A characterization of generalized nets using Weyl sums and its applications

Jan Baldeaux*, Josef Dick*, Friedrich Pillichshammer†

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Abstract

Point sets referred to as $(t, \alpha, \beta, n, m, s)$-nets were recently introduced and shown to generalize both digital $(t, \alpha, \beta, n \times m, s)$-nets and classical (t, m, s)-nets. Their definition captures the geometrical properties of their digital analogue, which has recently been shown to yield quadrature points for quasi-Monte Carlo rules which can achieve arbitrary high convergence rates of the integration error for sufficiently smooth functions. In this paper, we characterize $(t, \alpha, \beta, n, m, s)$-nets using Weyl sums generalizing the analogous result for (t, m, s)-nets.

As an application of this characterization we study numerical integration using such generalized nets. It is shown that for functions having square integrable mixed partial derivatives of order α in each variable, integration errors converge at a rate of $N^{-(\alpha-1+\delta)}$ for any $\delta > 0$, establishing that $(t, \alpha, \beta, n, m, s)$-nets can exploit the smoothness of the function under consideration.

As a further application, it can be used for the construction of new $(t, \alpha, \beta, n, m, s)$-nets itself: We introduce an analogue of the $(u, u+v)$-construction for digital $(t, \alpha, \beta, n \times m, s)$ nets and (t, m, s)-nets.

1 Introduction

Generalized digital nets and sequences were introduced in [6, 7], where it was also shown that point sets $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{b^m-1}$, obtained from a generalized digital

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net or sequence, can be used in a quasi-Monte Carlo rule $b^{-m}\sum_{h=0}^{b^m-1} f(\boldsymbol{x}_h)$ to approximate the integral $\int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$, and that the integration error can achieve an arbitrary high rate of convergence for sufficiently smooth functions.

In [1], the geometrical properties of those generalized digital nets and sequences, called digital $(t,\alpha,\beta,n\times m,s)$-nets and digital $(t,\alpha,\beta,\sigma,s)$-sequences, were analyzed. Point sets satisfying a certain geometrical property exhibited by the generalized digital nets and sequences are called (t,α,β,n,m,s)-nets, which include both digital $(t,\alpha,\beta,n\times m,s)$-nets, [7], and (t,m,s)-nets, [13, 14], as special cases. One motivation for studying the geometrical properties of generalized digital nets and sequences lies in the conjecture that non-digital nets and sequences may exist with better quality than their digital counterparts [15]. Studying the geometrical properties reveals the properties generalized non-digital, that is non-linear, nets and sequences need to have. This information can be used for the construction of new generalized (non-digital, i.e., non-linear) nets and sequences. Indeed, our results here also turn out to be applicable to constructing new generalized nets and sequences.

In this paper, we firstly show how to characterize (t,α,β,n,m,s)-nets using Weyl sums, in analogy to [12, Corollary 3], which provides the result for (t,m,s)-nets. This result also turns out to be useful for applications, which is the second contribution of the paper.

We study numerical integration in the Walsh space introduced in [7]. In particular, we show that if the function under consideration has square integrable mixed partial derivatives of order α in each variable, the integration errors resulting from approximating the integral with a quasi-Monte Carlo rule with a (t,α,β,n,m,s)-net as quadrature points, converge at a rate of $N^{-(\alpha-1)}$, multiplied by a $\log N$ factor, for sufficiently smooth functions. This bound is not optimal as one can obtain $N^{-\alpha}(\log N)^{\alpha s}$ with generalized digital nets [7] for example, but in Remark 2 we point out that for given concrete constructions optimal bounds may be obtained using further information of the construction.

We also generalize the $(u,u+v)$ construction, which is already used to construct (t,m,s)-nets [4] and digital $(t,\alpha,\beta,n\times m,s)$-nets [9], to the construction of (t,α,β,n,m,s)-nets. Again, the characterization of (t,α,β,n,m,s)-nets using Weyl sums turns out to be the appropriate tool to establish the result.

The main results of the paper are the following:

- Theorem 1, which shows that (t,α,β,n,m,s)-nets can be characterized using Weyl sums.

- Theorem 2, which shows that (t,α,β,n,m,s)-nets can achieve integration errors of order $N^{-(\alpha-1)}$ multiplied by a $\log N$ factor.

- Theorem 3, which shows how to obtain new $(t, \alpha, \beta, n, m, s)$-nets using an analogue of the $(u, u+v)$-construction.

The paper is structured as follows: In Section 2, we provide the definition of $(t, \alpha, \beta, n, m, s)$-nets and state some of their properties, recall the definition of Walsh functions and Weyl sums and give the basic features of the function space under consideration. The characterization of $(t, \alpha, \beta, n, m, s)$-nets in terms of Weyl sums is given in Section 3. The application of the characterization to numerical integration is given in Section 4 and the characterization is used to establish the $(u, u+v)$-construction for $(t, \alpha, \beta, n, m, s)$-nets in Section 5.

2 Basic definitions

In this section, we introduce $(t, \alpha, \beta, n, m, s)$-nets, Walsh functions, Weyl sums and the function space considered for numerical integration. In addition we also generalize the construction from [7, Section 4.4] (see also [6]).

Definition and construction of $(t, \alpha, \beta, n, m, s)$-nets. Before we can state the definition of $(t, \alpha, \beta, n, m, s)$-nets we need some notation.

Let $n, s \geq 1$, $b \geq 2$ be integers. For $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_s) \in \{0, \ldots, n\}^s$ let $|\boldsymbol{\nu}|_1 = \sum_{j=1}^s \nu_j$ and define $\boldsymbol{i}_{\boldsymbol{\nu}} = (i_{1,1}, \ldots, i_{1,\nu_1}, \ldots, i_{s,1}, \ldots, i_{s,\nu_s})$ with integers $1 \leq i_{j,\nu_j} < \ldots < i_{j,1} \leq n$ in case $\nu_j > 0$ and $\left\{i_{j,1}, \ldots, i_{j,\nu_j}\right\} = \emptyset$ in case $\nu_j = 0$, for $j = 1, \ldots, s$. For given $\boldsymbol{\nu}$ and $\boldsymbol{i}_{\boldsymbol{\nu}}$ let $\boldsymbol{a}_{\boldsymbol{\nu}} \in \{0, \ldots, b-1\}^{|\boldsymbol{\nu}|_1}$, which we write as $\boldsymbol{a}_{\boldsymbol{\nu}} = (a_{1,i_{1,1}}, \ldots, a_{1,i_{1,\nu_1}}, \ldots, a_{s,i_{s,1}}, \ldots, a_{s,i_{s,\nu_s}})$.

A *generalized elementary interval in base* b is a subset of $[0,1)^s$ of the form

$$J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}}) = \prod_{j=1}^s \bigcup_{\substack{a_{j,l}=0 \\ l \in \{1,\ldots,n\} \setminus \{i_{j,1},\ldots,i_{j,\nu_j}\}}}^{q-1} \left[\frac{a_{j,1}}{b} + \cdots + \frac{a_{j,n}}{b^n}, \frac{a_{j,1}}{b} + \cdots + \frac{a_{j,n}}{b^n} + \frac{1}{b^n}\right),$$

where $\left\{i_{j,1}, \ldots, i_{j,\nu_j}\right\} = \emptyset$ in case $\nu_j = 0$ for $1 \leq j \leq s$.

From [1, Lemmas 3.1 and 3.2] it is known that for $\boldsymbol{\nu} \in \{0, \ldots, n\}^s$ and $\boldsymbol{i}_{\boldsymbol{\nu}}$ defined as above and fixed, the generalized elementary intervals $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ for $\boldsymbol{a}_{\boldsymbol{\nu}} \in \{0, \ldots, b-1\}^{|\boldsymbol{\nu}|_1}$ form a partition of $[0,1)^s$ and the volume of $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ is $b^{-|\boldsymbol{\nu}|_1}$.

