# The construction of good extensible Korobov rules

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**Abstract**

In this paper we introduce construction algorithms for Korobov rules for numerical integration which work well for a given set of dimensions simultaneously. The existence of such rules was recently shown by Niederreiter. Here we provide a feasible construction algorithm and an upper bound on the worst-case error in certain reproducing kernel Hilbert spaces for such quadrature rules. The proof is based on a sieve principle recently used by the authors to construct extensible lattice rules. We treat classical lattice rules as well as polynomial lattice rules.

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## 1 Introduction

In many applications one has to evaluate a high dimensional integral of the form $I_s(F) = \int_{[0,1]^s} F(\boldsymbol{x}) \mathrm{d}x$. We are mainly interested in cases where the dimension $s$ is very large, for example in the hundreds or thousands. Such high-dimensional integrals are usually approximated by so-called *quasi-Monte Carlo (QMC) rules* of the form

$$Q_s(F;P) = \frac{1}{|P|} \sum_{x \in P} F(x)$$

where $P$ is a deterministically chosen point set in $[0,1)^s$ and where $|P|$ denotes the cardinality of $P$.

In this paper we consider two popular choices of the point set $P$. The first choice are integration lattices, introduced independently by Hlawka [14] and Korobov [18] and have been studied extensively in recent years by Sloan and his collaborators (see for example [13, 23, 27, 28, 31]). An integer vector $\boldsymbol{z}$, the *generating vector* of the lattice rule, is used to

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generate the $n$ points by $\{h\boldsymbol{z}/n + \boldsymbol{\Delta}\}$ for $h = 0, \ldots, n-1$. The braces indicate that we take the fractional part of each component. The shift $\boldsymbol{\Delta} \in [0,1)^s$ is either chosen $\boldsymbol{0}$ (for periodic functions) or i.i.d. (for non-periodic functions), in which case we call the points a *randomly shifted integration lattice*. QMC algorithms which use integration lattices as underlying point sets are called *lattice rules* or *randomly shifted lattice rule* if $\boldsymbol{\Delta} \in [0,1)^s$ is chosen i.i.d..

The second choice considered here are polynomial lattices, introduced by Niederreiter [22] (see also [23]). Those are similar to integration lattices, but here we use polynomial arithmetic instead of integer arithmetic. Polynomial lattices are special cases of so-called digital $(t,m,s)$-nets as introduced in this form by Niederreiter in [21] (see also [23] for an introduction to this topic). We give an exact definition of polynomial lattices in Section 4. As above, QMC algorithms which use polynomial lattices as underlying point sets are called polynomial lattice rules.

Korobov lattices and polynomial Korobov lattices on the other hand are generated by vectors whose components are successive powers of a single integer (or polynomial), i.e. where the generating vector is of the form $\boldsymbol{z} = (1, z, z^2, \ldots, z^{s-1})$. Such a choice for the generating vector was first proposed by Korobov [19]. A disadvantage of the Korobov construction is that it only works for a fixed given dimension $s$. However, in many applications it is desirable to have rules which work well for several given dimensions.

As explicit constructions of generating vectors are only known for dimension $s = 2$, one relies on computer search algorithms to find good generating vectors. As quality measure we use the worst-case error or root mean square worst-case error for QMC integration in special weighted Hilbert spaces of functions. Let $H_s$ be a Hilbert space of functions defined on $[0,1)^s$ equipped with the norm $\|\cdot\|_{H_s}$. The *worst-case error* of the QMC rule $Q_s(\cdot; P)$ is defined as its worst performance for integrands in the unit ball of $H_s$, i.e.,

$$e(P; H_s) = \sup_{\substack{F \in H_s \\ \|F\|_{H_s} \le 1}} |I_s(F) - Q_s(F; P)|,$$

where $P \subseteq [0,1)^s$ with $|P| = n$. When using randomized quadrature rules with quadrature points $P(\boldsymbol{\Delta})$, where $\boldsymbol{\Delta} \in [0,1)^s$ i.i.d. (for example a lattice rule where the shift $\boldsymbol{\Delta} \in [0,1)^s$ is chosen i.i.d.), then we will consider the root mean square worst-case error

$$\widehat{e}(P; H_s) := \left( \int_{[0,1]^s} e^2(P(\boldsymbol{\Delta}); H_s) \mathrm{d}\boldsymbol{\Delta} \right)^{1/2}. \tag{1}$$

Those criteria can be used in computer search algorithms since for certain spaces of functions the (root mean square) worst-case error can be computed quickly. Indeed, in many recent papers construction algorithms for lattices and polynomial lattices which yield a "small" (root mean square) worst-case error in appropriate Hilbert spaces of functions have been introduced, see for example [3, 5, 6, 7, 10, 15, 16, 17, 25, 26, 28, 29].

The existence of Korobov rules which work well for a given set of dimensions simultaneously was shown in [24]. In this paper we introduce algorithms for the construction of such rules and we prove upper bounds on the (root mean square) worst-case error for numerical integration in certain reproducing kernel Hilbert spaces. The proof technique used here was previously used in [4, 9] to construct both classical and polynomial lattice rules which are extensible in the modulus.

In the following section we introduce reproducing kernel Hilbert spaces and as particular examples thereof, weighted Sobolev spaces and weighted Korobov spaces. In Section 3 we treat the classical case of lattice rules. In this case we use the worst-case error of QMC integration in the weighted Korobov space and the root mean square worst-case error in the weighted Sobolev space as quality measure. In Section 4 we introduce polynomial lattices and in Section 5 we treat the polynomial case. Here, as quality measure we use the worst-case error of QMC integration in a weighted Hilbert space of functions which is based on Walsh functions (see Section 5 for an exact definition) and also the root mean square worst-case error in a weighted Sobolev space. Due to its similarity with the Korobov space (basically the trigonometric functions are replaced by Walsh functions) we call this function space Walsh space. Numerical results are presented in Section 6.

Despite our use of special measures for the quality of Korobov rules we stress that our algorithms also work for other quality measures such as the star discrepancy (via the quantity $R$).

# 2 Reproducing kernel Hilbert spaces

In this section we introduce classes of integrands for which we consider numerical integration. Reproducing kernel Hilbert spaces are nowadays widely used in numerical analysis and other areas and are also used here to define function classes of integrands. The theory of reproducing kernels was developed in [1], see also [12] where reproducing kernel Hilbert spaces where used to investigate numerical integration.

A reproducing kernel Hilbert space over $[0,1]$ is a Hilbert space $H$ admitting a function $K:[0,1]\times[0,1]\to\mathbb{R}$ such that $K(\cdot,y)\in H$ for all $y\in[0,1]$ and $\langle F,K(\cdot,y)\rangle_H=F(y)$ for all $y\in[0,1]$ and $F\in H$. A kernel function $K$ with these properties is unique and it can be shown that $K$ is also symmetric and positive definite. For dimensions $s>1$ we consider tensor products of one-dimensional spaces. It can be shown that the reproducing kernel for those spaces is just the product of the one-dimensional kernels, i.e., $K(\boldsymbol{x},\boldsymbol{y})=\prod_{j=1}^{s}K(x_j,y_j)$, where $\boldsymbol{x}=(x_1,\ldots,x_s)$ and $\boldsymbol{y}=(y_1,\ldots,y_s)$.

In the following we introduce the particular reproducing kernel Hilbert spaces in which numerical integration is frequently considered [3, 10, 12, 16, 17, 27, 28, 29, 32].

## 2.1 Weighted Sobolev spaces

We consider a tensor product Sobolev space $H_{s,\boldsymbol{\gamma}}$ of absolutely continuous functions whose partial mixed derivatives of order one in each variable are square integrable. The norm in the unanchored weighted Sobolev space $H_{s,\boldsymbol{\gamma}}$ [10] is given by

$$\|F\|_{H_{s,\gamma}} = \left( \sum_{u\subseteq\{1,\ldots,s\}} \prod_{j\in u} \gamma_j \int_{[0,1]^{|u|}} \left( \int_{[0,1]^{s-|u|}} \frac{\partial^{|u|}}{\partial \boldsymbol{x}_u} F(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}_{\{1,\ldots,s\}\setminus u} \right)^2 \mathrm{d}\boldsymbol{x}_u \right)^{1/2},$$

where $\partial^{|u|}/\partial\boldsymbol{x}_u F$ denotes the partial mixed derivative with respect to all variables $j \in u$. Here and in the rest of the paper the quantities $\gamma_j$ are non-negative real numbers called weights, which are introduced to modify the importance of different coordinate directions [30].

The reproducing kernel of the $s$-dimensional unanchored weighted Sobolev space [10] is given by

$$K_{s,\boldsymbol{\gamma}}(\boldsymbol{x},\boldsymbol{y}) = \prod_{j=1}^{s} \left( 1 + \gamma_j \left[ \frac{1}{2} B_2(|x_j - y_j|) + \left( x_j - \frac{1}{2} \right) \left( y_j - \frac{1}{2} \right) \right] \right),$$

where $B_2(\cdot)$ denotes the Bernoulli polynomial of degree 2, given by $B_2(x) = x^2 - x + 1/6$, which can also be written as

$$B_2(x) = x^2 - x + \frac{1}{6} = \frac{1}{2\pi^2} \sum_{h=-\infty}^{\infty}{}' \frac{\mathrm{e}^{2\pi \mathrm{i} hx}}{h^2} \quad \forall x \in [0,1]. \tag{2}$$

Here and throughout this paper the notation $\sum'$ indicates a summation with the zero term excluded.

