Periodic functions with bounded remainder

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Abstract

Let F be the class of all 1-periodic real functions with absolutely convergent Fourier series expansion and let $(x_n)_{n\geq 0}$ be the van der Corput sequence. In this paper results on the boundedness of $\sum_{n=0}^{N-1} f(x_n)$ for $f \in F$ are given. We give a criterion on the convergence rate of the Fourier coefficients of f such that the above sum is bounded independently of N. Further we show that our result is also best possible.

1 Introduction

In the theory of uniform distribution modulo 1 the van der Corput sequence is the prototype for a uniformly distributed sequence. This sequence, analyzed in a multitude of papers, is defined as follows:

Definition 1 Let $p \geq 2$ be an integer. For any $n \in \mathbb{N}_0$ with p-adic expansion $n = \sum_{i\geq 0} n_i p^i$ (note that this expansion is finite) the radical inverse function to the base p is defined as $\varphi_p(n) = \sum_{i\geq 0} n_i p^{-i-1}$. Now the *van der Corput sequence in base* p is the sequence $(x_n)_{n\geq 0}$ with $x_n = \varphi_p(n)$ for all $n \in \mathbb{N}_0$.

In this paper we consider the following question stated by Johannes Schoissengeier (private communication): let F be the class of all 1-periodic functions $f : \mathbb{R} \to \mathbb{R}$ with an absolutely convergent Fourier series expansion and $\int_0^1 f(x)\mathrm{d}x = 0$. Further let $(x_n)_{n\geq 0}$ be the van der Corput sequence in base p. Under which smoothness condition on the functions from F do we have

$$\left|\sum_{n=0}^{N-1} f(x_n)\right| \ll 1 \quad \forall f \in F? \tag{1}$$

For convenience we will call the sum in (1) the *remainder* of the function f.

We remark that it follows from the Koksma-Hlawka inequality, and since the star discrepancy D_N^* of the van der Corput sequence is of order $\log N$, that $\left|\sum_{n=0}^{N-1} f(x_n)\right| \ll \log N$ for any f with bounded total variation (see, for example, [1, 4, 6]).

*The first author is supported by the Australian Research Council under its Center of Excellence Program.

†The second author is supported by the Austrian Research Foundation (FWF), Project S9609 that is part of the Austrian National Research Network "Analytic Combinatorics and Probabilistic Number Theory".

It follows from a result of Hellekalek and Larcher [3, Theorem 2] that if F is the space of 1-periodic functions for which the first derivative is Lipschitz continuous, then we have

$$\left|\sum_{n=0}^{N-1} f(x_n)\right| \ll 1 \quad \forall f \in F.$$

(We remark that Hellekalek and Larcher also showed that the periodicity of the function is necessary.)

Recall that the Theorem of Rademacher states that every Lipschitz continuous function on an interval is continuously differentiable in almost every point of the interval (in the sense of Lebesgue measure). Hence for the functions considered in [3] we obtain (by using partial integration) that for $h \neq 0$ we have

$$\begin{aligned}\widehat{f}(h) &= \int_0^1 f(x)\mathrm{e}^{-2\pi \mathtt{i} hx}\mathrm{d}x \\ &= \frac{1}{2\pi \mathtt{i} h}\int_0^1 f'(x)\mathrm{e}^{-2\pi \mathtt{i} hx}\mathrm{d}x = \frac{1}{(2\pi \mathtt{i} h)^2}\int_0^1 f''(x)\mathrm{e}^{-2\pi \mathtt{i} hx}\mathrm{d}x,\end{aligned}$$

and therefore $|\widehat{f}(h)| \ll |h|^{-2}$.

This motivates the following

Definition 2 Let F_α be the class of all 1-periodic functions $f : \mathbb{R} \to \mathbb{R}$ with an absolutely convergent Fourier series expansion with $|\widehat{f}(h)| \ll |h|^{-\alpha}$ for all $h \neq 0$ and $\int_0^1 f(x)\mathrm{d}x = 0$.

