Strong tractability of multivariate integration of arbitrary high order using digitally shifted polynomial lattice rules

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Dedicated to Henryk Woźniakowski on the occasion of his 60th birthday.

Abstract

In this paper we proof the existence of digitally shifted polynomial lattice rules which achieve strong tractability results for Sobolev spaces of arbitrary high smoothness. The convergence rate is shown to be best possible up to a given degree of smoothness of the integrand. Indeed we even show the existence of polynomial lattice rules which automatically adjust themselves to the smoothness of the integrand up to a certain given degree.

Further we show that strong tractability under certain conditions on the weights can be obtained and that polynomial lattice rules exist for which the worst-case error can be bounded independently of the dimension. These results hold independent of the smoothness.

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1 Introduction

We are interested in the study of multivariate integration, more precisely, we want to approximate the s-dimensional integral $\int_{[0,1)^s} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}$ by a quadrature rule. This is done by calculating the average of the values $f(\boldsymbol{x}_h)$ for a point set $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{n-1}\}$. For Monte Carlo rules the point set is chosen randomly, whereas for quasi-Monte Carlo (QMC) rules the point set is chosen deterministically with the aim to obtain quadrature points which can fully exploit the smoothness of the integrand.

There are two major branches with respect to QMC rules. Lattice rules, which are due to Korobov [10] and Hlawka [9], are designed for the integration of high dimensional periodic functions. There they can, under suitable conditions on the integrand, achieve arbitrary high convergence rates (even exponential convergence) [14]. Though it has been shown that randomly shifted lattice rules can achieve a convergence of $N^{-1+\varepsilon}$ for any $\varepsilon > 0$, and if one also applies the Baker's transformation a convergence of $N^{-2+\varepsilon}$ for all $\varepsilon > 0$, see [8] and the references therein, for non-periodic functions higher convergence rates for smoother functions are not known.

Digital nets on the other hand have only been known to achieve a convergence of $N^{-1+\varepsilon}$ for all $\varepsilon > 0$ for functions with bounded variation [14]. In [3,4] these results have been extended to yield explicit constructions of generalized digital nets which can achieve arbitrary high convergence rates under suitable conditions on the integrands. The analysis in [3,4] is based on Walsh functions, in particular the behaviour of the Walsh coefficients of the reproducing kernel [3] and, in general, of smooth functions [4] was analyzed and used to obtain explicit constructions of generalized digital nets.

In this paper we use the insights obtained from [3,4] to also generalize polynomial lattice rules. Polynomial lattices first introduced in [13], which are the quadrature points used in a polynomial lattice rule, are very similar to lattice rules and have been shown to achieve the optimal rate of convergence for integration in Sobolev spaces with partial mixed derivatives up to order one square integrable [5]. In this paper we give the correct generalization of polynomial lattices which also achieve the optimal rate of convergence for Sobolev spaces with higher order mixed partial derivatives. Indeed we can even show the existence of polynomial lattice rules which automatically adjust themselves to the smoothness of the integrand in terms of the convergence of the integration error within a certain (arbitrary high) range. Note that an analogous result for lattice rules is not known, hence for the time being polynomial lattice rules have an upper hand for the integration of non-periodic smooth functions.

Strong tractability roughly means that the worst-case error in a sequence of spaces of increasing dimension goes to zero independently of the dimension. In [6] digital nets and in [5] polynomial lattice rules have already been shown to achieve strong tractability results in Sobolev spaces with partial mixed derivatives up to order one square integrable. Here we extend these results for higher order Sobolev spaces by showing the existence of polynomial lattice rules which also achieve strong tractability results in this case.

In the following section we generalize the classical definitions of digital nets and polynomial lattice rules. In Section 3 we briefly introduce Walsh functions and in Section 4 we consider numerical integration in Sobolev spaces. Section 5 finally deals with (strong) tractability.

2 Digital nets and polynomial lattice rules for arbitrary smooth functions

In this section we introduce digital nets and polynomial lattice rules which can achieve arbitrary high convergence rates of the integration error for suitably smooth functions, see [3,4]. This is achieved by a slight generalization of the classical definition of digital nets, see [12–14], and [15] for a very recent survey article on digital nets. The following generalization appeared first in [4].

Definition 2.1 (Digital nets) Let b be a prime and let $s \geq 1$ and $m, n \geq 1$ be integers. Let $C_1, \ldots, C_s$ be $n \times m$ matrices over the finite field $\mathbb{Z}_b$. We construct b^m points in $[0,1)^s$ in the following way: for $0 \leq h < b^m$ let $h = h_0 + h_1 b + \cdots + h_{m-1} b^{m-1}$ be the b-adic expansion of h. Identify h with the vector $\vec{h} = (h_0, \ldots, h_{m-1})^\top \in \mathbb{Z}_b^m$, where $\top$ means the transpose of the vector. For $1 \leq j \leq s$ multiply the matrix C_j by $\vec{h}$, i.e.,

$$C_j \vec{h} =: (y_{j,1}(h), \ldots, y_{j,n}(h))^\top \in \mathbb{Z}_b^n,$$

and set

$$x_{h,j} := \frac{y_{j,1}(h)}{b} + \cdots + \frac{y_{j,n}(h)}{b^n}.$$

We call the point set

$$\{\boldsymbol{x}_h = (x_{h,1}, \ldots, x_{h,s}) : 0 \leq h < b^m\}$$

a digital net over $\mathbb{Z}_b$, or shortly a digital net. The matrices $C_1, \ldots, C_s$ are called the generating matrices of the digital net.

In [13] (see also [14, Section 4.4]) Niederreiter introduced a special family of digital nets over $\mathbb{Z}_b$. Those nets are obtained from rational functions over finite fields. For a prime b let $\mathbb{Z}_b((x^{-1}))$ be the field of formal Laurent series over $\mathbb{Z}_b$.

Elements of $\mathbb{Z}_b((x^{-1}))$ are formal Laurent series,

$$L = \sum_{l=w}^{\infty} t_l x^{-l},$$

where w is an arbitrary integer and all $t_l \in \mathbb{Z}_b$. Note that $\mathbb{Z}_b((x^{-1}))$ contains the field of rational functions over $\mathbb{Z}_b$ as a subfield. Further let $\mathbb{Z}_b[x]$ be the set of all polynomials over $\mathbb{Z}_b$.

The following definition is a slight generalization of the definition from [13], see also [14]. As we will see later, polynomial lattice rules as defined below can achieve arbitrary high convergence rates and the generalization is based on results in [3,4].

Definition 2.2 (Polynomial lattice rules) Let b be prime and $1 \le m \le n$. Let υ_n be the map from $\mathbb{Z}_b((x^{-1}))$ to the interval $[0,1)$ defined by

$$\upsilon_n\left(\sum_{l=w}^{\infty} t_l x^{-l}\right) = \sum_{l=\max(1,w)}^{n} t_l b^{-l}.$$

For a given dimension $s \ge 1$, choose $p \in \mathbb{Z}_b[x]$ with $\deg(p) = n \ge 1$ and let $q_1, \ldots, q_s \in \mathbb{Z}_b[x]$. For $0 \le h < b^m$ let $h = h_0 + h_1 b + \cdots + h_{m-1} b^{m-1}$ be the b-adic expansion of h. With each such h we associate the polynomial

$$h(x) = \sum_{r=0}^{m-1} h_r x^r \in \mathbb{Z}_b[x].$$

Then $\mathcal{S}_{p,m,n}(\boldsymbol{q})$ is the point set consisting of the b^m points

$$\boldsymbol{x}_h = \left(\upsilon_n\left(\frac{h(x)q_1(x)}{p(x)}\right), \ldots, \upsilon_n\left(\frac{h(x)q_s(x)}{p(x)}\right)\right) \in [0,1)^s,$$

for $0 \le h < b^m$. A quasi-Monte Carlo rule using the point set $\mathcal{S}_{p,m,n}(\boldsymbol{q})$ is called a polynomial lattice rule.