We can associate a shift invariant kernel [12] with $K_{s,\boldsymbol{\gamma}}$ by setting

$$K^{\mathrm{sh}}_{s,\boldsymbol{\gamma}}(\boldsymbol{x},\boldsymbol{y}) = \int_{[0,1]^s} K_{s,\boldsymbol{\gamma}}(\{\boldsymbol{x}+\boldsymbol{\Delta}\},\{\boldsymbol{y}+\boldsymbol{\Delta}\}) \,\mathrm{d}\boldsymbol{\Delta}.$$

The shift-invariant kernel associated with $K_{s,\boldsymbol{\gamma}}$ is given by

$$K^{\mathrm{sh}}_{s,\boldsymbol{\gamma}}(\boldsymbol{x},\boldsymbol{y}) = \prod_{j=1}^{s} \left(1 + \gamma_j B_2(|x_j - y_j|)\right).$$

Using these definitions it follows that the mean square worst-case error (1) for the weighted Sobolev space using $P_{n,s} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{n-1}\} \subset [0,1)^s$ is given by [12]

$$\begin{aligned} \widehat{e}^2(P_{n,s}; H_{s,\boldsymbol{\gamma}}) \;=\; & \int_{[0,1]^{2s}} K^{\mathrm{sh}}_{s,\boldsymbol{\gamma}}(\boldsymbol{x},\boldsymbol{y}) \,\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} - \frac{2}{n} \sum_{k=0}^{n-1} \int_{[0,1]^s} K^{\mathrm{sh}}_{s,\boldsymbol{\gamma}}(\boldsymbol{x},\boldsymbol{x}_k) \mathrm{d}\boldsymbol{x} \\ & + \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{i=0}^{n-1} K^{\mathrm{sh}}_{s,\boldsymbol{\gamma}}(\boldsymbol{x}_i,\boldsymbol{x}_k), \end{aligned} \tag{3}$$

which in the case that $P_{n,s}$ is a randomly shifted (extensible) lattice rule can be simplified to (see for example [28])

$$\widehat{e}^2(P_{n,s}; H_{s,\boldsymbol{\gamma}}) = -1 + \frac{1}{n}\sum_{k=0}^{n-1} K_{s,\boldsymbol{\gamma}}^{\mathrm{sh}}(\boldsymbol{x}_k, \boldsymbol{0}) = -1 + \frac{1}{n}\sum_{k=0}^{n-1}\prod_{j=1}^{s}(1 + \gamma_j B_2(x_{k,j})), \tag{4}$$

where $x_{k,j}$ denotes the $j$-th component of the point $\boldsymbol{x}_k$. Note that the above formula can easily be evaluated using (2) for a given point set $P_{n,s}$.

The shift-invariant kernel $K_{s,\boldsymbol{\gamma}}^{\mathrm{sh}}$ is related to the reproducing kernel of a certain weighted Sobolev space of periodic functions which we introduce in the following.

## 2.2 Weighted Korobov spaces

The $s$-dimensional weighted Korobov space $H_{\mathrm{per},s,\alpha,\boldsymbol{\gamma}}$ has a reproducing kernel of the form [12]

$$\begin{aligned} K_{\mathrm{per},s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{x}, \boldsymbol{y}) &= \prod_{j=1}^{s}\left(1 + \gamma_j {\sum_{h=-\infty}^{\infty}}' \frac{\mathrm{e}^{2\pi \mathrm{i} h(x_j - y_j)}}{|h|^\alpha}\right) \\ &= \sum_{\boldsymbol{h}\in\mathbb{Z}^s} \frac{\mathrm{e}^{2\pi \mathrm{i}\boldsymbol{h}\cdot(\boldsymbol{x}-\boldsymbol{y})}}{r_\alpha(\boldsymbol{h}, \boldsymbol{\gamma})}, \end{aligned}$$

where $\mathbb{Z}$ denotes the set of integers and where for $\boldsymbol{h} = (h_1, \ldots, h_s)$,

$$r_\alpha(\boldsymbol{h}, \boldsymbol{\gamma}) = \prod_{j=1}^{s} r_\alpha(h_j, \gamma_j) \quad \text{and} \quad r_\alpha(h_j, \gamma_j) = \begin{cases} 1 & \text{if } h_j = 0, \\ \gamma_j^{-1}|h_j|^\alpha & \text{if } h_j \neq 0. \end{cases}$$

The parameter $\alpha$ restricts the convergence of the Fourier coefficients of the functions in the Korobov space. Throughout the paper we will assume that $\alpha > 1$.

Equation (3) can again be used to obtain a formula for the worst-case error in the Korobov space $H_{\mathrm{per},s,\alpha,\boldsymbol{\gamma}}$ when using an integration lattice $P_{n,s}(\boldsymbol{z})$,

$$\begin{aligned} e^2(P_{n,s}(\boldsymbol{z}); H_{\mathrm{per},s,\alpha,\boldsymbol{\gamma}}) &= -1 + \frac{1}{n}\sum_{k=0}^{n-1}\prod_{j=1}^{s}\left(1 + \gamma_j {\sum_{h=-\infty}^{\infty}}' \frac{\mathrm{e}^{2\pi \mathrm{i} k h z_j/n}}{|h|^\alpha}\right) && (5) \\ &= -1 + \frac{1}{n}\sum_{k=0}^{n-1}\sum_{\boldsymbol{h}\in\mathbb{Z}^s} \frac{\mathrm{e}^{2\pi \mathrm{i} k\boldsymbol{h}\cdot\boldsymbol{z}/n}}{r_\alpha(\boldsymbol{h}, \boldsymbol{\gamma})} \\ &= \sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^s\setminus\{\boldsymbol{0}\} \\ \boldsymbol{h}\cdot\boldsymbol{z}\equiv 0 \pmod{n}}} \frac{1}{r_\alpha(\boldsymbol{h}, \boldsymbol{\gamma})}. && (6) \end{aligned}$$

It follows from (2), (4) and (5) that

$$\widehat{e}(P_{n,s}(\boldsymbol{z}); H_{s,2\pi^2\boldsymbol{\gamma}}) = e(P_{n,s}(\boldsymbol{z}); H_{\mathrm{per},s,2,\boldsymbol{\gamma}}), \tag{7}$$

where $2\pi^2\boldsymbol{\gamma}$ denotes the sequence of weights $(2\pi^2\gamma_j)_{j\geq 1}$. Thus the results shown in the following are valid for the root mean square worst-case error for numerical integration in the Sobolev space as well as for the worst-case error for numerical integration in the Korobov space. Hence it is enough to state them only for $e(P_{n,s}(\boldsymbol{z}); H_{\mathrm{per},s,\alpha,\boldsymbol{\gamma}})$ (equation (7) can be used to obtain results also for $\widehat{e}(P_{n,s}(\boldsymbol{z}); H_{s,\boldsymbol{\gamma}})$). For short we will write $e_{n,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{z})$ instead of $e(P_{n,s}(\boldsymbol{z}); H_{\mathrm{per},s,\alpha,\boldsymbol{\gamma}})$.

# 3 Numerical integration using Korobov lattice rules

Recall that for $\alpha$ an even natural number the squared worst-case error $e^2_{n,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{z})$ can be calculated in $O(ns)$ operations (see, for example, [3]).

Let $\mathcal{Z}_n = \{1, 2, \ldots, n-1\}$ where $n$ is prime. We assume the generating vector $\boldsymbol{z}$ is of so-called *Korobov form*. That is, we take

$$\boldsymbol{z} = \boldsymbol{z}_s(a) = (1, a, a^2, \ldots, a^{s-1}) \pmod{n}, \quad \text{where} \quad a \in \mathcal{Z}_n.$$

A lattice rule which uses a lattice that is generated by a vector of Korobov form is called *Korobov lattice rule*. We will say that a generating vector is *good* if its worst-case error is in some sense small. In the subsequent lemma we recall some results for the worst-case error using Korobov lattice rules.

**Lemma 1** *Let $\alpha > 1$, $n$ prime and $s$ be a positive integer.*

1. *We have*

$$\frac{1}{n-1} \sum_{a \in \mathcal{Z}_n} e^2_{n,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{z}_s(a)) \leq E^2_{n,s,\alpha,\boldsymbol{\gamma}}$$

*where*

$$E^2_{n,s,\alpha,\boldsymbol{\gamma}} = \frac{s}{n-1} \prod_{j=1}^{s} (1 + 2\gamma_j \zeta(\alpha))$$

*and where $\zeta$ denotes the Riemann Zeta function.*

2. *Further there exists $\overline{a} \in \mathcal{Z}_n$ such that*

$$e^2_{n,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{z}_s(\overline{a})) \leq E^2_{n,s,\alpha,\boldsymbol{\gamma}}(\lambda)$$

*for any $\lambda \in (1/\alpha, 1]$ where*

$$E^2_{n,s,\alpha,\boldsymbol{\gamma}}(\lambda) = \frac{s^{1/\lambda}}{(n-1)^{1/\lambda}} \prod_{j=1}^{s} \left(1 + 2\gamma_j^{\lambda} \zeta(\alpha\lambda)\right)^{1/\lambda}.$$

**Proof.** A proof of these results can be found in [32]. $\square$

Let $\mu$ be the equiprobable measure on the set $\mathcal{Z}_n$. For a real $c \geq 1$ we define the set

$$\mathcal{C}_{n,s,\alpha,\boldsymbol{\gamma}}(c) = \left\{ a \in \mathcal{Z}_n \,:\, e^2_{n,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{z}_s(a)) \leq c E^2_{n,s,\alpha,\boldsymbol{\gamma}} \right\}.$$

In the subsequent lemma we give a lower bound on the measure of this set.

**Lemma 2** *Let $\alpha > 1$, $n$ prime and $s$ be a positive integer. Then for any $c \geq 1$ we have $\mu(\mathcal{C}_{n,s,\alpha,\boldsymbol{\gamma}}(c)) > 1 - c^{-1}$.*

**Proof.** This follows immediately from applying Markov's inequality to the first part of Lemma 1. $\square$

Furthermore, for a real $c \geq 1$ we define the set

$$\widetilde{\mathcal{C}}_{n,s,\alpha,\boldsymbol{\gamma}}(c) = \left\{ a \in \mathcal{Z}_n \,:\, e^2_{n,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{z}_s(a)) \leq c^{1/\lambda} E^2_{n,s,\alpha,\boldsymbol{\gamma}}(\lambda) \text{ for all } 1/\alpha < \lambda \leq 1 \right\}.$$

We obtain the following lemma.

**Lemma 3** *Let $\alpha > 1$, $n$ prime and $s$ be a positive integer. Then for any $c \geq 1$ we have $\mu\left(\widetilde{\mathcal{C}}_{n,s,\alpha,\boldsymbol{\gamma}}(c)\right) > 1 - c^{-1}$.*

**Proof.** Let $c \geq 1$ be given and choose $\lambda^* \in (1/\alpha, 1]$ such that $c^{1/\lambda^*} E^2_{n,s,\alpha\lambda^*,\boldsymbol{\gamma}^{\lambda^*}} \leq c^{1/\lambda} E^2_{n,s,\alpha\lambda,\boldsymbol{\gamma}^{\lambda}}$ for all $1/\alpha < \lambda \leq 1$.