In this paper we show that (1) holds for all functions $f \in F_\alpha$ whenever $\alpha > 1$. Further we show that there exists a function $g \in F_1$ such that $\left|\sum_{n=0}^{N-1} g(x_n)\right|$ is not bounded as N tends to infinity.

Moreover we consider functions $f \in F$ whose Fourier coefficients decrease with order $\prod_{j=0}^{L} \chi_j(h)$ where for $j \geq 1$, $\chi_j(h) := (\log_p \ldots \log_p h)^{-1}$, where we apply the $\log_p$ function j times on h, and for $j = 0$ we set $\chi_0(h) := h^{-1}$. Here $\log_p$ denotes the logarithm to the base p, $p \geq 2$.

The main results are stated in the subsequent Section 2 and the proofs are presented in Section 3 and Section 4.

Throughout the paper we assume that $f : [0,1] \to \mathbb{R}$ is a 1-periodic function with $\int_0^1 f(x)\mathrm{d}x = 0$ and $\sum_{h\in\mathbb{Z}} |\widehat{f}(h)| < \infty$.

2 The main results

Let the function class F_α be defined as in Section 1.

Theorem 1 *Let $(x_n)_{n\geq 0}$ be the van der Corput sequence in base p. For $\alpha > 1$ we have*

$$\left|\sum_{n=0}^{N-1} f(x_n)\right| \ll 1 \quad \forall f \in F_\alpha. \tag{2}$$

Remark 1 It can be shown that the above result remains true if the sequence $(x_n)_{n\geq 0}$ is a digital $(0,1)$-sequence over $\mathbb{Z}_p$, p a prime, which is generated by a non-singular upper triangular $\mathbb{N}\times\mathbb{N}$ matrix with entries from $\mathbb{Z}_p$ and with one entries in the main diagonal. (We call such sequences short NUT-sequence in base p.) For the definition of digital $(0,1)$-sequences over $\mathbb{Z}_p$ see [5] or [6].

Remark 2
1. Note that if f is a 1-periodic function whose first derivative satisfies a Hölder condition with coefficient $0<\lambda\leq 1$ then it follows from [8, Theorem 4.7] that the Fourier coefficients of f satisfy $|\widehat{f}(h)|\ll |h|^{-1-\lambda}$. Thus $f\in F_{1+\lambda}$ and the above theorem shows that the remainder of f is bounded.
2. Let F_{lin} be the class of all 1-periodic, continuous, piecewise linear functions $f:\mathbb{R}\to\mathbb{R}$ with $\int_0^1 f(x)\mathrm{d}x=0$. Then it can be easily shown that $\widehat{f}(h)\ll h^{-2}$ for any $f\in F_{\mathrm{lin}}$. Thus $F_{\mathrm{lin}}\subseteq F_2$ and Theorem 1 shows that any function in F_{lin} has bounded remainder.

We introduce some further notation. Let

$$\mathrm{supp}(\widehat{f})=\{h\in\mathbb{Z}:\widehat{f}(h)\neq 0\}$$

denote the support of the Fourier coefficients of f. Note that $\int_0^1 f(x)\mathrm{d}x=0$ implies that $0\notin\mathrm{supp}(\widehat{f})$. Further, as f is a real function it follows that $\widehat{f}(h)=\overline{\widehat{f}(-h)}$ and hence $h\in\mathrm{supp}(\widehat{f})$ implies that $-h\in\mathrm{supp}(\widehat{f})$. For a non-empty subset $E\subseteq\mathbb{Z}\setminus\{0\}$ we define $M(E)$ in the following way: if for every $M\in\mathbb{N}$ there is an $h\in E$ with $p^M|h$ we set $M(E)=\infty$, otherwise let $M(E)\geq 0$ denote the largest integer such that there exists an $h\in E$ with $p^M|h$. Further, for $p\geq 2$, $j\geq 0$ and $h\in\mathbb{Z}$ let

$$\delta_{p^j}(h):=\begin{cases}1 & \text{if } h\equiv 0 \pmod{p^j},\\ 0 & \text{if } h\not\equiv 0 \pmod{p^j}.\end{cases}$$