We can now recall the definition of $(t, \alpha, \beta, n, m, s)$-nets which is based on [1, Definition 3.1].

Definition 1 *Let n, m, s, $\alpha \geq 1$ and $b \geq 2$ be integers, let $0 < \beta \leq 1$ be a real number and let $0 \leq t \leq \beta n$ be an integer. Let $\mathcal{P} = (\boldsymbol{x}_n)_{n=0}^{b^m-1} \subseteq [0,1)^s$ be a point set in the s-dimensional unit cube. We say that $\mathcal{P}$ is a $(t, \alpha, \beta, n, m, s)$-net in base b, if for all integers $1 \leq i_{j,\nu_j} < \ldots < i_{j,1}$, where $\nu_j \geq 0$, with*

$$\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j,\alpha)} i_{j,l} \leq \beta n - t,$$

where for $\nu_j = 0$ we set the empty sum $\sum_{l=1}^{0} i_{j,l} = 0$, the generalized elementary interval $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ contains exactly $b^{m-|\boldsymbol{\nu}|_1}$ points of $\mathcal{P}$ for each $\boldsymbol{a}_{\boldsymbol{\nu}} \in \{0, \ldots, b-1\}^{|\boldsymbol{\nu}|_1}$.

Some remarks on the definition of $(t, \alpha, \beta, n, m, s)$-nets are in order (for more information see [1]).

Remark 1

1. We obtain the definition of a classical (t, m, s)-net (according to [13, 14]) from Definition 1 by setting $\alpha = \beta = 1$, $n = m$, and considering all $\nu_1, \ldots, \nu_s \geq 0$ so that $\sum_{j=1}^{s} \nu_j \leq m - t$, where we set $i_{j,k} = \nu_j - k + 1$ for $k = 1, \ldots, \nu_j$. Hence a $(t, 1, 1, m, m, s)$-net is a (t, m, s)-net.

2. Definition 1 says that for every generalized elementary interval $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ of volume $b^{-|\boldsymbol{\nu}|_1}$ we have

$$\frac{|\{0 \leq h < b^m : \boldsymbol{x}_h \in J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})\}|}{b^m} - \lambda_s(J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})) = 0,$$

where λ_s denotes the s dimensional Lebesgue measure.

For concrete constructions of $(t, \alpha, \beta, n, m, s)$-nets for various parameters see [7, Section 4.4], and also [1, 9] for bounds and further constructions of such nets. Most of these methods rely on the digital construction method, which is already well known for classical nets.

A method which does not necessarily use the digital construction scheme, but relies on classical (t, m, s)-nets instead, is as follows: for a fixed $d \in \mathbb{N}$, let $\{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{b^m-1}\}$ form a (t', m, sd)-net in base b. Let $\boldsymbol{x}_h = (x_{h,1}, \ldots, x_{h,sd})$, $x_{h,j} = \xi_{h,j,1} b^{-1} + \xi_{h,j,2} b^{-2} + \cdots$ for $h = 0, \ldots, b^m - 1$ and $1 \leq j \leq sd$. Then we construct a point set $\boldsymbol{y}_h = (y_{h,1}, \ldots, y_{h,s})$, $h = 0, \ldots, b^m - 1$, by

$$y_{h,j} = \sum_{i=1}^{m} \sum_{k=1}^{d} \xi_{h,(j-1)d+k,i} b^{-k-(i-1)d},$$

for any $1 \le j \le s$. It has been shown in [3] that for every $\alpha \ge 1$, the point set $\{\boldsymbol{y}_0, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_{b^m-1}\}$ forms a $(t, \alpha, \min(1, \frac{\alpha}{d}), dm, m, s)$-net in base b with

$$t = \min(d, \alpha) \min\left(m, t' + \left\lfloor \frac{s(d-1)}{2} \right\rfloor\right).$$

We remark that in Section 5, we will show how to combine two $(t, \alpha, \beta, n, m, s)$-nets to form another one using the $(u, u+v)$-construction.

Walsh functions and Weyl sums. In this subsection, we recall the concept of Weyl sums based on Walsh functions, see e.g. [12]; it turns out, see Section 3, that $(t, \alpha, \beta, n, m, s)$-nets can be characterized using Weyl sums.

Let, in the following, $\mathbb{N}_0$ denote the set of non-negative and $\mathbb{N}$ the set of positive integers and fix $b \in \mathbb{N}$, $b \ge 2$. Each $k \in \mathbb{N}_0$ has a unique b-adic representation $k = \sum_{i=0}^{a} \kappa_i b^i$, $\kappa_i \in \{0, \ldots, b-1\}$, where $\kappa_a \ne 0$. Each $x \in [0,1)$ has a b-adic representation $x = \sum_{i=1}^{\infty} \xi_i b^{-i}$, $\xi_i \in \{0, \ldots, b-1\}$, which is unique in the sense that infinitely many of the ξ_i must differ from $b-1$. We define the k-th Walsh function in base b, $\mathrm{wal}_k : [0,1) \to \mathbb{C}$ by

$$\mathrm{wal}_k(x) := \exp\left(\frac{2\pi \mathtt{i}}{b}\left(\xi_1 \kappa_0 + \cdots + \xi_{a+1} \kappa_a\right)\right).$$

For dimension $s \ge 2$ and vectors $\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ and $\boldsymbol{x} = (x_1, \ldots, x_s) \in [0,1)^s$ we define $\mathrm{wal}_{\boldsymbol{k}} : [0,1)^s \to \mathbb{C}$ by

$$\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) := \prod_{j=1}^{s} \mathrm{wal}_{k_j}(x_j).$$

It follows from the definition above that Walsh functions are piecewise constant functions. For more information on Walsh functions, see e.g. [5, 19].

We can now recall the concept of a Weyl sum.

Definition 2 *For a point set $\mathcal{P} = (\boldsymbol{x}_n)_{n=0}^{N-1} \in [0,1)^s$, $N \in \mathbb{N}_{\mathbf{0}}^s$ let*

$$S_N(f, \mathcal{P}) = \frac{1}{N} \sum_{n=0}^{N-1} f(\boldsymbol{x}_n).$$

If $f = \mathrm{wal}_{\boldsymbol{k}}$ for some $\boldsymbol{k} \in \mathbb{N}_{\mathbf{0}}^s$, then $S_N(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P})$ is called a Weyl sum (based on Walsh functions).

The function space $W_{\alpha,s,\boldsymbol{\gamma}}$. The function space under consideration in this paper is the space $W_{\alpha,s,\boldsymbol{\gamma}} \subseteq L_2([0,1)^s)$ as introduced in [7]. Here $\boldsymbol{\gamma} = (\gamma_j)_{j=1}^{\infty}$ is a sequence of positive, non-increasing weights, which are introduced to model the importance of different variables for our approximation problem, see [17]. Given a positive integer k with base b expansion $k = \kappa_1 b^{a_1-1} + \kappa_2 b^{a_2-1} + \cdots + \kappa_v b^{a_v-1}$, $1 \leq a_v < \ldots < a_1$, $v \geq 1$, we define

$$\mu_\alpha(k) := a_1 + \cdots + a_{\min(v,\alpha)}. \tag{1}$$

Furthermore, $\mu_\alpha(0) := 0$ and for $\boldsymbol{k} \in \mathbb{N}_0^s$, $\boldsymbol{k} = (k_1, \ldots, k_s)$, $\mu_\alpha(\boldsymbol{k}) = \sum_{j=1}^{s} \mu_\alpha(k_j)$.