From Lemma 2 we see that

$$\mu(\mathcal{C}_{n,s,\alpha\lambda^*,\boldsymbol{\gamma}^{\lambda^*}}(c)) > 1 - c^{-1}. \tag{8}$$

Now, if $a \in \mathcal{C}_{n,s,\alpha\lambda^*,\boldsymbol{\gamma}^{\lambda^*}}(c)$, then $e^2_{n,s,\alpha\lambda^*,\boldsymbol{\gamma}^{\lambda^*}}(\boldsymbol{z}_s(a)) \leq c E^2_{n,s,\alpha\lambda^*,\boldsymbol{\gamma}^{\lambda^*}}$. Combining the worst-case error in (6) with Jensen's inequality we see that for any $\lambda \in (1/\alpha, 1]$ we have

$$e^2_{n,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{z}_s(a)) \leq \left(e^2_{n,s,\alpha\lambda,\boldsymbol{\gamma}^{\lambda}}(\boldsymbol{z}_s(a))\right)^{1/\lambda},$$

and hence

$$\left(e^2_{n,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{z}_s(a))\right)^{\lambda^*} \leq c E^2_{n,s,\alpha\lambda^*,\boldsymbol{\gamma}^{\lambda^*}}.$$

This can be re-written as

$$e^2_{n,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{z}_s(a)) \leq \left(c E^2_{n,s,\alpha\lambda^*,\boldsymbol{\gamma}^{\lambda^*}}\right)^{1/\lambda^*} = c^{1/\lambda^*} E^2_{n,s,\alpha,\boldsymbol{\gamma}}(\lambda^*),$$

which implies that $a \in \widetilde{\mathcal{C}}_{n,s,\alpha,\boldsymbol{\gamma}}(c)$. This means that $\mathcal{C}_{n,s,\alpha\lambda^*,\boldsymbol{\gamma}^{\lambda^*}}(c) \subseteq \mathcal{C}_{n,s,\alpha,\boldsymbol{\gamma}}(c)$. Using (8) as a lower bound, we find that

$$\mu(\widetilde{\mathcal{C}}_{n,s,\alpha,\boldsymbol{\gamma}}(c)) \geq \mu(\mathcal{C}_{n,s,\alpha\lambda^*,\boldsymbol{\gamma}^{\lambda^*}}(c)) > 1 - c^{-1}.$$

$\square$

Now we want to construct an integer $a \in \mathcal{Z}_n$ such that the generating vector $\boldsymbol{z}_s(a)$ works well simultaneously for several choices of dimensions $s$. Let $\mathcal{S} = \{s_1, s_2, \ldots, s_d\}$ be a set of dimensions for which the generating vector should be *good*, where by *good*, we mean that

$$e^2_{n,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{z}_s(a)) \leq c_s^{1/\lambda} E^2_{n,s,\alpha,\boldsymbol{\gamma}}(\lambda)$$

for some $c_s \geq 1$, for all $1/\alpha < \lambda \leq 1$ and for each $s \in \mathcal{S}$. That is, we seek to find some $a \in \mathcal{Z}_n$ such that

$$a \in \bigcap_{s \in \mathcal{S}} \widetilde{\mathcal{C}}_{n,s,\alpha,\boldsymbol{\gamma}}(c_s).$$

In the theorem below, we see this is possible if we choose $c_s \geq 1$ large enough such that $\sum_{s \in \mathcal{S}} c_s^{-1} \leq 1$.

Here and throughout this paper for $A \subseteq \mathcal{Z}_n$ we denote by $A^c$ its complement in $\mathcal{Z}_n$. Further we denote the set of positive integers by $\mathbb{N}$ and the set of non-negative integers by $\mathbb{N}_0$.

**Theorem 1** *Let $\alpha > 1$, $n$ prime and $\mathcal{S}$ be a subset of $\mathbb{N}$. Let $c_s \geq 1$ for all $s \in \mathcal{S}$, such that $\sum_{s \in \mathcal{S}} c_s^{-1} \leq 1$. Then there exists an $a \in \mathcal{Z}_n$ such that*

$$e^2_{n,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{z}_s(a)) \leq c_s^{1/\lambda} E^2_{n,s,\alpha,\boldsymbol{\gamma}}(\lambda)$$

*for all $s \in \mathcal{S}$ and all $1/\alpha < \lambda \leq 1$.*

**Proof.** As

$$\begin{aligned} \mu\left(\bigcap_{s \in \mathcal{S}} \widetilde{\mathcal{C}}_{n,s,\alpha,\boldsymbol{\gamma}}(c_s)\right) &= 1 - \mu\left(\bigcup_{s \in \mathcal{S}} \widetilde{\mathcal{C}}^c_{n,s,\alpha,\boldsymbol{\gamma}}(c_s)\right) \\ &\geq 1 - \sum_{s \in \mathcal{S}} \mu(\widetilde{\mathcal{C}}^c_{n,s,\alpha,\boldsymbol{\gamma}}(c_s)) > 1 - \sum_{s \in \mathcal{S}} c_s^{-1} \geq 0, \end{aligned}$$

by Lemma 3 and our assumption on the choice of $c_s$, it follows that $\bigcap_{s \in \mathcal{S}} \widetilde{\mathcal{C}}_{n,s,\alpha,\boldsymbol{\gamma}}(c_s)$ is not empty and we are done. $\square$

**Remark 1** Note that it is always possible to choose $c_s$ of order $s^{1+\varepsilon}$ for some $\varepsilon > 0$. Hence the factor $c_s^{1/\lambda}$ in the above bound can be chosen such that it contributes at most another factor of $s^{(1+\varepsilon)/\lambda}$.

Assuming the conditions of Theorem 1, using Algorithm 1 one can find generating vectors of Korobov form for which the worst-case error satisfies the bound from Theorem 1 for a given set of dimensions.

**Algorithm 1** Search for $a \in \mathcal{Z}_n$ with small $e^2_{n,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{z}_s(a))$ for $s \in \mathcal{S}$

**Require:** $\mathcal{S} = \{s_1, \ldots, s_d\}$, $\alpha > 1$, a positive sequence of weights $\boldsymbol{\gamma}$, $n$ prime and the positive sequence $c_{s_1}, \ldots, c_{s_d} \geq 1$ such that $\sum_{s \in \mathcal{S}} c_s^{-1} \leq 1$.

$T_0 = \mathcal{Z}_n$.
**for** $k = 1$ to $d$ **do**
    Find at least $\lfloor (1 - \sum_{i=1}^k c_{s_i}^{-1})(n-1) \rfloor + 1$ elements to populate the set

$$T_k \subseteq \{a \in T_{k-1} : e^2_{n,s_k,\alpha,\boldsymbol{\gamma}}(\boldsymbol{z}_{s_k}(a)) \leq c_{s_k}^{1/\lambda} E^2_{n,s_k,\alpha,\boldsymbol{\gamma}}(\lambda),\ \forall\, 1/\alpha < \lambda \leq 1\}.$$

**end for**
Choose any $a \in T_d$.

**Theorem 2** *Let $\alpha > 1$, $n$ prime and $\mathcal{S}$ be a subset of $\mathbb{N}$. Let $c_s \geq 1$ for all $s \in \mathcal{S}$, such that $\sum_{s \in \mathcal{S}} c_s^{-1} \leq 1$. Then Algorithm 1 gives an element $a \in \mathcal{Z}_n$ such that*

$$e^2_{n,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{z}_s(a)) \leq c_s^{1/\lambda} E^2_{n,s,\alpha,\boldsymbol{\gamma}}(\lambda)$$

*for all $s \in \mathcal{S}$ and all $1/\alpha < \lambda \leq 1$.*

**Proof.** Let $\mathcal{S} = \{s_1, \ldots, s_d\}$. We show by induction on $d$ that the sets $T_{k-1}$, as constructed by Algorithm 1, contain at least $\lfloor (1 - \sum_{i=1}^k c_{s_i}^{-1})(n-1) \rfloor + 1$ elements $a$ such that $e^2_{n,s_k,\alpha,\boldsymbol{\gamma}}(\boldsymbol{z}_{s_k}(a)) \leq c_{s_k}^{1/\lambda} E^2_{n,s_k,\alpha,\boldsymbol{\gamma}}(\lambda)$ for all $1 \leq k \leq d$ and all $1/\alpha < \lambda \leq 1$.

Assume that $c_{s_1}^{-1} \leq 1$. Then it follows from Lemma 3 that there are at least $\lfloor (1 - c_{s_1}^{-1})(n-1) \rfloor + 1$ elements $a \in \mathcal{Z}_n = T_0$ such that $e^2_{n,s_1,\alpha,\boldsymbol{\gamma}}(\boldsymbol{z}_{s_1}(a)) \leq c_{s_1}^{1/\lambda} E^2_{n,s_1,\alpha,\boldsymbol{\gamma}}(\lambda)$. Hence the result is proved for $k = 1$.