From the proof of Theorem 1 we obtain actually the following result:

Theorem 2 *Let $(x_n)_{n\geq 0}$ be the van der Corput sequence in base p. If the Fourier coefficients of f satisfy*

$$\sum_{j=0}^{M(\mathrm{supp}(\widehat{f}))} p^j \sum_{h\in\mathrm{supp}(\widehat{f})} |\widehat{f}(h)|\delta_{p^j}(h)<\infty,$$

then

$$\left|\sum_{n=0}^{N-1} f(x_n)\right|\ll 1.$$

Remark 3
1. It follows from Theorem 2 that if $M(\mathrm{supp}(\widehat{f}))<\infty$ then $\sum_{h\in\mathrm{supp}(\widehat{f})}|\widehat{f}(h)|<\infty$ is enough to ensure that the remainder of f is bounded.

2. It follows also from Theorem 2 that the remainder of f is bounded if $|\widehat{f}(h)| \le \varphi(|h|)$, for all $h \in \mathbb{Z}$ with $|h| \in D_c := \{k \in \mathbb{N} : k \ge c\}$, where $c \in \mathbb{N}$ and $\varphi : D_c \subseteq \mathbb{N} \to \mathbb{R}_0^+$ is a function such that
$$\sum_{\substack{n,k=0\\ p^n k \in D_c}}^{\infty} p^n \varphi(p^n k) < \infty.$$
Of course $g(x) = x^{-\alpha}$, $\alpha > 1$, fulfills this condition, but also
$$g(x) = \frac{1}{x(\log x)^{\beta}} \qquad \text{with } \beta > 2 \qquad \text{for } x > 1.$$

In the following we shall show that the result of Theorem 1 is not true anymore if $\alpha = 1$.

Theorem 3 *Let $(x_n)_{n\ge 0}$ be the van der Corput sequence in base p. Then there exists a function $g \in F_1$ such that*
$$\limsup_{N\to\infty} \left| \sum_{n=0}^{N-1} g(x_n) \right| = \infty.$$

The result of Theorem 2 can be improved, we have

Theorem 4 *Let $(x_n)_{n\ge 0}$ be the van der Corput sequence in base p. For every $L \in \mathbb{N}$ there exists a function $g \in F$ with*
$$|\widehat{g}(h)| \le C \prod_{j=0}^{L} \chi_j(|h|)$$
for all $h \in \mathbb{Z}$ with $|h| \ge c(L)$, where $c(L)$ is the smallest natural number such that $1/\chi_L(c(L)) > 1$, and where $C > 0$ is an absolute constant, such that
$$\limsup_{N\to\infty} \left| \sum_{n=0}^{N-1} g(x_n) \right| = \infty.$$

Note that using Remark 3, item 1, one can easily construct a function which satisfies the conditions of Theorem 3 and Theorem 4 but for which the remainder is bounded. Hence those two theorems cannot hold for all 1-periodic functions which satisfy the assumptions in those theorems.

3 Functions with bounded remainder

For a sequence $\sigma = (x_n)_{n\ge 0}$, $h \in \mathbb{Z}$ and $N \in \mathbb{N}$ we define
$$S_N(\sigma, h) := \sum_{n=0}^{N-1} \mathrm{e}^{2\pi \mathrm{i} h x_n}.$$

Lemma 1 *Let $\sigma = (x_n)_{n\geq 0}$ the van der Corput sequence in base p (or a NUT-sequence over $\mathbb{Z}_p$ if p is prime). Let $h \in \mathbb{Z}$ and let $N \in \mathbb{N}$ with p-adic representation*

$$N = \sum_{j=0}^{\infty} N_j p^j,$$

where $N_j \in \{0, 1, \ldots, p-1\}$ for $j \geq 0$. Then we have

$$|S_N(\sigma, h)| \leq \sum_{j=0}^{\infty} N_j p^j \delta_{p^j}(h).$$

Proof. For the van der Corput sequence this lemma is proved in [7, Lemma 3]. The proof for the NUT-sequence is similar and can be found in [2, Lemma 3.3]. □

Lemma 2 *Let $n \in \mathbb{N}_0$ and $\alpha > 1$. Then we have*

$$\sum_{h=1}^{\infty} \frac{1}{h^\alpha} \delta_{p^n}(h) = \frac{\zeta(\alpha)}{p^{\alpha n}},$$

where $\zeta(\alpha)$ denotes the Riemann zeta function and $\delta_{p^j}(h)$ is defined as in Section 2.