Remark 2.3 The point set $\mathcal{S}_{p,m,n}(\boldsymbol{q})$ consists of the first b^m points of $\mathcal{S}_{p,n,n}(\boldsymbol{q})$, i.e., the first b^m points of a classical polynomial lattice. Hence the definition of a polynomial lattice in [13] is covered by choosing $n = m$ in the definition above.

Using similar arguments as for the classical case $n = m$, see [13,14], it can be shown that the point set $\mathcal{S}_{p,m,n}(\boldsymbol{q})$ is a digital net in the sense of Definition 2.1. The generating matrices $C_1, \ldots, C_s$ of this digital net can be obtained in the following way: for $1 \le j \le s$, consider the expansions

$$\frac{q_j(x)}{p(x)} = \sum_{l=w_j}^{\infty} u_l^{(j)} x^{-l} \in \mathbb{Z}_b((x^{-1}))$$

where $w_j \in \mathbb{Z}$. Then the elements $c_{i,r}^{(j)}$ of the $n \times m$ matrix C_j over $\mathbb{Z}_b$ are given by

$$c_{i,r}^{(j)} = u_{r+i}^{(j)} \in \mathbb{Z}_b, \tag{2.1}$$

for $1 \le j \le s$, $1 \le i \le n$, $0 \le r \le m-1$.

Let $x = \sum_{i=1}^{\infty} \frac{x_i}{b^i} \in [0,1)$ and let $\sigma = \sum_{i=1}^{\infty} \frac{\sigma_i}{b^i} \in [0,1)$, where $x_i, \sigma_i \in \{0,\dots,b-1\}$. We define the digital b-adic shifted point y by

$$y = x \oplus \sigma = \sum_{i=1}^{\infty} \tfrac{y_i}{b^i},$$

where $y_i = x_i + \sigma_i \in \mathbb{Z}_b$. For points $\boldsymbol{x} \in [0,1)^s$ and $\boldsymbol{\sigma} \in [0,1)^s$ the digital b-adic shift $\boldsymbol{x} \oplus \boldsymbol{\sigma}$ is defined component wise.

Definition 2.4 (Shifted digital nets and polynomial lattice rules) A digital net for which all points are digitally shifted by the same $\boldsymbol{\sigma} \in [0,1)^s$ is called a digitally shifted digital net or simply shifted digital net and a polynomial lattice rule for which the underlying quadrature points are digitally shifted by the same $\boldsymbol{\sigma} \in [0,1)^s$ is called a digitally shifted polynomial lattice rule or simply a shifted polynomial lattice rule.

Finally we introduce some notation: for arbitrary $\boldsymbol{k} = (k_1,\dots,k_s) \in \mathbb{Z}_b[x]^s$ and $\boldsymbol{q} = (q_1,\dots,q_s) \in \mathbb{Z}_b[x]^s$, we define the 'inner product'

$$\boldsymbol{k} \cdot \boldsymbol{q} = \sum_{j=1}^{s} k_j q_j \in \mathbb{Z}_b[x]$$

and we write $q \equiv 0 \pmod{p}$ if p divides q in $\mathbb{Z}_b[x]$. Further, for b prime we associate a non-negative integer $k = \kappa_0 + \kappa_1 b + \cdots + \kappa_a b^a$ with the polynomial $k(x) = \kappa_0 + \kappa_1 x + \cdots + \kappa_a x^a \in \mathbb{Z}_b[x]$ and vice versa.

3 Walsh functions

We recall the definition of Walsh functions. Henceforth let $\mathbb{N}_0$ denote the set of non-negative integers. We have the following definitions.

Definition 3.1 (Walsh functions) Let $b \ge 2$ be an integer. For a non-negative integer k with base b representation

$$k = \kappa_0 + \kappa_1 b + \cdots + \kappa_a b^a,$$

with $\kappa_i \in \{0, \ldots, b-1\}$, we define the Walsh function ${}_b\mathrm{wal}_k : [0,1) \longrightarrow \mathbb{C}$ by

$$ {}_b\mathrm{wal}_k(x) := \mathrm{e}^{2\pi \mathtt{i}(x_1\kappa_0 + \cdots + x_{a+1}\kappa_a)/b}, $$

for $x \in [0,1)$ with base b representation $x = \frac{x_1}{b} + \frac{x_2}{b^2} + \cdots$ (unique in the sense that infinitely many of the x_i must be different from $b-1$). If it is clear which base b is chosen we will simply write wal_k.

Definition 3.2 (Multivariate Walsh functions) Let $b \geq 2$ be an integer. For dimension $s \geq 2$, $x_1, \ldots, x_s \in [0,1)$ and $k_1, \ldots, k_s \in \mathbb{N}_0$ we define ${}_b\mathrm{wal}_{k_1,\ldots,k_s} : [0,1)^s \longrightarrow \mathbb{C}$ by

$$ {}_b\mathrm{wal}_{k_1,\ldots,k_s}(x_1, \ldots, x_s) := \prod_{j=1}^{s} {}_b\mathrm{wal}_{k_j}(x_j). $$

For vectors $\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ and $\boldsymbol{x} = (x_1, \ldots, x_s) \in [0,1)^s$ we write

$$ {}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) := {}_b\mathrm{wal}_{k_1,\ldots,k_s}(x_1, \ldots, x_s). $$

Again, if it is clear which base we mean we simply write $\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})$.

It is clear from the definitions that Walsh functions are piecewise constant. It can be shown that for any integers $s \geq 1$ and $b \geq 2$ the system $\{{}_b\mathrm{wal}_{k_1,\ldots,k_s} : k_1, \ldots, k_s \geq 0\}$ is a complete orthonormal system in $L_2([0,1)^s)$, see for example [2,11]. More information on Walsh functions can be found for example in [2,4,7,19].

We note that if Walsh functions, digital shifts, digital nets or polynomial lattice rules are used in conjunction with each other they are always in the same base b. Therefore we will often omit the b.

4 Numerical integration in Sobolev spaces

We consider the Sobolev space $\mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}$ for which $s \geq 1$ and $\alpha \geq 1$. For the 1 dimensional case the inner product is given by

$$ \begin{aligned} \langle f, g \rangle_{\mathcal{H}_{1,\alpha,(\gamma)}} = & \int_0^1 f(x) \,\mathrm{d}x \int_0^1 g(x) \,\mathrm{d}x + \gamma^{-1} \sum_{\tau=1}^{\alpha-1} \int_0^1 f^{(\tau)}(x) \,\mathrm{d}x \int_0^1 g^{(\tau)}(x) \,\mathrm{d}x \\ & + \gamma^{-1} \int_0^1 f^{(\alpha)}(x) g^{(\alpha)}(x) \,\mathrm{d}x, \end{aligned} \tag{4.1} $$

where $f^{(\tau)}$ denotes the τ-th derivative of f, $f^{(0)} = f$ and $\gamma > 0$ denotes the weight (see [18]). The corresponding norm in $\mathcal{H}_{1,\alpha,(\gamma)}$ is given by $\|f\|_{\mathcal{H}_{1,\alpha,(1)}} = \langle f, f \rangle_{\mathcal{H}_{1,\alpha,(\gamma)}}^{1/2}$.