Assume now that $\sum_{i=1}^{k+1} c_{s_i}^{-1} \leq 1$ and that for some integer $1 \leq k < d$ we have at least $\lfloor (1 - \sum_{i=1}^k c_{s_i}^{-1})(n-1) \rfloor + 1$ elements in the set

$$\{a \in T_{k-1} : e^2_{n,s_k,\alpha,\boldsymbol{\gamma}}(\boldsymbol{z}_{s_k}(a)) \leq c_{s_k}^{1/\lambda} E^2_{n,s_k,\alpha,\boldsymbol{\gamma}}(\lambda),\ \forall\, 1/\alpha < \lambda \leq 1\},$$

that is,

$$\mu\left(\{a \in T_{k-1} : e^2_{n,s_k,\alpha,\boldsymbol{\gamma}}(\boldsymbol{z}_{s_k}(a)) \leq c_{s_k}^{1/\lambda} E^2_{n,s_k,\alpha,\boldsymbol{\gamma}}(\lambda),\ \forall\, 1/\alpha < \lambda \leq 1\}\right) > 1 - \sum_{i=1}^k c_{s_i}^{-1}.$$

We show that

$$\mu\left(\{a \in T_k : e^2_{n,s_{k+1},\alpha,\boldsymbol{\gamma}}(\boldsymbol{z}_{s_{k+1}}(a)) \leq c_{s_{k+1}} E^2_{n,s_{k+1},\alpha,\boldsymbol{\gamma}}(\lambda),\ \forall\, 1/\alpha < \lambda \leq 1\}\right) > 1 - \sum_{i=1}^{k+1} c_{s_i}^{-1},$$

from which the result then follows as $1-\sum_{i=1}^{k+1} c_{s_i}^{-1} \geq 0$ and hence the above set is not empty. We have

$$
\begin{aligned}
& \{a \in T_k : e^2_{n,s_{k+1},\alpha,\boldsymbol{\gamma}}(\boldsymbol{z}_{s_{k+1}}(a)) \leq c_{s_{k+1}}^{1/\lambda} E^2_{n,s_{k+1},\alpha,\boldsymbol{\gamma}}(\lambda),\ \forall\, 1/\alpha < \lambda \leq 1\} \\
&= \{a \in T_{k-1} : e^2_{n,s_k,\alpha,\boldsymbol{\gamma}}(\boldsymbol{z}_{s_k}(a)) \leq c_{s_k}^{1/\lambda} E^2_{n,s_k,\alpha,\boldsymbol{\gamma}}(\lambda),\ \forall\, 1/\alpha < \lambda \leq 1\} \\
&\quad \cap \{a \in \mathcal{Z}_n : e^2_{n,s_{k+1},\alpha,\boldsymbol{\gamma}}(\boldsymbol{z}_{s_{k+1}}(a)) \leq c_{s_{k+1}}^{1/\lambda} E^2_{n,s_{k+1},\alpha,\boldsymbol{\gamma}}(\lambda),\ \forall\, 1/\alpha < \lambda \leq 1\}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
& \mu\left(\{a \in T_k : e^2_{n,s_{k+1},\alpha,\boldsymbol{\gamma}}(\boldsymbol{z}_{s_{k+1}}(a)) \leq c_{s_{k+1}}^{1/\lambda} E^2_{n,s_{k+1},\alpha,\boldsymbol{\gamma}}(\lambda),\ \forall\, 1/\alpha < \lambda \leq 1\}\right) \\
&= 1 - \mu\Big(\{a \in T_{k-1} : e^2_{n,s_k,\alpha,\boldsymbol{\gamma}}(\boldsymbol{z}_{s_k}(a)) \leq c_{s_k}^{1/\lambda} E^2_{n,s_k,\alpha,\boldsymbol{\gamma}}(\lambda),\ \forall\, 1/\alpha < \lambda \leq 1\}^c \\
&\qquad \cup \{a \in \mathcal{Z}_n : e^2_{n,s_{k+1},\alpha,\boldsymbol{\gamma}}(\boldsymbol{z}_{s_{k+1}}(a)) \leq c_{s_{k+1}}^{1/\lambda} E^2_{n,s_{k+1},\alpha,\boldsymbol{\gamma}}(\lambda),\ \forall\, 1/\alpha < \lambda \leq 1\}^c\Big) \\
&\geq 1 - \left(\sum_{i=1}^{k} c_{s_i}^{-1} + c_{s_{k+1}}^{-1}\right) \\
&= 1 - \sum_{i=1}^{k+1} c_{s_i}^{-1},
\end{aligned}
$$

where we used the induction hypothesis and Lemma 3. The result follows. □

# 4 Polynomial lattices

The construction of a polynomial lattice is quite similar to the construction of lattices, but now we use polynomial arithmetic over a finite field. Before we give the detailed definition we need to introduce some notation. Here we only consider polynomial lattices over the finite field $\mathbb{Z}_b$ where $b$ is a prime. This restriction simplifies the construction scheme a little bit and it is also useful for the forthcoming analysis. For an introduction of polynomial lattices in their full generality we refer to [22] or [23].

Let $b$ be a prime and let $\mathbb{Z}_b((x^{-1}))$ be the field of formal Laurent series over $\mathbb{Z}_b$. Elements of $\mathbb{Z}_b((x^{-1}))$ are formal Laurent series,

$$L = \sum_{l=w}^{\infty} t_l x^{-l},$$

where $w$ is an arbitrary integer and all $t_l \in \mathbb{Z}_b$. Note that $\mathbb{Z}_b((x^{-1}))$ contains the field of rational functions over $\mathbb{Z}_b$ as a subfield. Further let $\mathbb{Z}_b[x]$ be the set of all polynomials over $\mathbb{Z}_b$.

For an integer $m \geq 1$ let $\upsilon_m$ be the map from $\mathbb{Z}_b((x^{-1}))$ to the interval $[0,1)$ defined by

$$\upsilon_m\left(\sum_{l=w}^{\infty} t_l x^{-l}\right) = \sum_{l=\max(1,w)}^{m} t_l b^{-l},$$

where in case the sum is empty, i.e. $w > m$, we set $\upsilon_m\left(\sum_{l=w}^{\infty} t_l x^{-l}\right) = 0$.

With the above notations we can now introduce polynomial lattices. Choose $f \in \mathbb{Z}_b[x]$ and $\boldsymbol{g} = (g_1, \ldots, g_s) \in \mathbb{Z}_b[x]^s$. For $0 \leq h < b^m$ let $h = h_0 + h_1 b + \cdots + h_{m-1} b^{m-1}$ be the $b$-adic expansion of $h$. With each such $h$ we associate the polynomial

$$h(x) = \sum_{r=0}^{m-1} h_r x^r \in \mathbb{Z}_b[x].$$

Then $P(\boldsymbol{g}, f)$ is defined as the point set consisting of the $b^m$ points

$$\boldsymbol{x}_h = \left(\upsilon_m\left(\frac{h(x) g_1(x)}{f(x)}\right), \ldots, \upsilon_m\left(\frac{h(x) g_s(x)}{f(x)}\right)\right) \in [0,1)^s,$$

for $0 \leq h < b^m$. Note that the multiplication and division to compute $h(x)g_j(x)/f(x)$ is carried out in the field $\mathbb{Z}_b((x^{-1}))$. The point set $P(\boldsymbol{g}, f)$ is called a *polynomial lattice* and $\boldsymbol{g}$ is called the *generating vector* of the polynomial lattice.

In the following we will also consider randomly digitally shifted polynomial lattices. Choose a base $b \geq 2$ and let $x = \frac{x_1}{b} + \frac{x_2}{b^2} + \cdots$ and $\sigma = \frac{\sigma_1}{b} + \frac{\sigma_2}{b^2} + \cdots$ be the base $b$ representation of $x$ and $\sigma$. Then the digitally shifted point $y = x \oplus \sigma$ is given by $y = \frac{y_1}{b} + \frac{y_2}{b^2} + \cdots$, where $y_i = x_i + \sigma_i \in \mathbb{Z}_b$. For vectors $\boldsymbol{x}$ and $\boldsymbol{\sigma}$ we define the digitally shifted point $\boldsymbol{x} \oplus \boldsymbol{\sigma}$ component wise. Obviously, the shift depends on the base $b$. If the shift $\boldsymbol{\sigma} \in [0,1)^s$ is chosen i.i.d. and the same shift is applied to all points in $P(\boldsymbol{g}, f)$ we speak of a *randomly digitally shifted polynomial lattice.*

We introduce some notation which we use in polynomial arithmetic: for arbitrary $\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{Z}_b[x]^s$ and $\boldsymbol{g} = (g_1, \ldots, g_s) \in \mathbb{Z}_b[x]^s$, we define the 'inner product'

$$\boldsymbol{k} \cdot \boldsymbol{g} = \sum_{j=1}^{s} k_j g_j \in \mathbb{Z}_b[x]$$

and we write $g \equiv 0 \pmod{f}$ if $f$ divides $g$ in $\mathbb{Z}_b[x]$. Further, as already done above, we associate a non-negative integer $k = \kappa_0 + \kappa_1 b + \cdots + \kappa_a b^a$ with the polynomial $k(x) = \kappa_0 + \kappa_1 x + \cdots + \kappa_a x^a \in \mathbb{Z}_b[x]$ and vice versa.

# 5 Weighted Walsh spaces

Now we turn to Walsh spaces and develop an analogue theory as above for lattices. This was first considered in [8]. In analogy to lattices, we will use the worst-case error of QMC integration in a weighted Hilbert space of functions which is based on Walsh functions as quality measure. Further we also consider the root mean square worst-case error for randomly digitally shifted point sets in the weighted Sobolev space. First we introduce Walsh functions.

Let $b \geq 2$ be an integer. For a non-negative integer $k$ with base $b$ representation

$$k = \kappa_0 + \kappa_1 b + \cdots + \kappa_a b^a,$$

with $\kappa_i \in \{0, \ldots, b-1\}$, we define the $k$-th Walsh function ${}_b\mathrm{wal}_k : [0,1) \to \mathbb{C}$, periodic with period 1, by

$${}_b\mathrm{wal}_k(x) := \mathrm{e}^{2\pi \mathtt{i}(x_1\kappa_0 + \cdots + x_{a+1}\kappa_a)/b},$$

for $x \in [0,1)$ with base $b$ representation $x = \frac{x_1}{b} + \frac{x_2}{b^2} + \cdots$ (unique in the sense that infinitely many of the $x_i$ must be different from $b-1$). For vectors $\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ and reals $\boldsymbol{x} = (x_1, \ldots, x_s)$ we define

$${}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) = \prod_{j=1}^{s} {}_b\mathrm{wal}_{k_i}(x_i).$$

It is clear from the definitions that Walsh functions are piecewise constant. It can be shown that for any integers $s \geq 1$ and $b \geq 2$ the system $\{{}_b\mathrm{wal}_{\boldsymbol{k}} : \boldsymbol{k} \in \mathbb{N}_0^s\}$ is a complete orthonormal system in $L_2([0,1)^s)$, see for example [2, 20]. More information on Walsh functions can be found for example in [2, 11, 33]. Throughout this section we always consider a fixed base $b \geq 2$ and therefore usually we simply write wal instead of ${}_b\mathrm{wal}$.