Proof. This is easy calculation,

$$\sum_{h=1}^{\infty} \frac{1}{h^\alpha} \delta_{p^n}(h) = \sum_{\substack{h=1 \\ h \equiv 0 \pmod{p^n}}}^{\infty} \frac{1}{h^\alpha} = \sum_{h=1}^{\infty} \frac{1}{(p^n h)^\alpha} = \frac{\zeta(\alpha)}{p^{\alpha n}}.$$

□

Now we can give the

Proof of Theorem 1. We have $f(x) = \sum_{h \in \mathbb{Z}} \widehat{f}(h) \mathrm{e}^{2\pi \mathrm{i} h x}$. Hence

$$\sum_{n=0}^{N-1} f(x_n) = \sum_{h \in \mathbb{Z}} \widehat{f}(h) S_N(\sigma, h).$$

Let now $N = a_1 p^{n_1} + a_2 p^{n_2} + \cdots + a_s p^{n_s}$ with $0 \leq n_1 < n_2 < \cdots < n_s$ and $a_j \in \{1, \ldots, p-1\}$. Then from Lemma 1 we obtain

$$|S_N(\sigma, h)| \leq \sum_{j=1}^{s} a_j p^{n_j} \delta_{p^{n_j}}(h) \leq (p-1) \sum_{j=1}^{s} p^{n_j} \delta_{p^{n_j}}(h).$$

Since $f \in F_\alpha$, $\alpha > 1$, there is a constant $C > 0$ such that $|\widehat{f}(h)| \leq C/|h|^\alpha$. Therefore we have

$$\begin{aligned}
\left|\sum_{n=0}^{N-1} f(x_n)\right| &\leq \sum_{h\in\mathbb{Z}} |\widehat{f}(h)| \cdot |S_N(\sigma, h)| \\
&\leq (p-1)C \sum_{\substack{h\in\mathbb{Z}\\h\neq 0}} \frac{1}{|h|^\alpha} \sum_{j=1}^{s} p^{n_j} \delta_{p^{n_j}}(h) \\
&= 2(p-1)C \sum_{j=1}^{s} p^{n_j} \sum_{h=1}^{\infty} \frac{1}{h^\alpha} \delta_{p^{n_j}}(h) \\
&= 2(p-1)C\zeta(\alpha) \sum_{j=1}^{s} p^{n_j} \frac{1}{p^{\alpha n_j}} \\
&\leq 2(p-1)C\zeta(\alpha) \sum_{n=0}^{\infty} \frac{1}{p^{(\alpha-1)n}} = 2(p-1)C\zeta(\alpha)\frac{p^{\alpha-1}}{p^{\alpha-1}-1}.
\end{aligned}$$

4 Functions with unbounded remainder

We first give the

Proof of Theorem 3. Let

$$g(x) = \sum_{\substack{r=1\\r \text{ odd}}}^{\infty} p^{-r} \sin(2\pi p^r x), \quad x \in [0, 1).$$

First note that the sum converges absolutely and the function g is bounded, as

$$|g(x)| \leq \sum_{\substack{r=1\\r \text{ odd}}}^{\infty} p^{-r} |\sin(2\pi p^r x)| \leq \sum_{\substack{r=1\\r \text{ odd}}}^{\infty} p^{-r} = \frac{p}{p^2-1}.$$

Obviously the function g is periodic with period 1, $\int_0^1 g(x)\mathrm{d}x = 0$ and the Fourier coefficients satisfy $|\widehat{g}(h)| \ll |h|^{-1}$. Hence $g \in F_1$ but $g \notin F_\alpha$ for any $\alpha > 1$.