For $k \in \mathbb{N}_0$ and a weight $\gamma > 0$, we define a function

$$r_{\alpha,\gamma}(h) := \begin{cases} 1 & \text{if } h = 0, \\ \gamma b^{-\mu_\alpha(h)} & \text{otherwise.} \end{cases}$$

If we consider a vector $\boldsymbol{k} \in \mathbb{N}_0^s$, $\boldsymbol{k} = (k_1, \ldots, k_s)$, we set

$$r_{\alpha,s,\boldsymbol{\gamma}}(\boldsymbol{h}) := \prod_{j=1}^{s} r_{\alpha,\gamma_j}(h_j).$$

In this paper, we study integration errors resulting from the approximation of an integral based on $(t, \alpha, \beta, n, m, s)$-nets by considering the Walsh series of the integrand f; we remark that this approach has also been used when studying integration errors resulting from the application of digital and generalized digital nets, see e.g. [7, 10]. In particular, for $f \in L_2([0,1)^s)$, the Walsh series of f is given by

$$f(\boldsymbol{x}) \sim \sum_{\boldsymbol{k} \in \mathbb{N}_0^s} \widehat{f}(\boldsymbol{k}) \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}), \tag{2}$$

where the Walsh coefficients $\widehat{f}(\boldsymbol{k})$ are given by

$$\widehat{f}(\boldsymbol{k}) = \int_{[0,1]^s} f(\boldsymbol{x}) \overline{\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})} \, \mathrm{d}\boldsymbol{x}.$$

In general, the Walsh series given in Eq. (2) need not converge to f, however, for the space of Walsh series $W_{\alpha,s,\boldsymbol{\gamma}}$, which we define in the following, it does converge absolutely, see also [7].

The space $W_{\alpha,s,\boldsymbol{\gamma}}$ consists of all Walsh series $f = \sum_{\boldsymbol{k} \in \mathbb{N}_0^s} \widehat{f}(\boldsymbol{k}) \mathrm{wal}_{\boldsymbol{k}}$ for which the norm

$$\|f\|_{W_{\alpha,s,\gamma}} := \sup_{\boldsymbol{k} \in \mathbb{N}_0^s} \frac{\left|\widehat{f}(\boldsymbol{k})\right|}{r_{\alpha,s,\boldsymbol{\gamma}}(\boldsymbol{k})},$$

is finite. It follows immediately that for any $f \in W_{\alpha,s,\boldsymbol{\gamma}}$, and any $\boldsymbol{k} \in \mathbb{N}_0^s$,

$$\left|\widehat{f}(\boldsymbol{k})\right| \le \|f\|_{W_{\alpha,s,\boldsymbol{\gamma}}} r_{\alpha,s,\boldsymbol{\gamma}}(\boldsymbol{k}). \tag{3}$$

For $\alpha \ge 2$, the following property was shown in [7]: Let $f : [0,1]^s \to \mathbb{R}$ be such that all mixed partial derivatives up to order α in each variable are square integrable, then $f \in W_{\alpha,s,\boldsymbol{\gamma}}$. Furthermore, an inequality using a Sobolev type norm and the norm (3) has been shown, see also [6, 8]. Consequently, the results we are going to establish in the following for functions in $W_{\alpha,s,\boldsymbol{\gamma}}$ also apply automatically to smooth functions. The assumption $\alpha > 1$ is needed to ensure that the sum of the absolute values of the Walsh coefficients converges, the case $\alpha = 1$ requires a different analysis, which was carried out in [10] for numerical integration.

3 Characterization of $(t, \alpha, \beta, n, m, s)$-nets using Weyl sums

In this section, we characterize $(t, \alpha, \beta, n, m, s)$-nets using Weyl sums. Our results generalize [12, Lemmas 1 and 2 and Corollary 3].

Lemma 1 *Let $\mathcal{P} = (\boldsymbol{x}_n)_{n=0}^{b^m-1}$ be a $(t, \alpha, \beta, n, m, s)$-net in base $b \ge 2$, where $\alpha \ge 2$ is an integer, β a real number such that $0 < \beta \le 1$ and $n, m, s \in \mathbb{N}$. Then for all $\boldsymbol{k} \in \mathbb{N}_0^s$ satisfying $0 < \mu_\alpha(\boldsymbol{k}) \le \beta n - t$ we have*

$$S_{b^m}(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P}) = 0.$$

Proof. Let $\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$, be such that $0 < \mu_\alpha(\boldsymbol{k}) \le \beta n - t$ (hence $\boldsymbol{k} \neq \boldsymbol{0}$) and for $k_j \neq 0$ let

$$k_j = \kappa_{j,1} b^{i_{j,1}-1} + \cdots + \kappa_{j,\nu_j} b^{i_{j,\nu_j}-1},$$

with $\kappa_{j,l} \in \{1, \ldots, b-1\}$ be the b-adic expansion of k_j, $1 \le j \le s$. Then for j with $k_j \neq 0$ and $x = \sum_{l=1}^\infty \xi_l b^{-l} \in [0,1)$ we have

$$\mathrm{wal}_{k_j}(x) = \exp\left(\frac{2\pi \mathtt{i}}{b}(\kappa_{j,1}\xi_{i_{j,1}} + \cdots + \kappa_{j,\nu_j}\xi_{i_{j,\nu_j}})\right).$$

Hence, if we set $\boldsymbol{i}_{\boldsymbol{\nu}} = (i_{1,1}, \ldots, i_{1,\nu_1}, \ldots, i_{s,1}, \ldots, i_{s,\nu_s})$, which only depends on $\boldsymbol{k}$, then $\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})$ is constant on generalized elementary intervals of the form

$$J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}}) = \prod_{j=1}^s \bigcup_{\substack{a_{j,l}=0 \\ l \in \{1,\ldots,n\} \setminus \{i_{j,1},\ldots,i_{j,\nu_j}\}}}^{b-1} \left[\frac{a_{j,1}}{b} + \cdots + \frac{a_{j,n}}{b^n}, \frac{a_{j,1}}{b} + \cdots + \frac{a_{j,n}}{b^n} + \frac{1}{b^n}\right).$$

Furthermore we denote the value assumed by $\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})$ on $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ by $c_{\boldsymbol{a}_{\boldsymbol{\nu}}}$. As $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$, $\boldsymbol{a}_{\boldsymbol{\nu}} \in \{0, \ldots, b-1\}^{|\boldsymbol{\nu}|_1}$ is a partition of $[0,1)^s$ we obtain

$$\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) = \sum_{\boldsymbol{a}_{\boldsymbol{\nu}} \in \{0,\ldots,b-1\}^{|\boldsymbol{\nu}|_1}} c_{\boldsymbol{a}_{\boldsymbol{\nu}}} \mathbf{1}_{J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})}(\boldsymbol{x}),$$

where $\mathbf{1}_{J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})}$ denotes the characteristic function of $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$.

For $\boldsymbol{k} \neq \mathbf{0}$ we have $\int_{[0,1]^s} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} = 0$, and hence it follows that the sum $\sum_{\boldsymbol{a}_{\boldsymbol{\nu}} \in \{0,\ldots,b-1\}^{|\boldsymbol{\nu}|_1}} c_{\boldsymbol{a}_{\boldsymbol{\nu}}} = 0$, as the volume of $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ depends only on $\boldsymbol{\nu}$. Consequently,

$$S_{b^m}(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P}) = \sum_{\boldsymbol{a}_{\boldsymbol{\nu}} \in \{0,\ldots,b-1\}^{|\boldsymbol{\nu}|_1}} c_{\boldsymbol{a}_{\boldsymbol{\nu}}} S_{b^m}(\mathbf{1}_{J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})} - \lambda_s(J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})), \mathcal{P}).$$

As $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ is a generalized elementary interval of volume $b^{-|\boldsymbol{\nu}|_1}$ for which by assumption

$$\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j,\alpha)} i_{j,l} = \mu_\alpha(\boldsymbol{k}) \leq \beta n - t,$$

it follows that $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ contains $b^{m-|\boldsymbol{\nu}|_1}$ points of $\mathcal{P}$ and hence

$$S_{b^m}(\mathbf{1}_{J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})} - \lambda_s(J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})), \mathcal{P}) = \frac{1}{b^m}(b^{m-|\boldsymbol{\nu}|_1} - b^m \lambda_s(J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})) = 0$$

as desired. □

To establish the converse, we need the following lemma, which generalizes [11, Lemma 3(i)] and which can be proven along the same lines as [11, Remark (iv), Lemma 2(i) and Lemma 3(i)].