As in [6, 8], we consider the weighted Hilbert space of functions $H_{\mathrm{wal},s,\alpha,\boldsymbol{\gamma}}$ with reproducing kernel given by

$$K_{\mathrm{wal},s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{x}, \boldsymbol{y}) = \sum_{\boldsymbol{k} \in \mathbb{N}_0^s} \rho(\alpha, \boldsymbol{\gamma}, \boldsymbol{k}) \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) \overline{\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{y})},$$

where $\boldsymbol{k} = (k_1, \ldots, k_s)$ and where $\boldsymbol{x}$ and $\boldsymbol{y}$ are defined analogously. Further $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots)$ is a sequence of positive numbers which quantify the relative importance of successive variables [30] and $\rho(\alpha, \boldsymbol{\gamma}, \boldsymbol{k}) = \prod_{j=1}^{s} \rho(\alpha, \gamma_j, k_j)$, where

$$\rho(\alpha, \gamma, k) = \begin{cases} 1 & \text{if } k = 0, \\ \gamma b^{-\alpha \psi_b(k)} & \text{if } k \neq 0. \end{cases}$$

Here, for a natural number $k = \kappa_0 + \kappa_1 b + \ldots + \kappa_a b^a$, with $\kappa_a \neq 0$, let $\psi_b(k) = a$.

The space $H_{\mathrm{wal},s,\alpha,\boldsymbol{\gamma}}$ is equipped with the inner product

$$\langle f, g \rangle_{\mathrm{wal},s,\boldsymbol{\gamma}} = \sum_{\boldsymbol{k} \in \mathbb{N}_0^s} \rho(\alpha, \boldsymbol{\gamma}, \boldsymbol{k})^{-1} \widehat{f}_{\mathrm{wal}}(\boldsymbol{k}) \overline{\widehat{g}_{\mathrm{wal}}(\boldsymbol{k})},$$

with

$$\widehat{f}_{\mathrm{wal}}(\boldsymbol{k}) := \int_{[0,1]^s} f(\boldsymbol{x}) \overline{\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})} \mathrm{d}\boldsymbol{x}.$$

The space $H_{\mathrm{wal},s,\alpha,\boldsymbol{\gamma}}$ is called *weighted Walsh space.*

We need some further notation. For a non-negative integer $k$ with $b$-adic expansion $k = \kappa_0 + \kappa_1 b + \cdots$, we write

$$\mathrm{tr}_m(k) = \kappa_0 + \kappa_1 b + \cdots + \kappa_{m-1} b^{m-1},$$

thus the associated polynomial $\mathrm{tr}_m(k)(x) = \kappa_0 + \kappa_1 x + \cdots + \kappa_{m-1} x^{m-1} \in \mathbb{Z}_b[x]$ has degree $< m$. For a vector $\boldsymbol{k} \in \mathbb{N}_0^s$, $\mathrm{tr}_m(\boldsymbol{k})$ is defined component wise.

The worst-case error for QMC integration in the weighted Walsh space using a polynomial lattice $P(\boldsymbol{g}, f)$ is given in the following lemma.

**Lemma 4** *Let $\alpha > 1$, $b$ be prime, $s$ be a positive integer and $f \in \mathbb{Z}_b[x]$ with $\deg(f) = m \geq 1$ and let $\boldsymbol{g} = (g_1, \ldots, g_s) \in \mathbb{Z}_b[x]^s$. Then the square worst-case error for integration in the weighted Walsh space $H_{\mathrm{wal},b,s,\alpha,\gamma}$ using the polynomial lattice $P(\boldsymbol{g}, f)$ is given by*

$$e^2_{b^m,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{g}, f) := e^2(P(\boldsymbol{g}, f); H_{\mathrm{wal},s,\alpha,\boldsymbol{\gamma}}) = \sum_{\boldsymbol{k} \in \mathcal{D}} \rho(\alpha, \boldsymbol{\gamma}, \boldsymbol{k}),$$

*where*

$$\mathcal{D} = \{\boldsymbol{k} \in \mathbb{N}_0^s \setminus \{\boldsymbol{0}\} : \mathrm{tr}_m(\boldsymbol{k}) \cdot \boldsymbol{g} \equiv 0 \pmod{f}\}.$$

**Proof.** See [8]. □

In [8] (see also [6]) it is shown how one can compute the worst-case error $e^2_{b^m,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{g}, f)$ in $O(b^m s)$ operations.

Dick and Pillichshammer [8] introduced a 'digital shift invariant kernel' associated with a reproducing kernel. For an arbitrary reproducing kernel $K$ the associated digital shift invariant kernel $K_b^{\mathrm{ds}}$ in base $b$ is defined by

$$K_b^{\mathrm{ds}}(\boldsymbol{x}, \boldsymbol{y}) := \int_{[0,1]^s} K(\boldsymbol{x} \oplus \boldsymbol{\sigma}, \boldsymbol{y} \oplus \boldsymbol{\sigma}) \mathrm{d}\boldsymbol{\sigma},$$

where the digital shift is in base $b$, i.e., $\oplus$ denotes $b$-adic, digit-wise addition modulo $b$ (which is for vectors applied component-wise). They also showed that the digital shift invariant kernel in base $b$ of the weighted Sobolev space $H_{s,\boldsymbol{\gamma}}$ (as defined in Subsection 2.1) is given by

$$\begin{aligned} K_{b,\boldsymbol{\gamma}}^{\mathrm{ds}}(\boldsymbol{x}, \boldsymbol{y}) &= \prod_{j=1}^{s} \left( \sum_{k=0}^{\infty} \widehat{r}_b(\gamma_j, k) \, {}_b\mathrm{wal}_k(x_j) \, \overline{{}_b\mathrm{wal}_k(y_j)} \right) \\ &= \sum_{\boldsymbol{k} \in \mathbb{N}_0^s} \widehat{r}_b(\boldsymbol{\gamma}, \boldsymbol{k}) \, {}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) \, \overline{{}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{y})}, \end{aligned}$$

with $\widehat{r}_b(\boldsymbol{\gamma}, \boldsymbol{k}) = \prod_{j=1}^{s} \widehat{r}_b(\gamma_j, k_j)$, where

$$\widehat{r}_b(\gamma, k) = \begin{cases} 1 & \text{if } k = 0, \\ \frac{\gamma}{2b^{2(\psi_b(k)+1)}} \left( \frac{1}{\sin^2(\kappa_a \pi / b)} - \frac{1}{3} \right) & \text{if } k > 0. \end{cases}$$

Recall that for $k > 0$ we use the notation $k = \kappa_0 + \kappa_1 b + \cdots + \kappa_a b^a$ with $\kappa_a \neq 0$ and $\psi_b(k) = a$. Further, for $x = \frac{x_1}{b} + \frac{x_2}{b^2} + \cdots$ and $y = \frac{y_1}{b} + \frac{y_2}{b^2} + \cdots$ we define

$$\phi_{\mathrm{ds},b}(x, y) = \begin{cases} \frac{1}{6} & \text{if } x = y \\ \frac{1}{6} - \frac{|x_{i_0} - y_{i_0}|(b - |x_{i_0} - y_{i_0}|)}{b^{i_0+1}} & \text{if } x_1 = y_1, \ldots, x_{i_0-1} = y_{i_0-1}, \\ & \text{and } x_{i_0} \neq y_{i_0}. \end{cases} \tag{9}$$

Then the shift invariant kernel $K_{b,\boldsymbol{\gamma}}^{\mathrm{ds}}(\boldsymbol{x}, \boldsymbol{y})$ can be re-written as (see [8, Subsection 6.2])

$$K_{b,\boldsymbol{\gamma}}^{\mathrm{ds}}(\boldsymbol{x}, \boldsymbol{y}) = \prod_{j=1}^{s} (1 + \gamma_j \phi_{\mathrm{ds},b}(x_j, y_j)).$$

For a point set $P_{n,s} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{n-1}\} \subset [0,1)^s$ and a $\boldsymbol{\sigma} \in [0,1)^s$ let $P_{n,s,\boldsymbol{\sigma}} = \{\boldsymbol{x}_0 \oplus \boldsymbol{\sigma}, \ldots, \boldsymbol{x}_{n-1} \oplus \boldsymbol{\sigma}\}$ be the digitally shifted point set. To stress the dependence of the worst-case error in a reproducing kernel Hilbert space $H$ with reproducing kernel $K$ and a point set $P$ on the kernel $K$ we will write in the following lines $e(P; K)$ instead of $e(P; H)$. Let the mean square worst-case error $\widehat{e}^2(P_{n,s}; K)$ with respect to an i.i.d. random digital shift be given by

$$\widehat{e}^2(P_{n,s}; K) := \int_{[0,1]^s} e^2(P_{n,s,\boldsymbol{\sigma}}; K) \mathrm{d}\boldsymbol{\sigma}.$$

Then it was shown in [8, Theorem 7] that we have

$$\widehat{e}^2(P_{n,s}; K) = e^2(P_{n,s}; K_{\mathrm{ds}}).$$

Now with the same arguments as in [8, Theorem 8] one can show that the mean square worst-case error for multivariate integration in the weighted Sobolev space $H_{s,\boldsymbol{\gamma}}$ by using a random digital shift in base $b$ on the point set $P_{n,s} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{n-1}\} \subset [0,1)^s$, with $\boldsymbol{x}_h = (x_{h,1}, \ldots, x_{h,s})$, is given by

$$\begin{aligned} \widehat{e}^2(P_{n,s}; K_{s,\boldsymbol{\gamma}}) &= -1 + \frac{1}{n^2} \sum_{h,i=0}^{n-1} \sum_{\boldsymbol{k} \in \mathbb{N}_0^s} \widehat{r}_b(\boldsymbol{\gamma}, \boldsymbol{k}) \, {}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h) \, \overline{{}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_i)} \\ &= -1 + \frac{1}{n^2} \sum_{h,i=0}^{n-1} \prod_{j=1}^{s} (1 + \gamma_j \phi_{\mathrm{ds},b}(x_{h,j}, x_{i,j})), \end{aligned}$$

where the function $\phi_{\mathrm{ds},b}$ is given by (9). For the special case where the point set $P_{n,s}$ is a polynomial lattice, i.e. $P_{n,s} = P(\boldsymbol{g}, f)$, the mean-square worst case error can be written as

$$\widehat{e}^2(P_{n,s}; K_{s,\boldsymbol{\gamma}}) = \sum_{\boldsymbol{k} \in \mathcal{D}} \widehat{r}_b(\boldsymbol{\gamma}, \boldsymbol{k}),$$

where $\mathcal{D}$ is as defined in Lemma 4. Further we have

$$\widehat{e}^2(P(\boldsymbol{g},f);K_{s,\boldsymbol{\gamma}}) = -1+\frac{1}{n}\sum_{h=0}^{b^m-1}\prod_{j=1}^{s}\left(1+\gamma_j\phi_{\mathrm{ds},b}(x_{h,j},0)\right),$$

where $\phi_{\mathrm{ds},b}$ is given by (9). For a proof of those results see [8].