For $\eta \in \mathbb{N}$ let now $N = N(\eta) = p^{2\eta} + p^{2\eta-2} + p^{2\eta-4} + \cdots + 1$. We consider now

$$\sum_{n=0}^{N-1} g(x_n) = \sum_{\substack{r=1\\r \text{ odd}}}^{\infty} p^{-r} \sum_{n=0}^{N-1} \sin(2\pi p^r x_n).$$

We divide the above some in several parts, namely, for $0 \leq \nu < \eta$ we have

$$\begin{aligned}
\sum_{n=p^{2\eta}+\cdots+p^{2\eta-2\nu}}^{p^{2\eta}+\cdots+p^{2\eta-2(\nu+1)}-1} g(x_n) &= \sum_{\substack{r=1\\r \text{ odd}}}^{\infty} p^{-r} \sum_{n=p^{2\eta}+\cdots+p^{2\eta-2\nu}}^{p^{2\eta}+\cdots+p^{2\eta-2(\nu+1)}-1} \sin(2\pi p^r x_n) \\
&= \sum_{\substack{r=1\\r \text{ odd}}}^{\infty} p^{-r} \sum_{n=0}^{p^{2\eta-2(\nu+1)}-1} \sin(2\pi p^r (np^{-2\eta+2(\nu+1)} + p^{-2\eta+2\nu} + \cdots + p^{-2\eta})).
\end{aligned}$$

We now simplify the last sum. First observe that for $r \geq 2\eta$ we have

$$\sin(2\pi(np^{r-2\eta+2(\nu+1)} + p^{r-2\eta+2\nu} + \cdots + p^{r-2\eta})) = 0$$

for arbitrary $0 \leq n < p^{2\eta-2(\nu+1)}$. Let now $r < 2\eta - 2(\nu+1)$. Then we have $r - 2\eta + 2(\nu+1) < 0$ and hence

$$\begin{aligned}
&\sum_{n=0}^{p^{2\eta-2(\nu+1)}-1} \mathrm{e}^{2\pi\mathtt{i}(np^{r-2\eta+2(\nu+1)}+p^{r-2\eta+2\nu}+\cdots+p^{r-2\eta})} \\
&= \mathrm{e}^{2\pi\mathtt{i}(p^{r-2\eta+2\nu}+\cdots+p^{r-2\eta})} \sum_{n=0}^{p^{2\eta-2(\nu+1)}-1} (\mathrm{e}^{2\pi\mathtt{i}p^{r-2\eta+2(\nu+1)}})^n \\
&= 0.
\end{aligned}$$

As $\sin(\pi x) = (2\mathtt{i})^{-1}(\mathrm{e}^{\mathtt{i}\pi x} - \mathrm{e}^{-\mathtt{i}\pi x})$ we have in this case

$$\sum_{n=0}^{p^{2\eta-2(\nu+1)}-1} \sin(2\pi(np^{r-2\eta+2(\nu+1)} + p^{r-2\eta+2\nu} + \cdots + p^{r-2\eta})) = 0.$$

Note that from the above it also follows that

$$\sum_{n=0}^{2^\eta-1} g(x_n) = 0.$$

Thus we only need to consider odd r which satisfy $2\eta - 2(\nu+1) \leq r < 2\eta$. Note that in this case $r - 2\eta + 2(\nu+1) \geq 1$ and hence

$$\sin(2\pi(np^{r-2\eta+2(\nu+1)} + p^{r-2\eta+2\nu} + \cdots + p^{r-2\eta})) = \sin(2\pi(p^{r-2\eta+2\nu} + \cdots + p^{r-2\eta})).$$