The reproducing kernel (see [1] for more information about reproducing kernels) for this space is given by

$$\mathcal{K}_{1,\alpha,(1)}(x,y) = \sum_{\tau=0}^{\alpha} \frac{B_\tau(x)B_\tau(y)}{(\tau!)^2} + (-1)^{\alpha+1}\frac{B_{2\alpha}(|x-y|)}{(2\alpha)!},$$

where B_τ denotes the Bernoulli polynomial of degree τ. For example we have $B_0(x) = 1$, $B_1(x) = x - 1/2$, $B_2(x) = x^2 - x + 1/6$ and so on.

The reproducing kernel for the s dimensional weighted Sobolev space $\mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}$ is now given by

$$\mathcal{K}_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{x},\boldsymbol{y}) = \sum_{u\subseteq S} \gamma_u \prod_{j\in u}\left(\sum_{\tau=1}^{\alpha} \frac{B_\tau(x_j)B_\tau(y_j)}{(\tau!)^2} + (-1)^{\alpha+1}\frac{B_{2\alpha}(|x_j-y_j|)}{(2\alpha)!}\right).$$

For example if $\gamma_u = \prod_{j\in u}\gamma_j$ then the space $\mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}$ is a tensor product space of weighted one dimensional spaces. The inner product in this space is now the s-fold product of (4.1) and the corresponding norm in $\mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}$ is given by $\|f\|_{\mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}} = \langle f, f\rangle_{\mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}}$.

Note that numerical integration in the Sobolev space with $\alpha = 1$ using digital nets and polynomial lattice rules has already been considered in [5,6,16].

As $\mathcal{K}_{s,\alpha,\boldsymbol{\gamma}} \in \mathcal{L}_2([0,1)^{2s})$ it follows that $\mathcal{K}_{s,\alpha,\boldsymbol{\gamma}}$ can be represented by a Walsh series, i.e., we have

$$\mathcal{K}_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{x},\boldsymbol{y}) = \sum_{\boldsymbol{k},\boldsymbol{l}\in\mathbb{N}_0^s} \widehat{\mathcal{K}}_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k},\boldsymbol{l})\ \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})\overline{\mathrm{wal}_{\boldsymbol{l}}(\boldsymbol{y})},$$

where

$$\widehat{\mathcal{K}}_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k},\boldsymbol{l}) = \int_{[0,1)^{2s}} \mathcal{K}_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{x},\boldsymbol{y})\overline{\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})}\mathrm{wal}_{\boldsymbol{l}}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}.$$

Note that if $\gamma_u = 0$ for some u, then $\widehat{\mathcal{K}}_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k},\boldsymbol{l}) = 0$ if $k_j = l_j = 0$ for $j \notin u$ and $k_j, l_j \neq 0$ for $j \in u$.

For any $\alpha \geq 1$ there exists a constant $C_{b,\alpha} > 0$ independent of $\boldsymbol{k} \in \mathbb{N}_0^s$ such that

$$\left|\widehat{\mathcal{K}}_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k},\boldsymbol{k})\right| \leq C_{b,\alpha} r_{b,\alpha}^2(\boldsymbol{k}) \quad \text{for all } \boldsymbol{k} \in \mathbb{N}_0^s \tag{4.2}$$

where $r_{b,\alpha}(\boldsymbol{k}) = \prod_{j=1}^s r_{b,\alpha}(k_j)$ and $r_{b,\alpha}(0) = 1$ and for $k = \kappa_1 b^{a_1-1} + \cdots + \kappa_v b^{a_v-1}$ with $v \geq 1$, $0 < a_v < \cdots < a_1$ and $\kappa_i \in \{1,\ldots,b-1\}$ we set $r_{b,\alpha}(k) = b^{-a_1-\cdots-a_{\min(v,\alpha)}}$. For $\alpha = 1$ this follows from [6, Section 6] and for $\alpha > 1$ from [3,4].

In the following we consider the worst-case error for multivariate integration in the Sobolev space $\mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}$ for $s \geq 1$ and $\alpha \geq 1$, i.e.,

$$e(Q_{b^m,s}, \mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}) = \sup_{\substack{f \in \mathcal{H}_{s,\alpha,\boldsymbol{\gamma}} \\ \|f\|_{\mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}} \leq 1}} |I_s(f) - Q_{b^m,s}(f)| \,.$$

The initial error is given by

$$e(Q_{0,s}, \mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}) = \sup_{\substack{f \in \mathcal{H}_{s,\alpha,\boldsymbol{\gamma}} \\ \|f\|_{\mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}} \leq 1}} |I_s(f)| \,.$$

From [4, Theorem 15] we know that

$$e^2(Q_{0,s}, \mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}) = \gamma_\emptyset$$

and

$$\begin{aligned} e^2(Q_{b^m,s}, \mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}) &= -\gamma_\emptyset + \frac{1}{b^{2m}} \sum_{h,h'=0}^{b^m-1} \mathcal{K}_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{x}_h, \boldsymbol{x}_{h'}) \\ &= \frac{1}{b^{2m}} \sum_{h,h'=0}^{b^m-1} (\mathcal{K}_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{x}_h, \boldsymbol{x}_{h'}) - \gamma_\emptyset). \end{aligned}$$

For a digital net which has generating matrices $C_1, \ldots, C_s \in \mathbb{Z}_b^{n \times m}$ let $\mathcal{D} = \mathcal{D}(C_1, \ldots, C_s) = \{\boldsymbol{k} \in \mathbb{N}_0^s \setminus \{\boldsymbol{0}\} : C_1^\top \vec{k}_1 + \cdots + C_s^\top \vec{k}_s \equiv 0\}$, where $\boldsymbol{k} = (k_1, \ldots, k_s)$ and for $k_j = \kappa_0 + \kappa_1 b + \cdots$ let $\vec{k}_j = (\kappa_0, \ldots, \kappa_{n-1})^\top$. For $u \subseteq S$ let $\mathcal{D}_u = \mathcal{D}_u((C_j)_{j \in u})$ be the projection of the vectors in $\mathcal{D}$ to the coordinates in u and let $\mathcal{D}_u^* = \mathcal{D}_u^*((C_j)_{j \in u}) = \mathcal{D}_u \cap \mathbb{N}^{|u|}$.

Using the same arguments as in [6, Section 6] we can now obtain a formula for the mean square worst-case error $\widehat{e}^2(Q_{b^m,s}, \mathcal{H}_{s,\alpha,\boldsymbol{\gamma}})$ of randomly shifted digital nets (see Definition 2.4) where the expectation value of the square worst-case error is taken over all random i.i.d. $\boldsymbol{\sigma} \in [0,1)^s$, i.e.,

$$\widehat{e}^2(Q_{b^m,s}, \mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}) = \mathbb{E}_{\boldsymbol{\sigma}} \left[e^2(Q_{b^m,s}(\boldsymbol{\sigma}), \mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}) \right],$$

where $Q_{b^m,s}(\boldsymbol{\sigma})$ denotes the quadrature rule for which all quadrature points are digitally shifted by $\boldsymbol{\sigma} \in [0,1)^s$. Using [4, Theorem 15] together with the results from [6, Section 6] we obtain that for any $\alpha \geq 1$ we have

$$\widehat{e}^2(Q_{b^m,s}, \mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}) = \sum_{\emptyset \neq u \subseteq S} \gamma_u \sum_{\boldsymbol{k}_u \in \mathcal{D}_u^*} \widehat{\mathcal{K}}_{s,\alpha,\boldsymbol{\gamma}}((\boldsymbol{k}_u, \boldsymbol{0}), (\boldsymbol{k}_u, \boldsymbol{0}))$$

and by applying (4.2) we obtain for any $\alpha \geq 1$ that

$$\widehat{e}^2(Q_{b^m,s}, \mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}) \leq \sum_{\emptyset \neq u \subseteq S} \gamma_u C_{b,\alpha}^{|u|} \sum_{\boldsymbol{k}_u \in \mathcal{D}_u^*} r_{b,\alpha}^2(\boldsymbol{k}_u). \tag{4.3}$$

Compare this result with its deterministic version [4, Lemma 9].