As can be seen from (9), the function values of $\phi_{\mathrm{ds},b}$ can be computed easily for any $x$ and $y$ and therefore $\widehat{e}^2(P(\boldsymbol{g},f);K_{s,\boldsymbol{\gamma}})$ can be computed in $O(b^m s)$ operations for a given polynomial lattice $P(\boldsymbol{g},f)$ with cardinality $b^m$, where $m=\deg(f)$. Note because of the similarities between the worst-case error in the Walsh space and the root mean square worst-case error in the Sobolev space the results in the following will apply to both cases, though we will only state them for the Walsh space.

Define the set $R_{m,b}=\{g\in\mathbb{Z}_b[x]:\deg(g)<m\}$. We assume that the generating vector $\boldsymbol{g}$ is of *Korobov form*, i.e.,

$$\boldsymbol{g}=\boldsymbol{v}_s(g)=(1,g,g^2,\ldots,g^{s-1}) \pmod{f}, \quad \text{where} \quad g\in R_{m,b}.$$

A polynomial lattice rule which uses a polynomial lattice that is generated by a vector of Korobov form is called *polynomial Korobov lattice rule*. We will say that a generating vector is *good* if its worst-case error is in some sense small. In the subsequent lemma we recall some results for the worst-case error of polynomial Korobov lattice rules.

**Lemma 5** *Let $\alpha>1$, $b$ be prime, $s$ be a positive integer and $f\in\mathbb{Z}_b[x]$ irreducible with* $\deg(f)=m\geq 1$.

1. *We have*

$$\frac{1}{b^m-1}\sum_{g\in R_{m,b}} e^2_{b^m,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{v}_s(g),f)\leq F^2_{b^m,s,\alpha,\boldsymbol{\gamma}},$$

*where*

$$F^2_{b^m,s,\alpha,\boldsymbol{\gamma}}=\frac{s}{b^m-1}\prod_{j=1}^{s}(1+\gamma_j\mu(\alpha))$$

*and where* $\mu(\alpha)=\frac{b^\alpha(b-1)}{b^\alpha-b}$.

2. *There exists a polynomial $\overline{g}\in R_{m,b}$ such that*

$$e^2_{b^m,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{v}_s(\overline{g}),f)\leq F^2_{b^m,s,\alpha,\boldsymbol{\gamma}}(\lambda)$$

*for any $\lambda\in(1/\alpha,1]$, where*

$$F^2_{b^m,s,\alpha,\boldsymbol{\gamma}}(\lambda)=\frac{s^{1/\lambda}}{(b^m-1)^{1/\lambda}}\prod_{j=1}^{s}\left(1+\gamma_j^\lambda\mu(\alpha\lambda)\right)^{1/\lambda}.$$

**Proof.** A proof of these results can be found in [6]. $\square$

Let $\nu$ be the equiprobable measure on the set $R_{m,b}$. For a real $c \geq 1$ we define the set

$$\mathcal{E}_{b^m,s,\alpha,\boldsymbol{\gamma}}(c,f) = \left\{ g \in R_{m,b} \,:\, e^2_{b^m,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{v}_s(a),f) \leq cF^2_{b^m,s,\alpha,\boldsymbol{\gamma}} \right\}.$$

In the subsequent lemma we give a lower bound on the measure of this set.

**Lemma 6** *Let $\alpha > 1$, $b$ be prime, $s$ be a positive integer and $f \in \mathbb{Z}_b[x]$ irreducible with* $\deg(f) = m \geq 1$. *Then for any $c \geq 1$ we have $\nu(\mathcal{E}_{b^m,s,\alpha,\boldsymbol{\gamma}}(c,f)) > 1 - c^{-1}$.*

**Proof.** This follows immediately from applying Markov's inequality to the first part of Theorem 5. $\square$

Furthermore, for a real $c \geq 1$ we define the set

$$\widetilde{\mathcal{E}}_{b^m,s,\alpha,\boldsymbol{\gamma}}(c,f) = \left\{ g \in R_{m,b} \,:\, e^2_{b^m,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{v}_s(g),f) \leq c^{1/\lambda}F^2_{b^m,s,\alpha,\boldsymbol{\gamma}}(\lambda),\ \forall 1/\alpha < \lambda \leq 1 \right\}.$$

We obtain the following lemma.

**Lemma 7** *Let $\alpha > 1$, $b$ be prime, $s$ be a positive integer and $f \in \mathbb{Z}_b[x]$ irreducible with* $\deg(f) = m \geq 1$. *Then for any $c \geq 1$ we have $\nu\left(\widetilde{\mathcal{E}}_{b^m,s,\alpha,\boldsymbol{\gamma}}(c,f)\right) > 1 - c^{-1}$.*

**Proof.** The result follows by using analogous arguments as in the proof of Lemma 3. $\square$

We want to construct a polynomial $g \in R_{m,b}$ such that the generating vector $\boldsymbol{v}_s(g)$ works well simultaneously for several choices of $s$. Let $\mathcal{S} = \{s_1, s_2, \ldots, s_d\}$ be a set of dimensions for which the generating vector should be *good*, where by *good*, we mean that

$$e^2_{b^m,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{v}_s(g),f) \leq c_s^{1/\lambda}F^2_{b^m,s,\alpha,\boldsymbol{\gamma}}(\lambda)$$

for some $c_s \geq 1$, for all $1/\alpha < \lambda \leq 1$ and for each $s \in \mathcal{S}$. That is, we seek to find some $g \in R_{m,b}$ such that

$$g \in \bigcap_{s\in\mathcal{S}} \widetilde{\mathcal{E}}_{b^m,s,\alpha,\boldsymbol{\gamma}}(c_s,f).$$

In the theorem below, we see this is possible if we choose $c_s \geq 1$ large enough such that $\sum_{s\in\mathcal{S}} c_s^{-1} \leq 1$.

**Theorem 3** *Let $\alpha > 1$, $b$ be prime, $\mathcal{S}$ be a subset of $\mathbb{N}$ and $f \in \mathbb{Z}_b[x]$ irreducible with* $\deg(f) = m \geq 1$. *Let $c_s \geq 1$ for all $s \in \mathcal{S}$ such that $\sum_{s\in\mathcal{S}} c_s^{-1} \leq 1$. Then there exists $g \in R_{m,b}$ such that*

$$e^2_{b^m,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{v}_s(g),f) \leq c_s^{1/\lambda}F^2_{b^m,s,\alpha,\boldsymbol{\gamma}}(\lambda)$$

*for all $s \in \mathcal{S}$ and all $1/\alpha < \lambda \leq 1$.*

**Algorithm 2** Search for $g \in R_{m,b}$ with small $e^2_{b^m,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{v}_s(g))$ for $s \in \mathcal{S}$

**Require:** $\mathcal{S} = \{s_1, \ldots, s_d\}$, $\alpha > 1$, a positive sequence of weights $\boldsymbol{\gamma}$, $b$ prime, $f \in \mathbb{Z}_b[x]$ irreducible with $\deg(f) = m$ and the positive sequence $c_{s_1}, \ldots, c_{s_d} \geq 1$ such that $\sum_{s\in\mathcal{S}} c_s^{-1} \leq 1$.

1: $T_0 = R_{m,b}$.

2: **for** $k = 1$ to $d$ **do**

3: Find at least $\lfloor (1 - \sum_{i=1}^k c_{s_i}^{-1})(b^m - 1) \rfloor + 1$ elements to populate the set

$$T_k \subseteq \{g \in T_{k-1} : e^2_{b^m,s_k,\alpha,\boldsymbol{\gamma}}(\boldsymbol{v}_{s_k}(g), f) \leq c_{s_k}^{1/\lambda} F^2_{b^m,s_k,\alpha,\boldsymbol{\gamma}}(\lambda),\ \forall 1/\alpha < \lambda \leq 1\}.$$

4: **end for**

5: Choose any $g \in T_d$.

**Proof.** The proof follows exactly the lines of the proof of Theorem 1. □

Assuming the conditions of Theorem 3, using Algorithm 2 one can find generating vectors of Korobov form for which the worst-case error satisfies the bound from Theorem 3 for a given set of dimensions.

**Theorem 4** *Let $\alpha > 1$, $b$ be prime, $\mathcal{S}$ be a subset of $\mathbb{N}$ and $f \in \mathbb{Z}_b[x]$ irreducible with $\deg(f) = m \geq 1$. Let $c_s \geq 1$ for all $s \in \mathcal{S}$ such that $\sum_{s\in\mathcal{S}} c_s^{-1} \leq 1$. Then Algorithm 2 gives an element $g \in R_{m,b}$ such that*

$$e^2_{b^m,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{v}_s(g), f) \leq c_s^{1/\lambda} F^2_{b^m,s,\alpha,\boldsymbol{\gamma}}(\lambda)$$

*for all $s \in \mathcal{S}$ and all $1/\alpha < \lambda \leq 1$.*

**Proof.** The proof follows exactly the lines of the proof of Theorem 2. □

# 6 Numerical results

In this section we examine the quality of the Korobov form lattice rules which are good for multiple values of $s$. The experiment we perform constructs a generating vector of extensible Korobov form with small worst-case error for each $s \in \mathcal{S}$ where $\mathcal{S} = \{5, 10, 25, 50, 100\}$. We compare this worst-case error with the bound achieved in Theorems 2 and 4 and with the worst-case error of the non-extensible Korobov rule. Further, we compare the worst-case error with the worst-case error of the generating vector constructed by the CBC algorithm (see [3, 6, 16, 25, 28]). Each comparison is made for different values of $n$ and different sets of weights $\boldsymbol{\gamma}$.