Altogether we now obtain

$$\begin{aligned}
\sum_{n=p^{2\eta}+\cdots+p^{2\eta-2\nu}}^{p^{2\eta}+\cdots+p^{2\eta-2(\nu+1)}-1} g(x_n) &= \sum_{\substack{r=2\eta-2\nu-1 \\ r \text{ odd}}}^{2\eta-1} p^{-r} p^{2\eta-2(\nu+1)} \sin(2\pi(p^{r-2\eta+2\nu} + \cdots + p^{r-2\eta})) \\
&= \sum_{\substack{r=1 \\ r \text{ odd}}}^{2\nu+1} p^{-r} \sin(2\pi(p^{r-2} + \cdots + p^{r-2(\nu+1)})).
\end{aligned}$$

Thus we have

$$\begin{aligned}
\sum_{n=0}^{N(\eta)-1} g(x_n) &= \sum_{\nu=0}^{\eta-1} \sum_{\substack{r=1 \\ r \text{ odd}}}^{2\nu+1} p^{-r} \sin(2\pi(p^{r-2} + p^{r-4} + \cdots + p^{r-2(\nu+1)})) \\
&= \sum_{\nu=0}^{\eta-1} \sum_{\substack{r=1 \\ r \text{ odd}}}^{\infty} p^{-r} \sin(2\pi(p^{r-2} + p^{r-4} + \cdots + p^{r-2(\nu+1)})) \\
&= \sum_{\substack{r=1 \\ r \text{ odd}}}^{\infty} p^{-r} \sum_{\nu=0}^{\eta-1} \sin(2\pi(p^{r-2} + p^{r-4} + \cdots + p^{r-2(\nu+1)})) \\
&= \sum_{\substack{r=1 \\ r \text{ odd}}}^{2\eta-1} p^{-r} \sum_{\nu=(r-1)/2}^{\eta-1} \sin(2\pi(p^{r-2} + p^{r-4} + \cdots + p^{r-2(\nu+1)})).
\end{aligned}$$

We now estimate the value of the sine function. As $1 \leq r \leq 2\eta - 1$ is odd, we have

$$\sin(2\pi(p^{r-2} + p^{r-4} + \cdots + p^{r-2(\nu+1)})) \geq \sin(2\pi(p^{-1} + p^{-3} + \cdots)) = \sin(2\pi p/(p^2 - 1))$$

for $p = 2, 3, 4$ and

$$\sin(2\pi(p^{r-2} + p^{r-4} + \cdots + p^{r-2(\nu+1)})) \geq \sin(2\pi/p)$$

for $p \geq 5$. Let $c_p = \sin(2\pi p/(p^2 - 1))$ for $p = 2, 3, 4$ and $c_p = \sin(2\pi/p)$ for $p \geq 5$. Then we have

$$\sum_{n=0}^{N(\eta)-1} g(x_n) \geq c_p \sum_{\substack{r=1 \\ r \text{ odd}}}^{2\eta-1} p^{-r}(\eta - (r-1)/2) = \eta \frac{c_p p}{p^2 - 1} - \frac{c_p p(1 - p^{-2\eta})}{(p^2 - 1)^2}.$$

Thus we have

$$\lim_{\eta \to \infty} \sum_{n=0}^{N(\eta)-1} g(x_n) = \infty$$

and therefore $\left|\sum_{n=0}^{N-1} g(x_n)\right|$ cannot be bounded independently of N. This proves Theorem 3. $\square$

Remark 4 It follows from the above proof that for every $p \geq 2$ there are infinitely many $N \in \mathbb{N}$ such that

$$\sum_{n=0}^{N-1} g(x_n) \geq \log_p N \frac{c_p p}{2(p^2 - 1)} + \frac{c_p p}{2(p^2 - 1)^2}((p^2 - 1)\log_p(1 - p^{-2}) - (1 - N^{-1})/2).$$

The above counterexample can be improved to yield a stronger result. This is done in the following.