Lemma 2 *For given $\boldsymbol{\nu}$, $\boldsymbol{i}_{\boldsymbol{\nu}}$ and $\boldsymbol{a}_{\boldsymbol{\nu}}$ let*

$$J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}}) = \prod_{j=1}^{s} \bigcup_{\substack{a_{j,l}=0 \\ l \in \{1,\ldots,n\} \setminus \{i_{j,1},\ldots,i_{j,\nu_j}\}}}^{b-1} \left[\frac{a_{j,1}}{b} + \cdots + \frac{a_{j,n}}{b^n}, \frac{a_{j,1}}{b} + \cdots + \frac{a_{j,n}}{b^n} + \frac{1}{b^n}\right)$$

and let $f(\boldsymbol{x}) = \mathbf{1}_{J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})}(\boldsymbol{x}) - \lambda_s(J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}}))$. Define

$$\begin{aligned} \Delta_{\boldsymbol{i}_{\boldsymbol{\nu}}} \;:=\; & \Big\{\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{N}^s \,:\, k_j = \kappa_{j,1} b^{i_{j,1}-1} + \cdots + \kappa_{j,\nu_j} b^{i_{j,\nu_j}-1}; \\ & \kappa_{j,1}, \ldots \kappa_{j,\nu_j} \in \{1, \ldots, b-1\} \text{ if } \nu_j > 0 \text{ and } k_j = 0 \text{ for } \nu_j = 0\Big\}, \end{aligned}$$

Then for all $\boldsymbol{k} \notin \Delta_{\boldsymbol{i}_{\boldsymbol{\nu}}}$ we have $|\widehat{f}(\boldsymbol{k})| = 0$.

The following lemma generalizes [12, Lemma 2].

Lemma 3 *Let $\mathcal{P} = (\boldsymbol{x}_n)_{n=0}^{b^m-1}$ be a fixed sequence of b^m points in the s-dimensional unit cube $[0,1)^s$ and suppose that for each $\boldsymbol{k} \in \mathbb{N}_0^s$ satisfying $0 < \mu_\alpha(\boldsymbol{k}) \leq \beta n - t$ we have*

$$S_{b^m}(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P}) = 0.$$

Then $\mathcal{P}$ is a $(t, \alpha, \beta, n, m, s)$-net in base b.

Proof. Suppose that J is an arbitrary generalized elementary interval of the form

$$J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}}) = \prod_{j=1}^{s} \bigcup_{\substack{a_{j,l}=0 \\ l \in \{1,\ldots,n\} \setminus \{i_{j,1},\ldots,i_{j,\nu_j}\}}}^{b-1} \left[\frac{a_{j,1}}{b} + \cdots + \frac{a_{j,n}}{b^n}, \frac{a_{j,1}}{b} + \cdots + \frac{a_{j,n}}{b^n} + \frac{1}{b^n}\right),$$

with $\nu_j \geq 0$, $1 \leq i_{j,\nu_j} < \ldots < i_{j,1}$ so that $\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j,\alpha)} i_{j,l} \leq \beta n - t$. We define $f(\boldsymbol{x}) = \mathbf{1}_J(\boldsymbol{x}) - \lambda_s(J)$. In order to show that $\mathcal{P}$ is a $(t, \alpha, \beta, n, m, s)$-net in base b, it suffices to prove that $S_{b^m}(f, \mathcal{P}) = 0$. If $\widehat{\mathbf{1}}_J(\boldsymbol{k})$ denotes the $\boldsymbol{k}$-th Walsh coefficient of $\mathbf{1}_J$, then due to Lemma 2, for all $\boldsymbol{x} \in [0,1)^s$ we have

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in \Delta^*} \widehat{\mathbf{1}}_J(\boldsymbol{k}) \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})$$

and hence

$$S_{b^m}(f, \mathcal{P}) = \sum_{\boldsymbol{k} \in \Delta^*} \widehat{\mathbf{1}}_J(\boldsymbol{k}) S_{b^m}(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P}).$$

But $\boldsymbol{k} \in \Delta^*$ implies that $\mu_\alpha(\boldsymbol{k}) = \sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j,\alpha)} i_{j,l} \leq \beta n - t$, hence $S_{b^m}(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P}) = 0$. This implies that $S_{b^m}(f, \mathcal{P}) = 0$. □

Combining Lemma 1 and Lemma 3 we obtain the following characterization of $(t, \alpha, \beta, n, m, s)$-nets in terms of Weyl sums (for the Walsh function system).

Theorem 1 *Let $\mathcal{P} = (\boldsymbol{x}_n)_{n=0}^{b^m-1}$ be a finite sequence of b^m points in the s-dimensional unit cube $[0,1)^s$. Then $\mathcal{P}$ is a $(t, \alpha, \beta, n, m, s)$-net in base b if and only if for all $\boldsymbol{k} \in \mathbb{N}_0^s$ satisfying $0 < \mu_\alpha(\boldsymbol{k}) \leq \beta n - t$ we have*

$$S_{b^m}(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P}) = 0.$$

4 Application to numerical integration

In this section we establish that (t,α,β,n,m,s)-nets can exploit the smoothness α of a function $f \in W_{\alpha,s,\boldsymbol{\gamma}}$. We need to introduce some notation: Let $\mathcal{S} = \{1,\ldots,s\}$. For $\boldsymbol{k} = (k_1,\ldots,k_s) \in \mathbb{N}_0^s$ and for $\emptyset \neq \mathfrak{u} \subseteq \mathcal{S}$ let $\boldsymbol{k}_{\mathfrak{u}}$ be the vector in $\mathbb{N}_0^{|\mathfrak{u}|}$ which consists of all components of $\boldsymbol{k}$ whose index belongs to $\mathfrak{u}$. Furthermore let $(\boldsymbol{k}_{\mathfrak{u}}, \boldsymbol{0})$ be the vector $\boldsymbol{k}$ with all components whose index is not in $\mathfrak{u}$ replaced by 0. With this notation we have $\mu_\alpha(\boldsymbol{k}_{\mathfrak{u}}) = \mu_\alpha(\boldsymbol{k}_{\mathfrak{u}}, \boldsymbol{0})$. For a sequence $\boldsymbol{\gamma} = (\gamma_j)_{j\ge 1}$ we write $\boldsymbol{\gamma}_{\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \gamma_j$.

We need the following lemma.

Lemma 4 *Let $(\boldsymbol{x}_n)_{n=0}^{b^m-1}$ be a (t,α,β,n,m,s)-net in base b and let $f \in W_{\alpha,s,\boldsymbol{\gamma}}$, then*

$$\left| \int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \frac{1}{b^m}\sum_{n=0}^{b^m-1} f(\boldsymbol{x}_n) \right| \le \sum_{\emptyset\neq\mathfrak{u}\subseteq\mathcal{S}} \boldsymbol{\gamma}_{\mathfrak{u}} \sum_{\substack{\boldsymbol{k}_{\mathfrak{u}}\in\mathbb{N}^{|\mathfrak{u}|} \\ \mu_\alpha(\boldsymbol{k}_{\mathfrak{u}})>\beta n-t}} b^{-\mu_\alpha(\boldsymbol{k}_{\mathfrak{u}})}. \tag{4}$$