The subsequent lemma now states a similar result for polynomial lattice rules. From a slight generalization of [13, Lemma 4.40] we obtain that the analogous definition of the dual space for a polynomial lattice is given by

$$\mathcal{D} = \mathcal{D}_p(\boldsymbol{q}) = \{\boldsymbol{k} \in \mathbb{N}_0^s \setminus \{\boldsymbol{0}\} : \boldsymbol{q} \cdot \bar{\boldsymbol{k}} \equiv a \pmod{p} \text{ with } \deg(a) < n - m\},$$

where for $\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ we associate the vector of polynomials $\bar{\boldsymbol{k}} = (\bar{k}_1, \ldots, \bar{k}_s)^\top$ where for $k_j = \kappa_0 + \kappa_1 b + \cdots$ we define $\bar{k}_j(x) = \kappa_0 + \kappa_1 x + \cdots + \kappa_{n-1} x^{n-1}$ and where we set $\deg(0) = -1$. Hence for $m = n$ we obtain the usual definition of the dual space, see [5,14], and for $m < n$ we obtain a superset. As above, for any $u \subseteq S$, we also define the projections of the vectors in $\mathcal{D}$ to the coordinates in u by $\mathcal{D}_u = \mathcal{D}_{u,p}(\boldsymbol{q})$ and further we set $\mathcal{D}_u^* = \mathcal{D}_{u,p}^*(\boldsymbol{q}) = \mathcal{D}_u \cap \mathbb{N}^{|u|}$. A proof of the following lemma can be obtained by using a slight generalization of [14, Lemma 4.40] and (4.3).

Lemma 4.1 *Let b be a prime and $\alpha \geq 1$ be an integer. Then there exists a constant $C_{b,\alpha} > 0$ depending only on b and α (and not on s and m) such that the mean square worst-case error for multivariate integration in the Sobolev space $\mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}$ using a randomly shifted polynomial lattice rule $Q_{b^m,s}$ can be bounded by*

$$\widehat{e}^2(Q_{b^m,s}, \mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}) \leq \sum_{\emptyset \neq u \subseteq \{1,\ldots,s\}} \gamma_u C_{b,\alpha}^{|u|} \sum_{\boldsymbol{k}_u \in \mathcal{D}_{u,p}^*(\boldsymbol{q})} r_{b,\alpha}^2(\boldsymbol{k}_u).$$

Further, we need the following lemma.

Lemma 4.2 *Let $\alpha \geq 1$ be an integer. Then for every $\frac{1}{2\alpha} < \lambda \leq 1$ there exists a constant $0 < C_{b,\alpha,\lambda} < \infty$ such that*

$$\sum_{l=1}^{\infty} r_{b,\alpha}^{2\lambda}(l) \leq C_{b,\alpha,\lambda}.$$

Proof. Note that it is enough to show the result for λ satisfying $\frac{1}{2\alpha} < \lambda < \min\left(1, \frac{1}{2(\alpha-1)}\right)$ as $\sum_{l=1}^{\infty} r_{b,\alpha}^{2\lambda}(l)$ is a monotonically decreasing function in λ, i.e., we can use the constant $C_{b,\alpha,\lambda}$ to bound $\sum_{l=1}^{\infty} r_{b,\alpha}^{2\lambda'}(l)$ for all $\lambda < \lambda' \leq 1$.

In the following let $l = \lambda_1 b^{a_1-1} + \cdots + \lambda_v b^{a_v-1}$ where $v \geq 1$, $0 < a_v < \cdots < a_1$ and $\lambda_i \in \{1, \ldots, b-1\}$. We divide the sum over all $l \in \mathbb{N}$ into two parts, namely firstly where $1 \leq v \leq \alpha$ and secondly where $v > \alpha$. For the first part we have

$$
\begin{aligned}
&\sum_{v=1}^{\alpha}(b-1)^v \sum_{0<a_v<\cdots<a_1} b^{-2\lambda(a_1+\cdots+a_v)} \\
&= \sum_{v=1}^{\alpha}(b-1)^v \sum_{a_1=v}^{\infty} b^{-2\lambda a_1} \sum_{a_2=v-1}^{a_1-1} b^{-2\lambda a_2} \cdots \sum_{a_v=1}^{a_{v-1}-1} b^{-2\lambda a_v} \\
&\le \sum_{v=1}^{\alpha} \left(\frac{b-1}{b^{2\lambda}-1}\right)^v = \frac{(b-1)((b-1)^\alpha-(b^{2\lambda}-1)^\alpha)}{(b-b^{2\lambda})(b^{2\lambda}-1)^\alpha}.
\end{aligned}
$$

For the second part we have

$$
\begin{aligned}
&(b-1)^\alpha \sum_{0<a_\alpha<\cdots<a_1} b^{-2\lambda(a_1+\cdots+a_\alpha)} b^{a_\alpha-1} \\
&= (b-1)^\alpha b^{-1} \sum_{a_1=\alpha}^{\infty} b^{-2\lambda a_1} \sum_{a_2=\alpha-1}^{a_1-1} b^{-2\lambda a_2} \cdots \sum_{a_\alpha=1}^{a_{\alpha-1}-1} b^{-2\lambda a_\alpha} b^{a_\alpha}.
\end{aligned}
$$

All the above sums are geometric series and can therefore easily be simplified. Indeed we have

$$
\sum_{a_\alpha=1}^{a_{\alpha-1}-1} b^{-2\lambda a_\alpha} b^{a_\alpha} \le \frac{b^{(1-2\lambda)a_{\alpha-1}}}{b^{1-2\lambda}-1}.
$$

Next we can estimate the sum $\sum_{a_{\alpha-1}=2}^{a_{\alpha-2}-1} b^{-2\lambda a_{\alpha-1}} b^{(1-2\lambda)a_{\alpha-1}}$ in a similar way as above. By continuing in this way we obtain that the second part is bounded by

$$
(b-1)^\alpha \prod_{i=1}^{\alpha-1}(b^{1-2i\lambda}-1)^{-1} \sum_{a_1=1}^{\infty} b^{-2\lambda a_1} b^{(1-2(\alpha-1)\lambda)a_1}.
$$

Now the sum above can be written as $\sum_{a_1=1}^{\infty} b^{(1-2\alpha\lambda)a_1}$. This sum is finite as long as $1-2\alpha\lambda<0$, that is, as long as $\lambda>1/(2\alpha)$. In this case we have

$$
\sum_{a_1=1}^{\infty} b^{(1-2\alpha\lambda)a_1} = (b^{2\alpha\lambda-1}-1)^{-1}.
$$

The result now follows. □

From the proof above an explicit constant in Lemma 4.2 can easily be obtained.