Tables 2–7 contain the results of these experiments. In Tables 2, 4 and 6 the rows marked "Bound" contain the quantity

$$\min_{1/\alpha<\lambda\leq 1} c_s^{1/(2\lambda)} E_{n,s,2,\boldsymbol{\gamma}/2\pi^2}(\lambda),$$

where $E_{n,s,2,\boldsymbol{\gamma}/2\pi^2}$ is a bound on the root mean square worst-case error. The rows marked "Ext. Korobov" contain the root mean square worst-case error $\widehat{e}_{n,s,\boldsymbol{\gamma}}(\boldsymbol{z}_s(a))$ in the weighted Sobolev space, see Section 2.1, and where $a$ is constructed by Algorithm 1. The rows marked "Korobov" contain the root mean square worst-case error $\widehat{e}_{n,s,\boldsymbol{\gamma}}(\boldsymbol{z}_s(a))$, where $a$ is chosen to have the smallest root mean square worst-case error for that particular choice of $s$. Finally, the row marked "CBC" contains the root mean square worst-case error $\widehat{e}_{n,s,\boldsymbol{\gamma}}(\boldsymbol{z})$, where the generating vector $\boldsymbol{z}$ is chosen using the CBC algorithm. The choice of $a \in T_d$ in Line 5 of Algorithm 1 is taken to be

$$\operatorname*{argmin}_{a\in T_d} \sum_{k=1}^{d} \frac{\widehat{e}^2_{n,s_k,\boldsymbol{\gamma}}(\boldsymbol{z}_{s_k}(a))}{c_{s_k}^{1/\lambda^*} E^2_{n,s_k,2,\boldsymbol{\gamma}/2\pi^2}(\lambda^*)}$$

where $1/\alpha < \lambda^* \leq 1$ is the minimizer of $c_{s_k}^{1/\lambda} E^2_{n,s_k,2,\boldsymbol{\gamma}/2\pi^2}(\lambda)$. The constants $c_{s_k}$ are all taken to be 5.

Correspondingly, the rows marked "Bound" in Tables 3, 5 and 7 contain the quantity

$$\min_{1/\alpha<\lambda\leq 1} c_s^{1/(2\lambda)} G_{n,s,\boldsymbol{\gamma}}(\lambda),$$

where $n$ is some power of 2 and where $G_{n,s,\boldsymbol{\gamma}}$ is a bound on the root mean square worst-case error for randomly digitally shifted polynomial lattice rules

$$G_{n,s,\boldsymbol{\gamma}} = \frac{s^{1/\lambda}}{(b^m-1)^{1/\lambda}} \prod_{j=1}^{s} \left(1+\gamma_j^\lambda \tau_b(\lambda)\right)^{1/\lambda},$$

where $\tau_b(\lambda)$ is given by: $\tau_b(1) = 1/6$, and for $1/2 < \lambda < 1$ we define

$$\tau_2(\lambda) = \frac{1}{3^\lambda(2^{2\lambda}-2)} \quad \text{and} \quad \tau_b(\lambda) = \frac{(4b^2-9)^\lambda}{54^\lambda} \frac{b-1}{b^{2\lambda}-b} \quad \text{for } b>2.$$

This bound is the analogue for the root mean square worst-case error of the bound $F_{n,s,\alpha,\boldsymbol{\gamma}}(\lambda)$ on the worst-case error in the Walsh space. The rows marked "Ext. Korobov" contain the root mean square worst-case error $\widehat{e}_{n,s,\boldsymbol{\gamma}}(\boldsymbol{v}_s(g), f)$ where the $g$ is constructed by Algorithm 2 and $f$ is an irreducible polynomial of degree $m$, where $n = 2^m$. The rows marked "Korobov" contain the root mean square worst-case error $\widehat{e}_{n,s,\boldsymbol{\gamma}}(\boldsymbol{v}_s(g), f)$ where the $g$ is chosen to have the smallest root mean square worst-case error for that particular choice of $s$. Finally, the row marked "CBC" contains the root mean square worst-case error $e_{n,s,\boldsymbol{\gamma}}(\boldsymbol{g}, f)$, where the

| $m$ | $f$ |
|---|---|
| 8 | $x^8+x^5+x^4+x^3+1$ |
| 9 | $x^9+x^8+x^7+x^5+x^4+x^2+1$ |
| 10 | $x^{10}+x^7+x^3+x+1$ |
| 11 | $x^{11}+x^0+x^8+x^7+x^6+x^4+x^2+x+1$ |

Table 1: Irreducible polynomial $f \in \mathbb{Z}_2[x]$

generating vector $\boldsymbol{z}$ is chosen using the CBC algorithm. The choices of $f$ are listed in Table 1. The choice of $g \in T_d$ in Line 5 of Algorithm 2 is taken to be

$$\operatorname*{argmin}_{g \in T_d} \sum_{k=1}^{d} \frac{\widehat{e}^2_{n,s_k,\boldsymbol{\gamma}}(\boldsymbol{v}_{s_k}(g), f)}{c_{s_k}^{1/\lambda^*} G^2_{n,s_k,\boldsymbol{\gamma}}(\lambda^*)}$$

where $1/\alpha < \lambda^* \leq 1$ is the minimizer of $c_{s_k}^{1/\lambda} G^2_{n,s_k,\boldsymbol{\gamma}}(\lambda)$. Again, the constants $c_{s_k}$ are all taken to be 5.

The first observation we can make from the results in Tables 2–7 is that the extensible Korobov rule has a worst-case error much smaller than the bound in Theorems 2 and 4 suggests. This is similar to the results observed in [32]. The second observation we may make is that the worst-case error for the extensible Korobov rule is not much greater than that of either the Korobov rule for fixed dimension or the lattice rule constructed using the CBC algorithm. This is true for both the classical and polynomial versions. Hence from a practical point of view we obtain Korobov lattice rules which are useful for a range of dimensions.

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| $n$ | Method | $s=5$ | $s=10$ | $s=25$ | $s=50$ | $s=100$ |
|---|---|---|---|---|---|---|
| 257 | Bound | 3.36e-01 | 4.99e-01 | 7.93e-01 | 1.12e+00 | 1.59e+00 |
| | Ext. Korobov | 3.03e-03 | 3.71e-03 | 4.24e-03 | 4.51e-03 | 4.68e-03 |
| | Korobov | 3.03e-03 | 3.68e-03 | 4.24e-03 | 4.51e-03 | 4.68e-03 |
| | CBC | 2.88e-03 | 3.27e-03 | 3.60e-03 | 3.75e-03 | 3.83e-03 |
| 509 | Bound | 2.24e-01 | 3.45e-01 | 5.63e-01 | 7.98e-01 | 1.13e+00 |
| | Ext. Korobov | 1.52e-03 | 1.83e-03 | 2.71e-03 | 2.87e-03 | 2.93e-03 |
| | Korobov | 1.52e-03 | 1.83e-03 | 2.40e-03 | 2.59e-03 | 2.68e-03 |
| | CBC | 1.50e-03 | 1.72e-03 | 1.91e-03 | 2.00e-03 | 2.06e-03 |
| 1021 | Bound | 1.46e-01 | 2.32e-01 | 3.93e-01 | 5.63e-01 | 7.97e-01 |
| | Ext. Korobov | 8.48e-04 | 1.22e-03 | 1.59e-03 | 1.66e-03 | 1.78e-03 |
| | Korobov | 8.48e-04 | 1.07e-03 | 1.31e-03 | 1.50e-03 | 1.61e-03 |
| | CBC | 7.83e-04 | 9.14e-04 | 1.03e-03 | 1.08e-03 | 1.11e-03 |
| 2053 | Bound | 9.27e-02 | 1.52e-01 | 2.68e-01 | 3.93e-01 | 5.62e-01 |
| | Ext. Korobov | 4.30e-04 | 6.47e-04 | 8.14e-04 | 8.92e-04 | 9.23e-04 |
| | Korobov | 4.30e-04 | 5.75e-04 | 6.81e-04 | 7.71e-04 | 8.51e-04 |
| | CBC | 4.05e-04 | 4.81e-04 | 5.46e-04 | 5.76e-04 | 5.95e-04 |

Table 2: Comparison table for classical lattice rules with $\gamma_j = 1/j^2$

| $n$ | Method | $s=5$ | $s=10$ | $s=25$ | $s=50$ | $s=100$ |
|---|---|---|---|---|---|---|
| 256 | Bound | 3.35e-01 | 4.99e-01 | 7.95e-01 | 1.13e+00 | 1.59e+00 |
| | Ext. Korobov | 3.02e-03 | 3.65e-03 | 4.48e-03 | 4.71e-03 | 4.85e-03 |
| | Korobov | 3.02e-03 | 3.62e-03 | 4.28e-03 | 4.71e-03 | 4.85e-03 |
| | CBC | 2.78e-03 | 3.15e-03 | 3.46e-03 | 3.60e-03 | 3.68e-03 |
| 512 | Bound | 2.21e-01 | 3.42e-01 | 5.62e-01 | 7.95e-01 | 1.13e+00 |
| | Ext. Korobov | 1.63e-03 | 1.97e-03 | 2.56e-03 | 2.69e-03 | 2.81e-03 |
| | Korobov | 1.57e-03 | 1.95e-03 | 2.28e-03 | 2.46e-03 | 2.64e-03 |
| | CBC | 1.45e-03 | 1.66e-03 | 1.85e-03 | 1.93e-03 | 1.99e-03 |
| 1024 | Bound | 1.43e-01 | 2.29e-01 | 3.91e-01 | 5.62e-01 | 7.96e-01 |
| | Ext. Korobov | 8.59e-04 | 1.09e-03 | 1.50e-03 | 1.56e-03 | 1.75e-03 |
| | Korobov | 8.25e-04 | 1.08e-03 | 1.31e-03 | 1.41e-03 | 1.55e-03 |
| | CBC | 7.73e-04 | 8.95e-04 | 1.00e-03 | 1.05e-03 | 1.09e-03 |
| 2048 | Bound | 9.06e-02 | 1.50e-01 | 2.66e-01 | 3.93e-01 | 5.63e-01 |
| | Ext. Korobov | 4.41e-04 | 5.81e-04 | 7.74e-04 | 9.33e-04 | 9.65e-04 |
| | Korobov | 4.41e-04 | 5.67e-04 | 6.99e-04 | 7.77e-04 | 8.25e-04 |
| | CBC | 3.93e-04 | 4.65e-04 | 5.28e-04 | 5.58e-04 | 5.77e-04 |

