\substack{j=0 \\ lc\equiv j \bmod b}}^{b-1}\sum_{\substack{i=0 \\ \tilde{l}c\equiv i \bmod b}}^{b-1}\sum_{h=\max(j,i)+1}^{b} 1$$

$$= -(b-1)k^2 + \sum_{c=1}^{b-1}\sum_{l=0}^{k-1}\sum_{\tilde{l}=0}^{k-1}\sum_{h=\max([lc],[\tilde{l}c])+1}^{b} 1$$

$$= -(b-1)k^2 + \sum_{c=1}^{b-1}\sum_{l=0}^{k-1}\sum_{\tilde{l}=0}^{k-1}(b - \max([lc],[\tilde{l}c]))$$

$$= (b-1)^2 k^2 - \sum_{c=1}^{b-1}\sum_{l=0}^{k-1}\sum_{\tilde{l}=0}^{k-1}\max([lc],[\tilde{l}c]).$$

$\square$

Lemma 14. *For $k \geq 1$, b a prime, we have*

$$\sum_{c=1}^{b-1}\left(\sum_{h=1}^{b-1} A_{h,b,k,c}\right)^2 = (b-1)k(b^2 k - b + k - bk) - \sum_{c=1}^{b-1}\sum_{l=0}^{k-1}\sum_{\tilde{l}=0}^{k-1}(b([lc]+[\tilde{l}c]) - [lc][\tilde{l}c]).$$

Proof: Since $A_{b,b,k,c} = k$ we have

$$\sum_{c=1}^{b-1}\left(\sum_{h=1}^{b-1} A_{h,b,k,c}\right)^2 = \sum_{c=1}^{b-1}\left(-k + \sum_{h=1}^{b} A_{h,b,k,c}\right)^2$$

$$= (b-1)k^2 - 2k\sum_{c=1}^{b-1}\sum_{h=1}^{b} A_{h,b,k,c} + \sum_{c=1}^{b-1}\left(\sum_{h=1}^{b} A_{h,b,k,c}\right)^2.$$

We proceed as in the proof of Lemma 13 and get:

$$\sum_{c=1}^{b-1}\left(\sum_{h=1}^{b} A_{h,b,k,c}\right)^2 = \sum_{c=1}^{b-1}\sum_{h=1}^{b}\sum_{m=1}^{b}\sum_{j=0}^{h-1}\sum_{i=0}^{m-1}\sum_{\substack{l=0 \\ lc\equiv j \bmod b}}^{k-1}\sum_{\substack{\tilde{l}=0 \\ \tilde{l}c\equiv i \bmod b}}^{k-1} 1$$

$$= \sum_{c=1}^{b-1}\sum_{l=0}^{k-1}\sum_{\tilde{l}=0}^{k-1}\sum_{h=[lc]+1}^{b}\sum_{m=[\tilde{l}c]+1}^{b} 1$$

$$= b^2(b-1)k^2 - \sum_{c=1}^{b-1}\sum_{l=0}^{k-1}\sum_{\tilde{l}=0}^{k-1}(b([lc]+[\tilde{l}c]) - [lc][\tilde{l}c]).$$

Together with Lemma 12 this gives the desired result. $\qquad\square$

With Lemma 12, Lemma 13 and Lemma 14 we have now

$$\frac{1}{b-1}\sum_{c=1}^{b-1}\chi_b^c\left(\frac{k}{b}\right) = \frac{b}{b-1}(b-1)^2k^2 - \frac{b}{b-1}\sum_{c=1}^{b-1}\sum_{l=0}^{k-1}\sum_{\tilde{l}=0}^{k-1}\max([lc],[\tilde{l}c])$$

$$-\frac{2k}{b-1}\sum_{h=1}^{b-1}h(b+hk-h-k) + \frac{k^2}{b}\frac{(b-1)b(2b-1)}{6}$$

$$- k(b^2k-b+k-bk) + \frac{1}{b-1}\sum_{c=1}^{b-1}\sum_{l=0}^{k-1}\sum_{\tilde{l}=0}^{k-1}(b([lc]+[\tilde{l}c])-[lc][\tilde{l}c])$$