For an irreducible polynomial p in $\mathbb{Z}_b[x]$ we denote the mean square worst-case error using shifted polynomial lattice rules generated from the vector $\boldsymbol{q}$ by $\widehat{e}_p^2(\boldsymbol{q})$. We now define the average of $\widehat{e}_p^{2\lambda}(\boldsymbol{q})$ over all polynomials $q_1,\ldots,q_s$ in $G_{b,n}=\{q\in\mathbb{Z}_b[x]:\deg(q)<n\}$ by

$$
A_{m,n,s} = \frac{1}{b^{ns}} \sum_{q_1,\ldots,q_s\in G_{b,n}} \widehat{e}_p^{2\lambda}(\boldsymbol{q}),
$$

where $\boldsymbol{q}=(q_1,\ldots,q_s)$, $n=\deg(p)$ and $\frac{1}{2\alpha}<\lambda\le 1$.

With Lemma 4.1 together with Jensen's inequality we obtain

$$A_{m,n,s} \leq \sum_{\emptyset \neq u \subseteq S} \gamma_u^{\lambda} C_{b,\alpha}^{\lambda|u|} \frac{1}{b^{ns}} \sum_{q_1,\ldots,q_s \in G_{b,n}} \sum_{\boldsymbol{k}_u \in \mathcal{D}^*_{u,p}(\boldsymbol{q})} r_{b,\alpha}^{2\lambda}(\boldsymbol{k}_u).$$

In the following we estimate the term

$$\frac{1}{b^{ns}} \sum_{q_1,\ldots,q_s \in G_{b,n}} \sum_{\boldsymbol{k}_u \in \mathcal{D}^*_{u,p}(\boldsymbol{q})} r_{b,\alpha}^{2\lambda}(\boldsymbol{k}_u) = \sum_{\boldsymbol{k}_u \in \mathbb{N}^{|u|}} r_{b,\alpha}^{2\lambda}(\boldsymbol{k}_u) \frac{1}{b^{n|u|}} \sum_{\substack{\boldsymbol{q}_u \in G_{b,n}^{|u|} \\ \bar{\boldsymbol{k}}_u \cdot \boldsymbol{q}_u \equiv a \pmod{p} \\ \deg(a) < n-m}} 1. \quad (4.4)$$

The last sum is equal to the number of solutions $\boldsymbol{q}_u$ of the equation $\bar{\boldsymbol{k}}_u \cdot \boldsymbol{q}_u \equiv a \pmod{p}$ for some polynomial a with $\deg(a) < n - m$. This number depends of course on $\boldsymbol{k}_u$.

First consider the case where all components of $\bar{\boldsymbol{k}}_u$ are multiples of p. Then every $\boldsymbol{q}_u$ trivially satisfies the equation $\bar{\boldsymbol{k}}_u \cdot \boldsymbol{q}_u \equiv 0 \pmod{p}$. Hence in this case we have

$$\frac{1}{b^{n|u|}} \sum_{\substack{\boldsymbol{q}_u \in G_{b,n}^{|u|} \\ \bar{\boldsymbol{k}}_u \cdot \boldsymbol{q}_u \equiv a \pmod{p} \\ \deg(a) < n-m}} 1 = 1$$

and the sum over all $\boldsymbol{k}_u$ which satisfy this condition is therefore bounded by

$$\sum_{\substack{\boldsymbol{k}_u \in \mathbb{N}^{|u|} \\ \bar{k}_j \equiv 0 \pmod{p}}} r_{b,\alpha}^{2\lambda}(\boldsymbol{k}_u) = \left(\sum_{\substack{k=1 \\ p|\bar{k}}}^{\infty} r_{b,\alpha}^{2\lambda}(k) \right)^{|u|}.$$

Now we have

$$\sum_{\substack{k=1 \\ p|\bar{k}}}^{\infty} r_{b,\alpha}^{2\lambda}(k) = \sum_{l=1}^{\infty} r_{b,\alpha}^{2\lambda}(b^n l) + \sum_{l=0}^{\infty} \sum_{\substack{k=1 \\ p|\bar{k}}}^{b^n-1} r_{b,\alpha}^{2\lambda}(k + b^n l).$$

We note that for $l > 0$ we have $r_{b,\alpha}(b^n l) \leq b^{-n} r_{b,\alpha}(l)$. Further for $1 \leq k < b^n$ the polynomial p never divides $\bar{k}$ since $\deg(p) = n$. Hence

$$\sum_{\substack{k=1 \\ p|\bar{k}}}^{\infty} r_{b,\alpha}^{2\lambda}(k) = \sum_{l=1}^{\infty} r_{b,\alpha}^{2\lambda}(b^n l) \leq b^{-2\lambda n} \sum_{l=1}^{\infty} r_{b,\alpha}^{2\lambda}(l).$$

It remains to consider the case where there is at least one component of $\bar{\boldsymbol{k}}_u$

which is not a multiple of p. In this case we have

$$\frac{1}{b^{n|u|}} \sum_{\substack{\boldsymbol{q}_u \in G_{b,n}^{|u|} \\ \bar{\boldsymbol{k}}_u \cdot \boldsymbol{q}_u \equiv a \pmod{p} \\ \deg(a) < n-m}} 1 = \frac{1}{b^m}$$

and therefore this part of (4.4) is bounded by

$$b^{-m} \left(\sum_{l=1}^{\infty} r_{b,\alpha}^{2\lambda}(l) \right)^{|u|} .$$

Altogether we now obtain that

$$A_{m,n,s} \le \sum_{\emptyset \ne u \subseteq S} \gamma_u^{\lambda} C_{b,\alpha}^{\lambda|u|} \left(\sum_{l=1}^{\infty} r_{b,\alpha}^{2\lambda}(l) \right)^{|u|} \left(b^{-m} + b^{-2\lambda n|u|} \right) .$$

Using Lemma 4.2 we now obtain the following result.

Proposition 4.3 *Let $\alpha \ge 1$, $\frac{1}{2\alpha} < \lambda \le 1$ and $1 \le m \le n$. Then*

$$A_{m,n,s} \le \sum_{\emptyset \ne u \subseteq S} \gamma_u^{\lambda} C_{b,\alpha}^{\lambda|u|} C_{b,\alpha,\lambda}^{|u|} \left(b^{-m} + b^{-2\lambda n|u|} \right) .$$

The following theorem now establishes the existence of good shifted polynomial lattice rules.

Theorem 4.4 *Let $b \ge 2$ be prime, $\alpha \ge 1$, $1 \le m \le n$ be integers and let $p \in \mathbb{Z}_b[x]$ be an irreducible polynomial with $\deg(p) = n$. Then there exists a digitally shifted polynomial lattice rule $Q(\boldsymbol{q}^*)$ with generating vector $\boldsymbol{q}^* \in G_{b,n}^s$ such that*

$$e(Q(\boldsymbol{q}^*), \mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}) \le \frac{1}{b^{\min(\tau m, n)}} \left(2 \sum_{\emptyset \ne u \subseteq S} \gamma_u^{1/(2\tau)} C_{b,\alpha}^{|u|/(2\tau)} C_{b,\alpha,1/(2\tau)}^{|u|} \right)^{\tau}$$

for all $\frac{1}{2} \le \tau < \alpha$.