Table 3: Comparison table for polynomial lattice rules with $\gamma_j = 1/j^2$

| $n$ | Method | $s=5$ | $s=10$ | $s=25$ | $s=50$ | $s=100$ |
|---|---|---|---|---|---|---|
| 257 | Bound | 4.17e-01 | 7.03e-01 | 1.36e+00 | 2.03e+00 | 2.88e+00 |
| | Ext. Korobov | 1.12e-02 | 2.66e-02 | 5.19e-02 | 6.00e-02 | 6.08e-02 |
| | Korobov | 1.12e-02 | 2.58e-02 | 5.19e-02 | 6.00e-02 | 6.08e-02 |
| | CBC | 1.02e-02 | 2.45e-02 | 5.02e-02 | 5.80e-02 | 5.86e-02 |
| 509 | Bound | 2.96e-01 | 4.99e-01 | 9.68e-01 | 1.44e+00 | 2.04e+00 |
| | Ext. Korobov | 7.90e-03 | 1.74e-02 | 3.34e-02 | 3.95e-02 | 4.00e-02 |
| | Korobov | 5.92e-03 | 1.59e-02 | 3.34e-02 | 3.93e-02 | 3.98e-02 |
| | CBC | 5.78e-03 | 1.51e-02 | 3.19e-02 | 3.73e-02 | 3.77e-02 |
| 1021 | Bound | 2.07e-01 | 3.52e-01 | 6.83e-01 | 1.02e+00 | 1.44e+00 |
| | Ext. Korobov | 4.67e-03 | 1.05e-02 | 2.19e-02 | 2.65e-02 | 2.68e-02 |
| | Korobov | 3.45e-03 | 9.69e-03 | 2.19e-02 | 2.61e-02 | 2.65e-02 |
| | CBC | 3.31e-03 | 9.01e-03 | 2.01e-02 | 2.37e-02 | 2.40e-02 |
| 2053 | Bound | 1.42e-01 | 2.48e-01 | 4.82e-01 | 7.16e-01 | 1.02e+00 |
| | Ext. Korobov | 3.34e-03 | 6.20e-03 | 1.33e-02 | 1.58e-02 | 1.77e-02 |
| | Korobov | 1.93e-03 | 5.76e-03 | 1.33e-02 | 1.58e-02 | 1.66e-02 |
| | CBC | 1.78e-03 | 5.37e-03 | 1.27e-02 | 1.51e-02 | 1.53e-02 |

Table 4: Comparison table for classical lattice rules with $\gamma_j = 0.9^j$

| $n$ | Method | $s=5$ | $s=10$ | $s=25$ | $s=50$ | $s=100$ |
|---|---|---|---|---|---|---|
| 256 | Bound | 4.18e-01 | 7.05e-01 | 1.37e+00 | 2.03e+00 | 2.88e+00 |
| | Ext. Korobov | 1.11e-02 | 2.64e-02 | 5.39e-02 | 6.21e-02 | 6.27e-02 |
| | Korobov | 1.08e-02 | 2.50e-02 | 5.10e-02 | 5.96e-02 | 6.04e-02 |
| | CBC | 9.84e-03 | 2.36e-02 | 4.87e-02 | 5.66e-02 | 5.72e-02 |
| 512 | Bound | 2.95e-01 | 4.98e-01 | 9.65e-01 | 1.44e+00 | 2.04e+00 |
| | Ext. Korobov | 6.13e-03 | 1.56e-02 | 3.17e-02 | 3.94e-02 | 3.99e-02 |
| | Korobov | 5.90e-03 | 1.49e-02 | 3.17e-02 | 3.86e-02 | 3.92e-02 |
| | CBC | 5.56e-03 | 1.45e-02 | 3.08e-02 | 3.61e-02 | 3.65e-02 |
| 1024 | Bound | 2.05e-01 | 3.52e-01 | 6.82e-01 | 1.01e+00 | 1.44e+00 |
| | Ext. Korobov | 4.52e-03 | 9.63e-03 | 2.13e-02 | 2.53e-02 | 2.59e-02 |
| | Korobov | 3.22e-03 | 9.35e-03 | 2.06e-02 | 2.50e-02 | 2.53e-02 |
| | CBC | 3.13e-03 | 8.66e-03 | 1.96e-02 | 2.31e-02 | 2.34e-02 |
| 2048 | Bound | 1.40e-01 | 2.49e-01 | 4.82e-01 | 7.17e-01 | 1.02e+00 |
| | Ext. Korobov | 1.88e-03 | 5.54e-03 | 1.37e-02 | 1.61e-02 | 1.64e-02 |
| | Korobov | 1.86e-03 | 5.40e-03 | 1.28e-02 | 1.56e-02 | 1.59e-02 |
| | CBC | 1.73e-03 | 5.14e-03 | 1.24e-02 | 1.47e-02 | 1.50e-02 |

Table 5: Comparison table for polynomial lattice rules with $\gamma_j = 0.9^j$

| $n$ | Method | $s=5$ | $s=10$ | $s=25$ | $s=50$ | $s=100$ |
|---|---|---|---|---|---|---|
| 257 | Bound | 2.57e-01 | 4.43e-01 | 7.75e-01 | 1.22e+00 | 2.12e+00 |
| | Ext. Korobov | 1.09e-03 | 1.99e-03 | 5.37e-03 | 1.37e-02 | 3.52e-02 |
| | Korobov | 9.31e-04 | 1.74e-03 | 5.04e-03 | 1.32e-02 | 3.52e-02 |
| | CBC | 9.29e-04 | 1.70e-03 | 5.27e-03 | 1.36e-02 | 3.53e-02 |
| 509 | Bound | 1.60e-01 | 2.95e-01 | 5.50e-01 | 8.63e-01 | 1.50e+00 |
| | Ext. Korobov | 7.16e-04 | 1.47e-03 | 4.11e-03 | 8.73e-03 | 2.17e-02 |
| | Korobov | 4.66e-04 | 9.10e-04 | 3.00e-03 | 8.00e-03 | 2.17e-02 |
| | CBC | 4.68e-04 | 8.75e-04 | 3.05e-03 | 8.09e-03 | 2.23e-02 |
| 1021 | Bound | 9.76e-02 | 1.92e-01 | 3.87e-01 | 6.09e-01 | 1.06e+00 |
| | Ext. Korobov | 4.31e-04 | 1.01e-03 | 2.54e-03 | 5.57e-03 | 1.36e-02 |
| | Korobov | 2.44e-04 | 5.02e-04 | 1.71e-03 | 4.90e-03 | 1.36e-02 |
| | CBC | 2.43e-04 | 4.73e-04 | 1.69e-03 | 4.75e-03 | 1.38e-02 |
| 2053 | Bound | 5.86e-02 | 1.23e-01 | 2.67e-01 | 4.30e-01 | 7.48e-01 |
| | Ext. Korobov | 1.36e-04 | 2.64e-04 | 1.14e-03 | 3.08e-03 | 8.72e-03 |
| | Korobov | 1.23e-04 | 2.64e-04 | 9.49e-04 | 2.84e-03 | 8.72e-03 |
| | CBC | 1.23e-04 | 2.49e-04 | 9.27e-04 | 2.88e-03 | 8.73e-03 |

Table 6: Comparison table for classical lattice rules with $\gamma_j = 0.05$

| $n$ | Method | $s=5$ | $s=10$ | $s=25$ | $s=50$ | $s=100$ |
|---|---|---|---|---|---|---|
| 256 | Bound | 2.54e-01 | 4.42e-01 | 7.77e-01 | 1.22e+00 | 2.12e+00 |
| | Ext. Korobov | 1.09e-03 | 1.93e-03 | 5.50e-03 | 1.30e-02 | 3.47e-02 |
| | Korobov | 9.29e-04 | 1.69e-03 | 5.17e-03 | 1.30e-02 | 3.39e-02 |
| | CBC | 9.14e-04 | 1.65e-03 | 5.20e-03 | 1.31e-02 | 3.43e-02 |
| 512 | Bound | 1.57e-01 | 2.92e-01 | 5.49e-01 | 8.61e-01 | 1.50e+00 |
| | Ext. Korobov | 5.13e-04 | 9.60e-04 | 2.80e-03 | 8.48e-03 | 2.17e-02 |
| | Korobov | 4.75e-04 | 8.70e-04 | 2.80e-03 | 7.61e-03 | 2.13e-02 |
| | CBC | 4.67e-04 | 8.57e-04 | 2.90e-03 | 7.98e-03 | 2.16e-02 |
| 1024 | Bound | 9.54e-02 | 1.89e-01 | 3.86e-01 | 6.08e-01 | 1.06e+00 |
| | Ext. Korobov | 3.76e-04 | 6.82e-04 | 1.99e-03 | 4.65e-03 | 1.33e-02 |
| | Korobov | 2.42e-04 | 4.93e-04 | 1.67e-03 | 4.50e-03 | 1.33e-02 |
| | CBC | 2.38e-04 | 4.67e-04 | 1.64e-03 | 4.69e-03 | 1.36e-02 |
| 2048 | Bound | 5.72e-02 | 1.21e-01 | 2.67e-01 | 4.30e-01 | 7.48e-01 |
| | Ext. Korobov | 1.41e-04 | 3.14e-04 | 9.97e-04 | 3.11e-03 | 8.55e-03 |
| | Korobov | 1.25e-04 | 2.60e-04 | 9.46e-04 | 2.70e-03 | 8.25e-03 |
| | CBC | 1.21e-04 | 2.47e-04 | 9.08e-04 | 2.81e-03 | 8.55e-03 |

Table 7: Comparison table for polynomial lattice rules with $\gamma_j = 0.05$

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