$$+ k\sum_{h=1}^{b-1}(b+hk-h-k) - \frac{k^2(b^2-1)}{4}$$

$$=\frac{(b+1)k(2b-bk-k)}{12} + \frac{1}{b-1}\sum_{c=1}^{b-1}\sum_{l=0}^{k-1}\sum_{\tilde{l}=0}^{k-1}(b\min([lc],[\tilde{l}c])-[lc][\tilde{l}c])$$

$$=\frac{(b+1)k(2b-bk-k)}{12} + \frac{1}{b-1}\sum_{c=1}^{b-1}\sum_{l=1}^{k-1}\sum_{\tilde{l}=1}^{k-1}(b\min([c],[\tilde{l}l^{-1}c])-[c][\tilde{l}l^{-1}c]).$$

We will need the following lemma, which will be proved afterwards.

Lemma 15. *Let b be a prime and $a \in \{1,\ldots,b-1\}$. We have*

$$\sum_{c=1}^{b-1}(b\min([c],[ac])-[c][ac]) = \begin{cases} \frac{b(b^2-1)}{6} & a=1, \\ \frac{b(b^2-1)}{12} & a\neq 1. \end{cases}$$

With this last lemma we get

$$\frac{1}{b-1}\sum_{c=1}^{b-1}\chi_b^c\left(\frac{k}{b}\right) = \frac{(b+1)k(2b-bk-k)}{12} + \frac{k(k-1)}{b-1}\frac{b(b^2-1)}{12}$$

$$= \frac{(b+1)k(b-k)}{12} = \psi_b\left(\frac{k}{b}\right),$$

where we have used [8, Lemma 13] for the last equality. This finishes the proof of Proposition 5. $\qquad\square$

Proof of Lemma 15: The case $a=1$ is easy:

$$\sum_{c=1}^{b-1}(b\min([c],[c])-[c][c]) = \sum_{c=1}^{b-1}(bc-c^2) = \frac{b(b^2-1)}{6}.$$

From now on let $a \in \{2,\ldots,b-1\}$ be fixed and let $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ denote the floor and ceiling function respectively. We have

$$\Sigma := \sum_{c=1}^{b-1}(b\min([c],[ac])-[c][ac])$$

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$$= \sum_{c=1}^{b-1} \min([c][-ac], [-c][ac])$$

$$= \sum_{l=0}^{a-1} \sum_{\frac{lb}{a} \le c < \frac{(l+1)b}{a}} \min(c(b - ac + lb), (b - c)(ac - lb))$$

$$= \sum_{l=0}^{a-1} \left(\sum_{\frac{lb}{a} \le c < \frac{lb}{a-1}} (b - c)(ac - lb) + \sum_{\frac{lb}{a-1} \le c < \frac{(l+1)b}{a}} c(b - ac + lb) \right)$$

$$= -\sum_{c=1}^{b-1} ac^2 + \sum_{l=0}^{a-1} \left(\sum_{\frac{lb}{a} \le c < \frac{lb}{a-1}} \left((l+a)bc - lb^2\right) + \sum_{\frac{lb}{a-1} \le c < \frac{(l+1)b}{a}} (l+1)bc \right).$$

We set now $x_l := \left\lceil \frac{lb}{a} \right\rceil$ and $y_l := \left\lceil \frac{lb}{a-1} \right\rceil$ and note that $x_0 = 0, y_0 = 0$ and $x_a = b, y_{a-1} = b$. We have

$$\Sigma = -\frac{a(b-1)b(2b-1)}{6} + \sum_{l=0}^{a-1} \left(\sum_{c=x_l}^{y_l - 1} (l+a)bc - lb^2 + \sum_{c=y_l}^{x_{l+1} - 1} (l+1)bc \right)$$

$$= -\frac{a(b-1)b(2b-1)}{6} + \sum_{l=0}^{a-1} \left((l+a)b\frac{(x_l + y_l - 1)(y_l - x_l)}{2} - lb^2(y_l - x_l) \right.$$

$$\left. + (l+1)b\frac{(x_{l+1} + y_l - 1)(x_{l+1} - y_l)}{2} \right)$$

$$= -\frac{a(b-1)b(2b-1)}{6} + \sum_{l=0}^{a-1} \left(\frac{(a-1)b}{2} y_l(y_l - 1) - lb^2(y_l - x_l) \right.$$