Proof. For a given irreducible polynomial p with $\deg(p) = n$ let $\boldsymbol{q}^* \in G_{b,n}^s$ satisfy $\widehat{e}_p(\boldsymbol{q}^*) \le \widehat{e}_p(\boldsymbol{q})$ for all $\boldsymbol{q} \in G_{b,n}^s$. Then follows from Proposition 4.3 that for every $\frac{1}{2\alpha} < \lambda \le 1$ we have

$$\widehat{e}_p^{2\lambda}(\boldsymbol{q}^*) \le \frac{1}{b^{ns}} \sum_{\boldsymbol{q} \in G_{b,n}^s} \widehat{e}_p^{2\lambda}(\boldsymbol{q}) \le \sum_{\emptyset \ne u \subseteq S} \gamma_u^{\lambda} C_{b,\alpha}^{\lambda|u|} C_{b,\alpha,\lambda}^{|u|} \left(b^{-m} + b^{-2\lambda n|u|} \right) .$$

By using the estimation $b^{-m}+b^{-2\lambda n|u|} \le 2\max(b^{-m}, b^{-2\lambda n})$ we obtain

$$\widehat{e}_p(\boldsymbol{q}^*) \le 2^{1/(2\lambda)} \max(b^{-m/(2\lambda)}, b^{-n}) \left(\sum_{\emptyset \ne u \subseteq S} \gamma_u^\lambda C_{b,\alpha}^{\lambda|u|} C_{b,\alpha,\lambda}^{|u|} \right)^{1/(2\lambda)} .$$

As the root mean square worst-case error $\widehat{e}_p(\boldsymbol{q}^*)$ taken over all digital shifts satisfies the above bound it is clear that there must exist a shift $\boldsymbol{\sigma}^*$ such that the worst-case error using the $\boldsymbol{\sigma}^*$-shifted polynomial lattice rule generated from $\boldsymbol{q}^*$ satisfies this bound as well.

The result now follows by a change of variables together with the fact that $\max(b^{-\tau m}, b^{-n}) = b^{-\min(\tau m, n)}$. $\square$

Remark 4.5 Note that the upper bound in the above theorem is essentially best possible which follows from the lower bound in [17].

The polynomial lattice rule considered in the above theorem is only shown to work for a fixed $\alpha \ge 1$. In the following we also show the existence of polynomial lattice rules which work well for a range of possible α's.

Let ν be the equiprobable measure on the set $G_{b,n}^s$, i.e., $\nu(\boldsymbol{q}) = b^{-ns}$. For $c \ge 1$ and $\frac{1}{2} \le \tau < \alpha$ we define

$$\mathcal{C}_{b,\alpha}(c,\tau) = \left\{ \boldsymbol{q} \in G_{b,n}^s : \widehat{e}_p(\boldsymbol{q}) \le \frac{c^\tau}{b^{\min(\tau m, n)}} \left(2 \sum_{\emptyset \ne u \subseteq S} \gamma_u^{1/(2\tau)} C_{b,\alpha}^{|u|/(2\tau)} C_{b,\alpha,1/(2\tau)}^{|u|} \right)^\tau \right\}.$$

We obtain the following result.

Lemma 4.6 *Let $c \ge 1$ and $\frac{1}{2} \le \tau < \alpha$. Then we have*

$$\nu(\mathcal{C}_{b,\alpha}(c,\tau)) > 1 - c^{-1}.$$

Proof. The result follows from the fact that for any $\frac{1}{2\alpha} < \lambda \le 1$ we have

$$A_{m,n,s} > \frac{c}{b^{\min(m, 2\lambda n)}} \left(2 \sum_{\emptyset \ne u \subseteq S} \gamma_u^\lambda C_{b,\alpha}^{\lambda|u|} C_{b,\alpha,\lambda}^{|u|} \right) \nu\left(G_{b,n}^s \setminus \mathcal{C}_{b,\alpha}\left(c, \tfrac{1}{2\lambda}\right)\right)$$

together with Proposition 4.3 and ideas from the proof of Theorem 4.4. $\square$

The above lemma shows that, for any given $\alpha \ge 1$, there are many good polynomial lattice rules. Hence it is not surprising that there also exists a polynomial lattice rule which works well for a range of α's. This is shown in the following theorem.

Theorem 4.7 *Let $\alpha, m \geq 1$. Then there exists a $\boldsymbol{q}^* \in G_{b,\alpha m}^s$ such that*

$$\widehat{e}_{p,\beta}(\boldsymbol{q}^*) \leq \frac{\alpha^{\tau_\beta}}{b^{\tau_\beta m}} \left(2 \sum_{\emptyset \neq u \subseteq S} \gamma_u^{1/(2\tau_\beta)} C_{b,\alpha}^{|u|/(2\tau_\beta)} C_{b,\alpha,1/(2\tau_\beta)}^{|u|} \right)^{\tau_\beta}$$

for all $1 \leq \beta \leq \alpha$ and all $\frac{1}{2} \leq \tau_\beta < \beta$. Here $\widehat{e}_{p,\beta}(\boldsymbol{q}^*)$ means the root mean square worst-case error $\widehat{e}_p(\boldsymbol{q}^*)$ for integration in the space $\mathcal{H}_{s,\beta,\boldsymbol{\gamma}}$.

Proof. Let $0 < \varepsilon < \frac{1}{2}$. By choosing $c = \alpha$ in Lemma 4.6 we obtain that

$$\nu(\mathcal{C}_{b,\beta}(\alpha, \beta - \varepsilon)) > 1 - \alpha^{-1}.$$

Thus it follows that

$$\nu\left(\bigcap_{\beta=1}^{\alpha} \mathcal{C}_{b,\beta}(\alpha, \beta - \varepsilon) \right) > 0.$$

By choosing $n = \alpha m$ we have now shown that for a given $0 < \varepsilon < 1/2$ there exists a $\boldsymbol{q}^* \in G_{b,\alpha m}^s$ for which

$$\widehat{e}_{p,\beta}(\boldsymbol{q}^*) \leq \frac{\alpha^{\beta-\varepsilon}}{b^{(\beta-\varepsilon)m}} \left(2 \sum_{\emptyset \neq u \subseteq S} \gamma_u^{1/(2(\beta-\varepsilon))} C_{b,\alpha}^{|u|/(2(\beta-\varepsilon))} C_{b,\alpha,1/(2(\beta-\varepsilon))}^{|u|} \right)^{\beta-\varepsilon}.$$

Since $G_{b,m}^s$ is a finite set it follows that there exists a $\boldsymbol{q}^*$ which works for all choices of $0 < \varepsilon < 1/2$. Thus the result follows. □

In the following we also show the existence of deterministic quadrature rules which work well for all spaces up to smoothness α.

Let μ be the Lebesgue measure on the set $[0,1)^s$. Let $\boldsymbol{q}^*$ be taken from Theorem 4.4. For $\boldsymbol{\sigma} \in [0,1)^s$ let $e_p(\boldsymbol{q}^*, \boldsymbol{\sigma})$ denote the worst-case error of a polynomial lattice rule with generating vector $\boldsymbol{q}^*$ which is digitally shifted by $\boldsymbol{\sigma}$. For $c \geq 1$ we define

$$\mathcal{E}_{b,\alpha}(c) = \left\{ \boldsymbol{\sigma}' \in [0,1)^s : e_p(\boldsymbol{q}^*, \boldsymbol{\sigma}') \leq c \cdot \widehat{e}_p(\boldsymbol{q}^*) \right\}.$$

Further let

$$\mathcal{F}_{b,\alpha}(c) = \Bigg\{ \boldsymbol{\sigma}' \in [0,1)^s : e_p(\boldsymbol{q}^*, \boldsymbol{\sigma}') \leq \frac{c}{b^{\min(\tau m, n)}} \times \left(2 \sum_{\emptyset \neq u \subseteq S} \gamma_u^{1/(2\tau)} C_{b,\alpha}^{|u|/(2\tau)} C_{b,\alpha,1/(2\tau)}^{|u|} \right)^{\tau} \text{ for all } \frac{1}{2} \leq \tau < \alpha \Bigg\}.$$

Then we have $\mathcal{E}_{b,\alpha}(c) \subseteq \mathcal{F}_{b,\alpha}(c)$. This follows from the proof of Theorem 4.4. From standard arguments from probability theory we obtain the following result.