Proof. For $f \in W_{\alpha,s,\boldsymbol{\gamma}}$ and $(\boldsymbol{x}_n)_{n=0}^{b^m-1}$ a (t,α,β,n,m,s)-net, we can write

$$\begin{aligned}
\left| \int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \frac{1}{b^m}\sum_{n=0}^{b^m-1} f(\boldsymbol{x}_n) \right| &= \left| \widehat{f}(\boldsymbol{0}) - \frac{1}{b^m}\sum_{n=0}^{b^m-1}\sum_{\boldsymbol{k}\in\mathbb{N}_{\boldsymbol{0}}^s} \widehat{f}(\boldsymbol{k})\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_n)\right| \\
= \left| \widehat{f}(\boldsymbol{0}) - \sum_{\boldsymbol{k}\in\mathbb{N}_{\boldsymbol{0}}^s} \widehat{f}(\boldsymbol{k})\frac{1}{b^m}\sum_{n=0}^{b^m-1}\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_n)\right| &= \left| \sum_{\boldsymbol{k}\in\mathbb{N}_{\boldsymbol{0}}^s\setminus\{\boldsymbol{0}\}} \widehat{f}(\boldsymbol{k})\frac{1}{b^m}\sum_{n=0}^{b^m-1}\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_n)\right| \\
= \left| \sum_{\substack{\boldsymbol{k}\in\mathbb{N}_{\boldsymbol{0}}^s\setminus\{\boldsymbol{0}\} \\ \mu_\alpha(\boldsymbol{k})>\beta n-t}} \widehat{f}(\boldsymbol{k})\frac{1}{b^m}\sum_{n=0}^{b^m-1}\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_n)\right|, & \qquad (5)
\end{aligned}$$

where we used Lemma 1. Using the triangular inequality it now follows that

$$\begin{aligned}
\left| \int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \frac{1}{b^m}\sum_{n=0}^{b^m-1} f(\boldsymbol{x}_n) \right| &\le \sum_{\substack{\boldsymbol{k}\in\mathbb{N}_{\boldsymbol{0}}^s\setminus\{\boldsymbol{0}\} \\ \mu_\alpha(\boldsymbol{k})>\beta n-t}} \left|\widehat{f}(\boldsymbol{k})\right| \\
\le \|f\|_{W_{\alpha,s,\boldsymbol{\gamma}}} \sum_{\substack{\boldsymbol{k}\in\mathbb{N}_{\boldsymbol{0}}^s \\ \mu_\alpha(\boldsymbol{k})>\beta n-t}} r_{\alpha,s,\boldsymbol{\gamma}}(\boldsymbol{k}) &= \|f\|_{W_{\alpha,s,\boldsymbol{\gamma}}} \sum_{\emptyset\neq\mathfrak{u}\subseteq\mathcal{S}} \boldsymbol{\gamma}_{\mathfrak{u}} \sum_{\substack{\boldsymbol{k}_{\mathfrak{u}}\in\mathbb{N}^{|\mathfrak{u}|} \\ \mu_\alpha(\boldsymbol{k}_{\mathfrak{u}})>\beta n-t}} b^{-\mu_\alpha(\boldsymbol{k}_{\mathfrak{u}})}
\end{aligned}$$

as desired. □

Remark 2 Comparing Eq. (4) to [7, Eq. (5.1)], we note that in Eq. (4), the second sum runs over all $\boldsymbol{k}_{\mathfrak{u}} \in \mathbb{N}^{|\mathfrak{u}|}$ for which $\mu_\alpha(\boldsymbol{k}_{\mathfrak{u}}) > \beta n - t$, whereas in [7, Eq. (5.1)], the corresponding sum is over all $\boldsymbol{k}_{\mathfrak{u}}$ in the dual space corresponding to the set $\mathfrak{u}$. We obtain this estimation as we estimate the absolute value of the character sum $b^{-m} \sum_{n=0}^{b^m-1} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_n)$ in (5) by 1. Given concrete constructions, better estimations of this sum may be obtained, as is the case for digital nets and sequences.

To establish the main result of this section, we need the following lemma.

Lemma 5 *Let $l \geq 1$ and $\alpha \geq 2$ be integers. Then*

$$\sum_{\substack{k \in \mathbb{N} \\ \mu_\alpha(k) = l}} 1 \leq 2 \binom{l+\alpha-1}{\alpha-1} (b-1)^\alpha b^{\lfloor l/\alpha \rfloor}.$$

Proof. For $k \in \mathbb{N}$ let ν_k denote the number of non-zero digits in the base b representation of k. We represent $k \in \mathbb{N}$ in the form

$$k = \kappa_1 b^{a_1 - 1} + \cdots + \kappa_{\nu_k} b^{a_{\nu_k} - 1},$$

where $\kappa_1, \ldots, \kappa_{\nu_k} \in \{1, \ldots, b-1\}$ and $a_1 > \ldots > a_{\nu_k} \geq 1$. We firstly consider those $k \in \mathbb{N}$ for which $\nu_k \leq \alpha$. In that case, we put a bound on the number of k for which $\mu_\alpha(k) = a_1 + \cdots + a_{\nu_k} = l$. Then we have

$$\begin{aligned}
&|\{(a_1, \ldots, a_{\nu_k}) : a_1 + \cdots + a_{\nu_k} = l, a_1 > \ldots > a_{\nu_k} \geq 1\}| \\
&\quad \leq \; |\{(a_1, \ldots, a_{\nu_k}) : a_1 + \cdots + a_{\nu_k} = l, a_1 \geq 0, \ldots, a_{\nu_k} \geq 0\}| \\
&\quad \leq \; |\{(a_1, \ldots, a_\alpha) : a_1 + \cdots + a_\alpha = l, a_1 \geq 0, \ldots, a_\alpha \geq 0\}| \\
&\quad \leq \; \binom{l+\alpha-1}{\alpha-1}.
\end{aligned}$$

The coefficients $\kappa_1, \ldots, \kappa_{\nu_k}$ take values in the set $\{1, \ldots, b-1\}$, such that there are $(b-1)^{\nu_k} \leq (b-1)^\alpha$ possibilities, hence

$$\sum_{\substack{k \in \mathbb{N} \\ \mu_\alpha(k) = l, \nu_k \leq \alpha}} 1 \leq (b-1)^\alpha \binom{l+\alpha-1}{\alpha-1}.$$

We now consider those k for which $\nu_k > \alpha$. Then

$$k = \kappa_1 b^{a_1 - 1} + \cdots + \kappa_\alpha b^{a_\alpha - 1} + \kappa_{\alpha+1} b^{a_{\alpha+1} - 1} + \cdots + \kappa_{\nu_k} b^{a_{\nu_k} - 1},$$

and we put a bound on the number of k for which $\mu_\alpha(k) = a_1 + \cdots + a_\alpha = l$. Now,

$$\begin{aligned}
&|\{(a_1, \ldots, a_\alpha) \,:\, a_1 + \cdots + a_\alpha = l, a_1 > \cdots > a_\alpha \geq 1\}| \\
&\quad \leq \; |\{(a_1, \ldots, a_\alpha) \,:\, a_1 + \cdots + a_\alpha = l, a_1 \geq 0, \ldots, a_\alpha \geq 0\}| \\
&\quad \leq \; \binom{l + \alpha - 1}{\alpha - 1}.
\end{aligned}$$

Regarding the coefficients, it is clear that $\kappa_1, \ldots, \kappa_{\nu_k} \in \{1, \ldots, b-1\}$, so the first α coefficients, $\kappa_1, \ldots, \kappa_\alpha$ can assume $(b-1)^\alpha$ different values. Regarding the sum

$$\kappa_{\alpha+1} b^{a_{\alpha+1}-1} + \cdots + \kappa_{\nu_k} b^{a_{\nu_k}-1}, \tag{6}$$

where $\kappa_{\alpha+1}, \ldots, \kappa_{\nu_k} \in \{1, \ldots, b-1\}$ and $a_{\alpha+1} > \ldots > a_{\nu_k} \geq 1$, it is clear that the number of different values that the sum in Eq. (6) can assume is bounded by $b^{a_\alpha - 1}$. But by assumption, $a_\alpha + \cdots + a_1 = l$, hence $a_\alpha \leq \lfloor l/\alpha \rfloor$, so we conclude that

$$\sum_{\substack{k \in \mathbb{N} \\ \mu_\alpha(k) = l, \nu_k > \alpha}} 1 \leq (b-1)^\alpha b^{\lfloor l/\alpha \rfloor} \binom{l + \alpha - 1}{\alpha - 1}$$

and the result follows by summing up the two cases. $\square$

The next theorem establishes that $(t, \alpha, \beta, n, m, s)$-nets can exploit the smoothness α of a function $f \in W_{\alpha,s,\boldsymbol{\gamma}}$.