$$\left. - \frac{(l+a)b}{2} x_l(x_l - 1) + \frac{(l+1)b}{2} x_{l+1}(x_{l+1} - 1) \right)$$

$$= -\frac{a(b-1)b(2b-1)}{6} + \sum_{l=0}^{a-1} \left(\frac{(a-1)b}{2} y_l(y_l - 1) - lb^2(y_l - x_l) - \frac{(l+a)b}{2} x_l(x_l - 1) \right)$$

$$+ \sum_{l=1}^{a} \frac{lb}{2} x_l(x_l - 1)$$

$$= \frac{ab(b^2 - 1)}{6} + \sum_{l=1}^{a-1} \left(\frac{(a-1)b}{2} y_l(y_l - 1) - lb^2(y_l - x_l) - \frac{ab}{2} x_l(x_l - 1) \right).$$

We define now

$$g(b, a) := \sum_{l=1}^{a-1} -\frac{ab}{2} \left\lceil \frac{lb}{a} \right\rceil \left(\left\lceil \frac{lb}{a} \right\rceil - 1 \right) + lb^2 \left\lceil \frac{lb}{a} \right\rceil$$

$$= \sum_{l=1}^{a-1} -\frac{ab}{2} \left\lfloor \frac{lb}{a} \right\rfloor \left(\left\lfloor \frac{lb}{a} \right\rfloor + 1 \right) + lb^2 \left\lfloor \frac{lb}{a} \right\rfloor + lb^2.$$

Note that $\frac{lb}{a}$ is not an integer for $l \in \{1, \ldots, a-1\}$ and therefore we can write $\lceil \cdot \rceil = \lfloor \cdot \rfloor + 1$.

So far we have

$$\Sigma = -\frac{b(b+1)(a-3b+2ab)}{6} + g(b,a) - g(b,a-1). \qquad (11)$$

Our goal is now to find a closed formula for $g(b,a)$. We will need the following lemma.

Lemma 16. *Let m,n be positive and co-prime and let $i(l,m,n) \in \{0,\dots,n-1\}$ such that $i(l,m,n) \equiv lm \mod n$. Then we have*

1. $\displaystyle\sum_{l=1}^{n-1} \left\lfloor \frac{lm}{n} \right\rfloor = \frac{(m-1)(n-1)}{2},$

2. $\displaystyle\sum_{l=1}^{n-1} \left\lfloor \frac{lm}{n} \right\rfloor^2 = \frac{(m^2-1)(n-1)(2n-1)}{6n} - \frac{2}{n}\sum_{l=1}^{n-1} i(l,m,n)\left\lfloor \frac{lm}{n} \right\rfloor,$

3. $\displaystyle\sum_{l=1}^{n-1} lm \left\lfloor \frac{lm}{n} \right\rfloor = \frac{(m^2-1)(n-1)(2n-1)}{6} - \sum_{l=1}^{n-1} i(l,m,n)\left\lfloor \frac{lm}{n} \right\rfloor.$

Proof: Use that $\{i(l,m,n) : l = 1,\dots,n-1\} = \{1,\dots,n-1\}$, since m and n are co-prime, and $\frac{lm}{n} = \left\lfloor \frac{lm}{n} \right\rfloor + \frac{i(l,m,n)}{n}$. Then the above results follow by easy calculations. $\square$

With the above lemma we get now

$$g(b,a) = -\frac{b(b^2-1)(a-1)(2a-1)}{12} + b\sum_{l=1}^{a-1} i(l,b,a)\left\lfloor \frac{lb}{a} \right\rfloor$$
$$-\frac{ab(a-1)(b-1)}{4} + \frac{b(b^2-1)(a-1)(2a-1)}{6}$$
$$-b\sum_{l=1}^{a-1} i(l,b,a)\left\lfloor \frac{lb}{a} \right\rfloor + \frac{b^2(a-1)a}{2}$$
$$= \frac{b(b+1)(a-1)(1+a-b+2ab)}{12}$$

and hence from equation (11) it follows that

$$\sum_{c=1}^{b-1} (b\min([c],[ac]) - [c][ac]) = \frac{b(b^2-1)}{12}.$$

This finishes the proof of Lemma 15. $\square$

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