Proof of Theorem 4. Let $\boldsymbol{\gamma} = (\gamma_1, \gamma_3, \gamma_5, \ldots)$ be a sequence of non-negative real numbers. Then we set

$$g_{\boldsymbol{\gamma}}(x) = \sum_{\substack{r=1 \\ r \text{ odd}}}^{\infty} p^{-r} \gamma_r \sin(2\pi p^r x), \quad x \in [0, 1).$$

By proceeding the same way as above we obtain

$$\sum_{n=0}^{N(\eta)-1} g_{\boldsymbol{\gamma}}(x_n) \geq c_p \sum_{\substack{r=1 \\ r \text{ odd}}}^{2\eta-1} p^{-r} \sum_{\nu=(r-1)/2}^{\eta-1} \gamma_{2\eta-2(\nu+1)+r} \geq \frac{c_p}{p} \sum_{\nu=1}^{\eta} \gamma_{2\nu-1}.$$

From this it follows that if $\sum_{\nu=1}^{\infty} \gamma_{2\nu-1}$ is not bounded then $\sum_{n=0}^{N(\eta)-1} g_{\boldsymbol{\gamma}}(x_n)$ cannot be bounded independently of N.

The Fourier series representation of $g_{\boldsymbol{\gamma}}$ is given by

$$g_{\boldsymbol{\gamma}}(x) = \sum_{h \in \mathbb{Z}} \widehat{g}_{\boldsymbol{\gamma}}(h) \mathrm{e}^{2\pi \mathrm{i} h x},$$

where

$$\widehat{g}_{\boldsymbol{\gamma}}(h) = \begin{cases} -\frac{\mathtt{i}}{2h}\gamma_r & \text{if } h = p^r, r > 0 \text{ odd}, \\ -\frac{\mathtt{i}}{2h}\gamma_r & \text{if } h = -p^r, r > 0 \text{ odd}, \\ 0 & \text{otherwise}. \end{cases}$$

For $|h| = p^r$, $r > 0$ odd, we have $r = \log_p |h|$. By choosing for example $\gamma_\nu = \nu^{-1}$ for odd ν we see that $\sum_{n=0}^{N-1} g_{\boldsymbol{\gamma}}(x_n)$ cannot be bounded independently of N. In this case we have

$$\widehat{g}_{\boldsymbol{\gamma}}(h) = -\frac{\mathtt{i}}{2h \log_p |h|}$$

for infinitely many h.

Hence we have just shown that in the space of functions f for which

$$|\widehat{f}(h)| \ll \frac{1}{|h| \log_p |h|}$$

there is a function g for which $\left|\sum_{n=0}^{N-1} g(x_n)\right|$ cannot be bounded independently of N.

For $j \geq 0$ let $\chi_j(h)$ be defined as in Section 1. Then by using a similar argument as above we obtain that, for every $L \in \mathbb{N}$, the function space for which

$$|\widehat{f}(h)| \leq C \prod_{j=0}^{L} \chi_j(|h|) \quad \text{for all } h \in \mathbb{Z} \text{ with } |h| \geq c(L),$$

where $c(L)$ is the smallest natural number such that $\chi_L^{-1}(c(L)) > 1$, there is a function g for which $\left|\sum_{n=0}^{N-1} g(x_n)\right|$ cannot be bounded independently of N. This proves Theorem 4.□

Remark 5 For example it follows from the above proof, that in the space of functions for which $|\widehat{f}(h)| \ll (|h| \log_p |h|)^{-1}$ there is a function g and a constant $C_p > 0$ for which

$$\left|\sum_{n=0}^{N-1} g(x_n)\right| \geq C_p \log_p \log_p N$$

for infinitely many values of $N \in \mathbb{N}$. More generally, for every $L \in \mathbb{N}$ the space of functions for which $|\widehat{f}(h)| \ll \prod_{j=0}^{L-1} \chi_j(|h|)$, for all h such that $\chi_{L-1}^{-1}(|h|) > 1$, contains a function g such that there is a constant $C_{p,L} > 0$ with

$$\left|\sum_{n=0}^{N-1} g(x_n)\right| \geq C_{p,L} \chi_L^{-1}(N)$$

for infinitely many values of $N \in \mathbb{N}$.

Acknowledgement: The authors would like to thank Johannes Schoissengeier for making us aware of the initial question stated in Section 1. Further we thank Gerhard Larcher for valuable discussions and suggestions.

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