Theorem 2 *Let $(\boldsymbol{x}_n)_{n=0}^{b^m-1}$ be a $(t, \alpha, \beta, n, m, s)$-net in base b and let $f \in W_{\alpha,s,\boldsymbol{\gamma}}$. Then*

$$\begin{aligned}
&\left| \int_{[0,1]^s} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} - \frac{1}{b^m} \sum_{n=0}^{b^m-1} f(\boldsymbol{x}_n) \right| \\
&\quad \leq \; b^{-(1-1/\alpha)(\lfloor \beta n - t \rfloor + 1)} \sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{S}} \boldsymbol{\gamma}_{\mathfrak{u}} \left(\frac{b^{1-1/\alpha}(b-1)}{(b^{1-1/\alpha} - 1)} \right)^{\alpha|\mathfrak{u}|} \frac{(\lfloor \beta n - t \rfloor + \alpha|\mathfrak{u}|)!}{(|\mathfrak{u}| - 1)!(\lfloor \beta n - t \rfloor + 1)!}.
\end{aligned}$$

Proof. Lemma 4 established that

$$\left| \int_{[0,1]^s} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} - \frac{1}{b^m} \sum_{n=0}^{b^m-1} f(\boldsymbol{x}_n) \right| \leq \sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{S}} \boldsymbol{\gamma}_{\mathfrak{u}} \sum_{\substack{\boldsymbol{k}_{\mathfrak{u}} \in \mathbb{N}^{|\mathfrak{u}|} \\ \mu_\alpha(\boldsymbol{k}_{\mathfrak{u}}) > \beta n - t}} b^{-\mu_\alpha(\boldsymbol{k}_{\mathfrak{u}})}. \tag{7}$$

For a given $\emptyset \neq \mathfrak{u} \subseteq \mathcal{S}$ we rewrite

$$\sum_{\substack{\boldsymbol{k}_{\mathfrak{u}} \in \mathbb{N}^{|\mathfrak{u}|} \\ \mu_\alpha(\boldsymbol{k}_{\mathfrak{u}}) > \beta n - t}} b^{-\mu_\alpha(\boldsymbol{k}_{\mathfrak{u}})} = \sum_{l=\lfloor \beta n - t\rfloor + 1}^{\infty} b^{-l} \sum_{\substack{\boldsymbol{k}_{\mathfrak{u}} \in \mathbb{N}^{|\mathfrak{u}|} \\ \mu_\alpha(\boldsymbol{k}_{\mathfrak{u}}) = l}} 1.$$

Using Lemma 5 we obtain

$$\begin{aligned}
\sum_{\substack{\boldsymbol{k}_{\mathfrak{u}} \in \mathbb{N}^{|\mathfrak{u}|} \\ \mu_\alpha(\boldsymbol{k}_{\mathfrak{u}}) = l}} 1 &= \sum_{l_1 + \cdots + l_{|\mathfrak{u}|} = l} \prod_{j=1}^{|\mathfrak{u}|} \sum_{\substack{k_j \in \mathbb{N} \\ \mu_\alpha(k_j) = l_j}} 1 \\
&\leq \sum_{l_1 + \cdots + l_{|\mathfrak{u}|} = l} \prod_{j=1}^{|\mathfrak{u}|} \left[2 \binom{l_j + \alpha - 1}{\alpha - 1} (b-1)^\alpha b^{\lfloor l_j/\alpha \rfloor} \right] \\
&\leq 2^{|\mathfrak{u}|} (b-1)^{\alpha |\mathfrak{u}|} b^{l/\alpha} \sum_{l_1 + \cdots + l_{|\mathfrak{u}|} = l} \prod_{j=1}^{|\mathfrak{u}|} \binom{l_j + \alpha - 1}{\alpha - 1}.
\end{aligned}$$

For any $1 \leq j \leq |\mathfrak{u}|$ we have $\binom{l_j+\alpha-1}{\alpha-1} \leq (1+l_j)^{\alpha-1}$. Since $l_1, \ldots, l_{|\mathfrak{u}|} \geq 1$ and $\alpha \geq 2$ and $l_1 + \cdots + l_{|\mathfrak{u}|} = l$ we have $1 + l_j \leq l$ and therefore $\binom{l_j+\alpha-1}{\alpha-1} \leq l^{\alpha-1}$. Hence we obtain

$$\begin{aligned}
& 2^{|\mathfrak{u}|} (b-1)^{\alpha |\mathfrak{u}|} b^{l/\alpha} \sum_{l_1 + \cdots + l_{|\mathfrak{u}|} = l} \prod_{j=1}^{|\mathfrak{u}|} \binom{l_j + \alpha - 1}{\alpha - 1} \\
\leq\ & 2^{|\mathfrak{u}|} (b-1)^{\alpha |\mathfrak{u}|} b^{l/\alpha} \sum_{l_1 + \cdots + l_{|\mathfrak{u}|} = l} l^{(\alpha-1)|\mathfrak{u}|} \\
& \leq 2^{|\mathfrak{u}|} (b-1)^{\alpha |\mathfrak{u}|} b^{l/\alpha} l^{(\alpha-1)|\mathfrak{u}|} \binom{l + |\mathfrak{u}| - 1}{|\mathfrak{u}| - 1}.
\end{aligned}$$

Hence

$$\sum_{l=\lfloor \beta n - t\rfloor + 1}^{\infty} b^{-l} \sum_{\substack{\mathfrak{u} \in \mathbb{N}^{|\mathfrak{u}|} \\ \mu_\alpha(\boldsymbol{k}_{\mathfrak{u}}) = l}} 1 \leq 2^{|\mathfrak{u}|} (b-1)^{\alpha |\mathfrak{u}|} \sum_{l=\lfloor \beta n - t\rfloor + 1}^{\infty} b^{-l} b^{l/\alpha} l^{(\alpha-1)|\mathfrak{u}|} \binom{l + |\mathfrak{u}| - 1}{|\mathfrak{u}| - 1}.$$

Invoking the inequality $l^{(\alpha-1)|\mathfrak{u}|}\binom{l+|\mathfrak{u}|-1}{|\mathfrak{u}|-1} \le \binom{l+\alpha|\mathfrak{u}|-1}{\alpha|\mathfrak{u}|-1}\frac{(\alpha|\mathfrak{u}|-1)!}{(|\mathfrak{u}|-1)!}$ we get

$$\begin{aligned}
&2^{|\mathfrak{u}|}(b-1)^{\alpha|\mathfrak{u}|}\sum_{l=\lfloor\beta n-t\rfloor+1}^{\infty} b^{-l}b^{l/\alpha}l^{(\alpha-1)|\mathfrak{u}|}\binom{l+|\mathfrak{u}|-1}{|\mathfrak{u}|-1} \\
&\le\; 2^{|\mathfrak{u}|}(b-1)^{\alpha|\mathfrak{u}|}\frac{(\alpha|\mathfrak{u}|-1)!}{(|\mathfrak{u}|-1)!}\sum_{l=\lfloor\beta n-t\rfloor+1}^{\infty} b^{-(1-1/\alpha)l}\binom{l+\alpha|\mathfrak{u}|-1}{\alpha|\mathfrak{u}|-1} \\
&\le\; 2^{|\mathfrak{u}|}\left(\frac{b^{1-1/\alpha}(b-1)}{(b^{1-1/\alpha}-1)}\right)^{\alpha|\mathfrak{u}|}\frac{(\alpha|\mathfrak{u}|-1)!}{(|\mathfrak{u}|-1)!}b^{-(1-1/\alpha)(\lfloor\beta n-t\rfloor+1)}\binom{\lfloor\beta n-t\rfloor+\alpha|\mathfrak{u}|}{\alpha|\mathfrak{u}|-1},
\end{aligned}$$