Lemma 4.8 *Let $c \geq 1$ and $\alpha \in \mathbb{N}$. Then we have*

$$\mu(\mathcal{F}_{b,\alpha}(c)) \geq \mu(\mathcal{E}_{b,\alpha}(c)) > 1 - c^{-2}.$$

We can now also show that there exists a digital shift which can be used for a range of choices of α's.

Theorem 4.9 *Let $\alpha, m \geq 1$. Then there exists a $\boldsymbol{q}^* \in G_{b,\alpha m}^s$ and a $\boldsymbol{\sigma}^* \in [0,1)^s$ such that the worst-case error for the polynomial lattice rule with generating vector $\boldsymbol{q}^*$ and shifted by $\boldsymbol{\sigma}^*$ is bounded by*

$$e_{p,\beta}(\boldsymbol{q}^*, \boldsymbol{\sigma}^*) \leq b^{-\tau_\beta m} \sqrt{\alpha} \left(2 \sum_{\emptyset \neq u \subseteq S} \gamma_u^{1/(2\tau_\beta)} C_{b,\alpha}^{|u|/(2\tau_\beta)} C_{b,\alpha,1/(2\tau_\beta)}^{|u|} \right)^{\tau_\beta}$$

for all $1 \leq \beta \leq \alpha$ and all $\frac{1}{2} \leq \tau_\beta < \beta$.

This result follows from Lemma 4.8. We omit a detailed proof since it is very similar to the proof of Theorem 4.7.

Remark 4.10 The above results can also be shown for the digital nets introduced in Definition 2.1, in fact the proofs are simpler for this case. Polynomial lattice rules for which the generating vector $\boldsymbol{q}$ is of the form $(1, q, \ldots, q^{s-1})$ with $q \in G_{b,m}$ are called Korobov polynomial lattice rules. Similar results as for polynomial lattice rules can also be shown for this case, with the difference that we have an additional factor of $s-1$ in the upper bounds of the above results (see also [5]). Further, instead of considering $1 \leq \beta \leq \alpha$, one could also consider a finite set $\mathcal{A} \subset \mathbb{N}_0^s$ and obtain the existence of a digitally shifted polynomial lattice rule which works well for all choices $\alpha \in \mathcal{A}$.

5 Tractability

In this section we study the dependence of the worst-case error on the dimension. This commonly goes by the name of tractability [18].

We would like to reduce the initial error of QMC integration in the Sobolev space $\mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}$ by a factor of $\varepsilon \in (0,1)$. For $\varepsilon \in (0,1)$ let $n(\varepsilon, \mathcal{H}_{s,\alpha,\boldsymbol{\gamma}})$ denote the minimal number of sample points used by a QMC-algorithm such that the initial error is reduced by a factor of ε, i.e.,

$$n(\varepsilon, \mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}) := \min\{n : \exists Q_{n,s} \text{ such that } e(Q_{n,s}, \mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}) \leq \varepsilon e(Q_{0,s}, \mathcal{H}_{s,\alpha,\boldsymbol{\gamma}})\}.$$

Definition 5.1 (Tractability) (1) We say that multivariate integration in the sequence of spaces $\{\mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}\}_{s \geq 1}$ is QMC-tractable if there exist non-negative C, p and q such that

$$n(\varepsilon, \mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}) \leq C\, s^q\, \varepsilon^{-p}$$

holds for all dimensions $s = 1, 2, \ldots$ and for all $\varepsilon \in (0, 1)$.

(2) We say that multivariate integration in the sequence of spaces $\{\mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}\}_{s\geq 1}$ is strongly QMC-tractable if the inequality above holds with $q = 0$.

(3) The minimal (infimum) q and p are called the s-exponent and the ε-exponent of (strong) QMC-tractability.

For $\tau \in [1/2, \alpha)$ and $q \geq 0$ define

$$B_{\tau,q} := \sup_{s=1,2,\ldots} \left(\frac{1}{s^q} \sum_{\emptyset \neq u \subseteq S} \gamma_u^{1/(2\tau)} C^{|u|} \right),$$

where $C = C_{b,\alpha}^{1/(2\tau)} C_{b,\alpha,1/(2\tau)}$ is from the bound in Theorem 4.9.

Theorem 5.2 *Let $\alpha \geq 1$. We have:*

(1) For some $\tau \in [1/2, \alpha)$ assume that

$$B_{\tau,0} < \infty. \tag{5.1}$$

Then the integration problem in the sequence of spaces $\{\mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}\}_{s\geq 1}$ is strongly QMC-tractable. Let τ_0 be the supremum over all τ which satisfy (5.1). Then the ε-exponent of strong tractability lies in the interval $[1/\alpha, 1/\tau_0]$. If (5.1) holds for all $\tau \in [1/2, \alpha)$, then the ε-exponent of strong tractability has the value $1/\alpha$ (which is optimal).

(2) Under the assumption

$$B_{1/2,q} < \infty \tag{5.2}$$

for some non-negative q we obtain that the integration problem in the sequence of spaces $\{\mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}\}_{s\geq 1}$ is QMC-tractable. If $B_{\tau,q} < \infty$, then the ε-exponent of tractability is the interval $[1/\alpha, 1/\tau]$ and the s-exponent is at most q.

Moreover the corresponding upper bounds on the worst-case error can be achieved by digitally shifted polynomial lattice rules.

Proof. Note that here the initial error $e(Q_{0,s}, \mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}) = \sqrt{\gamma_\emptyset}$, which is chosen in advance and can therefore be viewed as a constant. Let $\alpha \geq 1$. For any $\tau \in [1/2, \alpha)$ and $q \geq 0$ we know from Theorem 4.9 the existence of a quasi-Monte Carlo integration rule $Q_{n,s}$ such that

$$e(Q_{n,s}, \mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}) \leq \frac{1}{n^\tau} \sqrt{\alpha} 2^\tau (B_{\tau,q} s^q)^\tau \qquad \forall s \geq 1,$$

and for all $n = b^m$.

(1) If $B_{\tau,0} < \infty$, then we obtain $e(Q_{n,s}, \mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}) \le c \cdot n^{\tau}$ for some $c > 0$ independent of s and n. Therefore the integration problem in the sequence of spaces $\{\mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}\}_{s\ge 1}$ is strongly QMC-tractable. From this it is clear, that if τ_0 is the supremum over all τ which satisfy (5.1), then the ε-exponent of strong tractability lies in the interval $[1/\alpha, 1/\tau_0]$. If (5.1) holds for all $\tau \in [1/2, \alpha)$, then $\tau_0 = \alpha$ which proves the last assertion of item (1).

(2) If $B_{1/2,q} < \infty$ for some non-negative q, then we have $e(Q_{n,s}, \mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}) \le c \cdot s^{q/2} \cdot n^{-1/2}$ for some $c > 0$ independent of s and n and it follows that the integration problem in the sequence of spaces $\{\mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}\}_{s\ge 1}$ is QMC-tractable. If $B_{\tau,q} < \infty$, then we have $e(Q_{n,s}, \mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}) \le c \cdot s^{q\tau} \cdot n^{-\tau}$ and the assertion concerning ε- and s-exponent follows.

As the proof is based on the result from Theorem 4.9 it is clear that the corresponding bounds on the worst-case error can be achieved by digitally shifted polynomial lattice rules. $\square$

In the sequel we will consider a special choice of weights, namely so-called product weights. Here we have a sequence $\gamma_1, \gamma_2, \ldots$ of non-negative reals and the weight corresponding to the projection given by $u \subseteq \{1, \ldots, s\}$ is given by $\gamma_u = \prod_{j\in u} \gamma_j$ for $u \ne \emptyset$ and $\gamma_\emptyset = 1$. In this case for any $\tau < \alpha$ it follows from Theorem 4.9 that there exists a digitally shifted polynomial lattice rule such that

$$e(Q_{n,s}, \mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}) \le \frac{1}{n^{\tau}} \sqrt{\alpha} 2^{\tau} \left(-1 + \prod_{j=1}^{s} \left(1 + C \gamma_j^{1/(2\tau)} \right) \right)^{\tau}, \tag{5.3}$$

where $C = C_{b,\alpha}^{1/(2\tau)} C_{b,\alpha,1/(2\tau)}$ is from the bound in Theorem 4.9 and where $n = b^m$.

Theorem 5.3 *Let $\alpha \ge 1$. We have:*

(1) For some $\tau \in [1/2, \alpha)$ assume that

$$\sum_{j=1}^{\infty} \gamma_j^{1/(2\tau)} < \infty. \tag{5.4}$$

Then the integration problem in the sequence of spaces $\{\mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}\}_{s\ge 1}$ is strongly QMC-tractable. Let τ_0 be the supremum over all τ which satisfy (5.4). Then the ε-exponent of strong tractability lies in the interval $[1/\alpha, 1/\tau_0]$. If (5.4) holds for all $\tau \in [1/2, \alpha)$, then the ε-exponent of strong tractability has the value $1/\alpha$ (which is optimal).

(2) Under the assumption

$$A := \limsup_{s\to\infty} \frac{\sum_{j=1}^{\infty} \gamma_j}{\log s} < \infty$$

we obtain that the integration problem in the sequence of spaces $\{\mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}\}_{s\ge 1}$

is QMC-tractable. If

$$A_\tau := \limsup_{s\to\infty} \frac{\sum_{j=1}^\infty \gamma_j^{1/(2\tau)}}{\log s} < \infty,$$

then the ε-exponent of tractability is in the interval $[1/\alpha, 1/\tau]$ and the s-exponent is at most $C \cdot A_\tau$.

Moreover the corresponding upper bounds on the worst-case error can be achieved by digitally shifted polynomial lattice rules.

Proof. (1) This part of the theorem follows from Theorem 5.2, part (1), since for product weights we have

$$B_{\tau,0} \le \exp\left(C \sum_{j=1}^\infty \gamma_j^{1/(2\tau)}\right),$$

if the sum in the above expression is finite.

(2) For any $\delta > 0$ there exists a positive s_δ such that

$$\sum_{j=1}^s \gamma_j^{1/(2\tau)} \le (A_\tau + \delta) \log s \qquad \forall s \ge s_\delta.$$

From (5.3) we obtain

$$\begin{aligned} e(Q_{n,s}, \mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}) &\le n^{-\tau} \sqrt{\alpha} 2^\tau s^{\tau \sum_{j=1}^s \log\left(1+C\gamma_j^{1/(2\tau)}\right)/\log s} \\ &\le n^{-\tau} \sqrt{\alpha} 2^\tau s^{\tau C\left(\sum_{j=1}^s \gamma_j^{1/(2\tau)}\right)/\log s} \\ &\le n^{-\tau} \sqrt{\alpha} 2^\tau s^{\tau C(A_\tau+\delta)} \end{aligned}$$

for any $\delta > 0$ and all $s \ge s_\delta$. The result follows.

As the proof is based on the result from Theorem 4.9 it is clear that the corresponding bounds on the worst-case error can be achieved by digitally shifted polynomial lattice rules. The result follows. □

Remark 5.4 Note that the conditions for (strong) tractability in the case of product weights are independent of the smoothness parameter α.

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References

[1] N. Aronszajn, Theory of reproducing kernels. Trans. Amer. Math. Soc., 68 (1950), 337–404.

[2] H.E. Chrestenson, A class of generalized Walsh functions. Pacific J. Math., 5 (1955), 17–31.

[3] J. Dick, Explicit constructions of quasi-Monte Carlo rules for the numerical integration of high dimensional periodic functions. Submitted.

[4] J. Dick, Walsh spaces containing smooth functions and quasi-Monte Carlo rules of arbitrary high order. Submitted.

[5] J. Dick, F.Y. Kuo, F. Pillichshammer and I.H. Sloan, Construction algorithms for polynomial lattice rules for multivariate integration. Math. Comp., 74 (2005), 1895–1921.

[6] J. Dick and F. Pillichshammer, Multivariate integration in weighted Hilbert spaces based on Walsh functions and weighted Sobolev spaces. J. Complexity, 21 (2005), 149–195.

[7] N.J. Fine, On the Walsh functions. Trans. Amer. Math. Soc., 65 (1949), 372–414.

[8] F.J. Hickernell, Obtaining $O(N^{-2+\varepsilon})$ convergence for lattice quadrature rules. In: K.T. Fang, F.J. Hickernell and H. Niederreiter, eds., *Monte Carlo and Quasi-Monte Carlo Methods 2000*, Springer Verlag, Berlin, 2002, 274–289.

[9] E. Hlawka, Zur angenäherten Berechnung mehrfacher Integrale. Monatsh. Math., 66 (1962), 140–151.

[10] N.M. Korobov, The approximate computation of multiple integrals. Dokl. Akad. Nauk SSSR 124 (1959), 1207–1210.

[11] K. Niederdrenk, *Die endliche Fourier- und Walshtransformation mit einer Einführung in die Bildverarbeitung.* Vieweg, Braunschweig, 1982.

[12] H. Niederreiter, Point sets and sequences with small discrepancy. Monatsh. Math., 104 (1987), 273–337.

[13] H. Niederreiter, Low-discrepancy point sets obtained by digital constructions over finite fields. Czechoslovak Math. J., 42 (1992), 143–166.

[14] H. Niederreiter, *Random Number Generation and Quasi-Monte Carlo Methods.* No. 63 in CBMS-NSF Series in Applied Mathematics. SIAM, Philadelphia, 1992.

[15] H. Niederreiter, Constructions of (t, m, s)-nets and (t, s)-sequences. Finite Fields Appl., 11 (2005), 578–600.

[16] G. Pirsic, J. Dick and F. Pillichshammer, Cyclic digital nets, hyperplane nets, and multivariate integration in Sobolev spaces. SIAM J. Numer. Anal., 44 (2006), 385–411.

[17] I.F. Sharygin, A lower estimate for the error of quadrature formulas for certain classes of functions, Zh. Vychisl. Mat. i Mat. Fiz., 3 (1963), 370–376.

[18] I.H. Sloan and H. Woźniakowski, When are quasi-Monte Carlo algorithms efficient for high dimensional integrals? J. Complexity, 14 (1998), 1–33.

[19] J.L. Walsh, A closed set of normal orthogonal functions. Amer. J. Math., 55 (1923), 5–24.