where we used an inequality which was for example also used in the proof of [7, Lemma 5.2]. Hence

$$\begin{aligned}
&\sum_{\emptyset\neq\mathfrak{u}\subseteq\mathcal{S}}\gamma_{\mathfrak{u}}\sum_{\substack{\boldsymbol{k}_{\mathfrak{u}}\in\mathbb{N}^{|u|}\\ \mu_\alpha(\boldsymbol{k}_{\mathfrak{u}})>\beta n-t}} b^{-\mu_\alpha(\boldsymbol{k}_{\mathfrak{u}})} \\
&\le\; b^{-(1-1/\alpha)(\lfloor\beta n-t\rfloor+1)} \\
&\qquad\sum_{\emptyset\neq\mathfrak{u}\subseteq\mathcal{S}}\gamma_{\mathfrak{u}}2^{|\mathfrak{u}|}\left(\frac{b^{1-1/\alpha}(b-1)}{(b^{1-1/\alpha}-1)}\right)^{\alpha|\mathfrak{u}|}\frac{(\alpha|\mathfrak{u}|-1)!}{(|\mathfrak{u}|-1)!}\binom{\lfloor\beta n-t\rfloor+\alpha|\mathfrak{u}|}{\alpha|\mathfrak{u}|-1} \\
&=\; b^{-(1-1/\alpha)(\lfloor\beta n-t\rfloor+1)} \\
&\qquad\sum_{\emptyset\neq\mathfrak{u}\subseteq\mathcal{S}}\gamma_{\mathfrak{u}}2^{|\mathfrak{u}|}\left(\frac{b^{1-1/\alpha}(b-1)}{(b^{1-1/\alpha}-1)}\right)^{\alpha|\mathfrak{u}|}\frac{(\lfloor\beta n-t\rfloor+\alpha|\mathfrak{u}|)!}{(|\mathfrak{u}|-1)!(\lfloor\beta n-t\rfloor+1)!},
\end{aligned}$$

which establishes the result. □

Remark 3 For $\beta n = \alpha m$, we obtain a convergence rate of the integration error of $N^{-(\alpha-1)}$ multiplied by a $\log N$ factor. This rate, although not optimal, see [6, 7], does establish that (t,α,β,n,m,s)-nets can exploit the smoothness of functions lying in $W_{\alpha,s,\boldsymbol{\gamma}}$. This was not possible with the classical concept of (t,m,s)-nets.

5 The $(u, u+v)$ construction

In this section, we will generalize the $(u,u+v)$-construction from coding theory, which seems to stem from [18], to (t,α,β,n,m,s)-nets. We remark that the $(u,u+v)$-construction has already been used to construct (t,m,s)-nets, see [4], and recently to construct generalized digital nets, see [9]. As in Section 4, the main tool in proving the result is Theorem 1. We now outline the $(u,u+v)$-construction.

Assume we are given a $(t_1, \alpha, \beta_1, n_1, m_1, s_1)$-net $\mathcal{P}_1$ denoted by $(\boldsymbol{x}_i)_{i=0}^{b^{m_1}-1}$ and a $(t_2, \alpha, \beta_2, n_2, m_2, s_2)$-net $\mathcal{P}_2$ denoted by $(\boldsymbol{y}_i)_{i=0}^{b^{m_2}-1}$, where we assume $s_1 \le s_2$. W.l.o.g. we may assume that $\boldsymbol{x}_i = (x_{i,1}, \ldots, x_{i,s_1})$ with $x_{i,j} = \xi_{i,j,1}/b + \cdots + \xi_{i,j,n_1}/b^{n_1}$ and $\boldsymbol{y}_i = (y_{i,1}, \ldots, y_{i,s_2})$ with $y_{i,j} = \eta_{i,j,1}/b + \cdots + \eta_{i,j,n_2}/b^{n_2}$ (if there are digits $\xi_{i,j,r} \neq 0$ for $r > n_1$ or $\eta_{i,j,r} \neq 0$ for $r > n_2$ we can slightly change $\mathcal{P}_1$, $\mathcal{P}_2$ by setting $\xi_{i,j,r} = 0$ for $r > n_1$ and $\eta_{i,j,r} = 0$ for $r > n_2$, without changing the $(t_w, \alpha, \beta_w, n_w, m_w, s_w)$-net property of $\mathcal{P}_w$, $w = 1, 2$).

We now define a new point set $\mathcal{P} = (\boldsymbol{z}_i)_{i=1}^{b^{m_1+m_2}-1}$, $\boldsymbol{z}_i = (z_{i,1}, \ldots, z_{i,s_1+s_2})$, consisting of $b^{m_1+m_2}$ points in $[0,1)^{s_1+s_2}$ as follows: first we set

$$\ell := \min(2\beta_1 n_1 - 2t_1 + 1, \beta_2 n_2 - t_2).$$

We denote the addition modulo b by $\oplus$ and the subtraction modulo b by $\ominus$ (for short we use $\ominus x := 0 \ominus x$).

- For $j = 1, \ldots, s_1$, $h = 0, \ldots, b^{m_2} - 1$ and $i = 0, \ldots, b^{m_1} - 1$ we set

$$\begin{aligned} z_{hb^{m_1}+i,j} \quad = \quad & \frac{\xi_{i,j,1} \ominus \eta_{h,j,1}}{b} + \cdots + \frac{\xi_{i,j,\min(\ell,n_1)} \ominus \eta_{h,j,\min(\ell,n_1)}}{b^{\min(\ell,n_1)}} \\ & + \left(\frac{\xi_{i,j,\ell+1}}{b^{\ell+1}} + \cdots + \frac{\xi_{i,j,n_1}}{b^{n_1}} \right) \mathbf{1}_{n_1 \ge \ell} \\ & + \left(\frac{\ominus \eta_{h,j,n_1+1}}{b^{n_1+1}} + \cdots + \frac{\ominus \eta_{h,j,\ell}}{b^{\ell}} \right) \mathbf{1}_{n_1 < \ell}. \end{aligned}$$

- For $j = s_1 + 1, \ldots, s_1 + s_2$, $h = 0, \ldots, b^{m_2} - 1$ and $i = 0, \ldots, b^{m_1} - 1$ we set

$$z_{hb^{m}+i,j} = y_{h,j-s_1}.$$

Note that for every component of $\boldsymbol{z}_i$ at most the first $\max(n_1, n_2) \le n_1 + n_2 =: n$ digits in its b-adic expansion are non-zero.

In the following we analyze the Weyl sum $S_{b^{m_1+m_2}}(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P})$ for $\boldsymbol{k} \in \mathbb{N}_0^{s_1+s_2}$ satisfying $\mu_\alpha(\boldsymbol{k}) \le \ell$. For this analysis we need to introduce some notation: For vectors $\boldsymbol{k}, \boldsymbol{l} \in \mathbb{N}_0^s$, $\boldsymbol{k} = (k_1, \ldots, k_s)$, $\boldsymbol{l} = (l_1, \ldots, l_s)$, $\boldsymbol{k} \oplus \boldsymbol{l} := (k_1 \oplus l_1, k_2 \oplus l_2, \ldots, k_s \oplus l_s)$.

We embed a vector $\boldsymbol{u} \in \mathbb{N}_0^{s_1}$ into $\mathbb{N}_0^{s_2}$ by filling up the remaining components with zeros. This vector will be denoted by $(\boldsymbol{u}, \mathbf{0}) \in \mathbb{N}_0^{s_2}$. In the following we will represent a vector $\boldsymbol{k} \in \mathbb{N}_0^{s_1+s_2}$ in the form $\boldsymbol{k} = (\boldsymbol{u}, (\boldsymbol{u}, \mathbf{0}) \oplus \boldsymbol{v})$, where $\boldsymbol{u} \in \mathbb{N}_0^{s_1}$, $\boldsymbol{v} \in \mathbb{N}_0^{s_2}$, i.e., $\boldsymbol{k}$ is the concatenation of the two vectors $\boldsymbol{u} \in \mathbb{N}_0^{s_1}$ and $(\boldsymbol{u}, \mathbf{0}) \oplus \boldsymbol{v} \in \mathbb{N}_0^{s_2}$.

Lemma 6 *For $\boldsymbol{k} \in \{0, \ldots, b^\ell - 1\}^{s_1+s_2}$ and for $\mathcal{P}_1, \mathcal{P}_2$ and $\mathcal{P}$ given above we have*

$$S_{b^{m_1+m_2}}(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P}) = S_{b^{m_1}}(\mathrm{wal}_{\boldsymbol{u}}, \mathcal{P}_1) S_{b^{m_2}}(\mathrm{wal}_{\boldsymbol{v}}, \mathcal{P}_2)$$

Proof. For $\boldsymbol{y}_h \in [0,1)^{s_2}$ we denote its projection to the first s_1 components by $\boldsymbol{y}_h^{(s_1)}$. Then we have

$$\begin{aligned}
\frac{1}{b^{m_1+m_2}} \sum_{h'=0}^{b^{m_1+m_2}-1} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{z}_{h'}) &= \frac{1}{b^{m_1+m_2}} \sum_{h=0}^{b^{m_2}-1} \sum_{i=0}^{b^{m_1}-1} \mathrm{wal}_{(\boldsymbol{u},(\boldsymbol{u},\boldsymbol{0})\oplus\boldsymbol{v})}(\boldsymbol{z}_{hb^{m_1}+i}) \\
&= \frac{1}{b^{m_1+m_2}} \sum_{h=0}^{b^{m_2}-1} \sum_{i=0}^{b^{m_1}-1} \mathrm{wal}_{\boldsymbol{u}}(\boldsymbol{x}_i \ominus \boldsymbol{y}_h^{(s_1)}) \mathrm{wal}_{(\boldsymbol{u},\boldsymbol{0})\oplus\boldsymbol{v}}(\boldsymbol{y}_h) \\
&= \frac{1}{b^{m_1}} \sum_{i=0}^{b^{m_1}-1} \mathrm{wal}_{\boldsymbol{u}}(\boldsymbol{x}_i) \frac{1}{b^{m_2}} \sum_{h=0}^{b^{m_2}-1} \mathrm{wal}_{\boldsymbol{v}}(\boldsymbol{y}_h).
\end{aligned}$$

The last two equalities use the assumption that $\boldsymbol{k} \in \{0,\ldots,b^\ell - 1\}^{s_1+s_2}$, which means that for all components of $\boldsymbol{k}$ at most the first ℓ digits in their b-adic expansion are different from zero. $\square$

We need the following lemma, which is [2, Lemma 5].

Lemma 7 *For $\alpha \geq 2$, $\boldsymbol{k},\boldsymbol{l} \in \mathbb{N}_0^s$ we have $\mu_\alpha(\boldsymbol{k}\oplus\boldsymbol{l}) \geq \mu_\alpha(\boldsymbol{k}) - \mu_\alpha(\boldsymbol{l})$.*

The following theorem establishes the main result of this section.

Theorem 3 *Let $b \in \mathbb{N}$, $b \geq 2$, let $\mathcal{P}_1$ be a $(t_1,\alpha,\beta_1,n_1,m_1,s_1)$-net in base b, and $\mathcal{P}_2$ be a $(t_2,\alpha,\beta_2,n_2,m_2,s_2)$-net in base b. Then $\mathcal{P}$ defined as above is a (t,α,β,n,m,s)-net in base b, where $n = n_1+n_2$, $m = m_1+m_2$, $s = s_1+s_2$ and*

$$\beta = \min(\beta_1,\beta_2), \quad t = \beta n - \ell.$$

Proof. We will use Theorem 1 to establish the result, i.e., we need to show that for all $\boldsymbol{k} \in \mathbb{N}_0^{s_1+s_2}$ satisfying $0 < \mu_\alpha(\boldsymbol{k}) \leq \beta n - t$ we have

$$S_{b^{m_1+m_2}}(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P}) = 0.$$

For $\boldsymbol{k} \in \mathbb{N}_0^{s_1+s_2}$ satisfying $0 < \mu_\alpha(\boldsymbol{k}) \leq \beta n - t = \ell$ we necessarily have that $\boldsymbol{k} \in \{0,\ldots,b^\ell-1\}^{s_1+s_2}$. Hence we may use Lemma 6 which states that

$$S_{b^{m_1+m_2}}(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P}) = S_{b^{m_1}}(\mathrm{wal}_{\boldsymbol{u}}, \mathcal{P}_1) S_{b^{m_2}}(\mathrm{wal}_{\boldsymbol{v}}, \mathcal{P}_2).$$

We proceed in a manner very similar to the proof of [16, Theorem 5.3] and distinguish three cases.

Case 1: We firstly assume that $\boldsymbol{v} \neq \boldsymbol{0}$ and $\mu_\alpha(\boldsymbol{k}) \leq \beta n - t$. We want to show that $0 < \mu_\alpha(\boldsymbol{v}) \leq \beta_2 n_2 - t_2$, in which case we obtain $S_{b^{m_2}}(\mathrm{wal}_{\boldsymbol{v}}, \mathcal{P}_2) = 0$ by Theorem 1. As $\boldsymbol{v} \neq \boldsymbol{0}$ we have $\mu_\alpha(\boldsymbol{v}) > 0$. Also, using Lemma 7,

$$\mu_\alpha(\boldsymbol{v}) \leq \mu_\alpha((\boldsymbol{u},\boldsymbol{0})\oplus\boldsymbol{v}) + \mu_\alpha(\boldsymbol{u}) = \mu_\alpha(\boldsymbol{k}) \leq \beta n - t \leq \beta_2 n_2 - t_2.$$

Case 2: We now assume that $\boldsymbol{v} = \mathbf{0}$, $\boldsymbol{u} \neq \mathbf{0}$ and $0 < \mu_\alpha(\boldsymbol{k}) \leq \beta n - t$. We want to show that $0 < \mu_\alpha(\boldsymbol{u}) \leq \beta_1 n_1 - t_1$, in which case we obtain $S_{b^{m_1}}(\mathrm{wal}_{\boldsymbol{u}}, \mathcal{P}_1) = 0$ by Theorem 1. As $\boldsymbol{u} \neq \mathbf{0}$ we have $\mu_\alpha(\boldsymbol{u}) > 0$. Also,

$$2(\beta_1 n_1 - t_1) + 1 \geq \beta n - t \geq \mu_\alpha(\boldsymbol{k}) = \mu_\alpha((\boldsymbol{u}, \mathbf{0}) \oplus \boldsymbol{v}) + \mu_\alpha(\boldsymbol{u}) = 2\mu_\alpha(\boldsymbol{u}).$$

Hence $\mu_\alpha(\boldsymbol{u}) \leq \beta_1 n_1 - t_1$, as $\mu_\alpha(\boldsymbol{u})$ is an integer.

Case 3: We now assume that $\boldsymbol{v} = \mathbf{0}$ and $\boldsymbol{u} = \mathbf{0}$ and $0 < \mu_\alpha(\boldsymbol{k}) \leq \beta n - t$. However, as $\boldsymbol{v} = 0$ and $\boldsymbol{u} = 0$, it follows that $\mu_\alpha(\boldsymbol{k}) = 0$ such that this case need not be considered.

Thus we have $S_{b^{m_1+m_2}}(\mathrm{wal}_{\boldsymbol{k}}, \mathcal{P}) = 0$ whenever $0 < \mu_\alpha(\boldsymbol{k}) \leq \beta n - t$ and this completes the proof. $\square$

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