On the Component by Component Construction of Polynomial Lattice Point Sets for Numerical Integration in Weighted Sobolev Spaces

Peter Kritzer and Friedrich Pillichshammer*

Abstract

Polynomial lattice point sets are polynomial versions of classical lattice point sets and among the most widely used classes of node sets for quasi-Monte Carlo integration. In this paper, we study the worst-case integration error of polynomial lattice point sets and give step by step construction algorithms to obtain polynomial lattices that achieve a low worst-case error in certain weighted Sobolev spaces. Furthermore, under certain conditions on the weights, we achieve that there is only a polynomial or even no dependence of the worst-case error on the dimension of the integration problem. The construction algorithm is a so-called component by component algorithm, choosing one component of the relevant point set at a time, and involving a digital shift.

Keywords: Quasi-Monte Carlo, polynomial lattice rules, weighted Sobolev spaces.

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1 Introduction

We study the problem of approximating the value of an integral $I_s(F) := \int_{[0,1]^s} F(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$ of a function $F : [0,1]^s \to \mathbb{R}$. One way of numerically approximating $I_s(F)$ is to employ a quasi-Monte Carlo (QMC) rule,

$$Q_{N,s}(F) := \frac{1}{N}\sum_{n=0}^{N-1} F(\boldsymbol{x}_n),$$

where $\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N-1}$ are deterministically chosen points in $[0,1)^s$. We refer to a collection of integration nodes as a "point set", by which we mean a multi-set, i.e., points may occur repeatedly. It is well known (see, e.g., [4, 5, 9, 14, 18]) that point sets which are in some way evenly distributed in the unit cube yield a low integration error when applying a QMC rule for approximating $I_s(F)$.

An essential question in the theory of QMC methods is how the node set of a QMC integration rule should be chosen. One very prominent class of point sets are polynomial lattices, as proposed by Niederreiter in [13, 14]. These point sets are polynomial versions of classical lattice point sets in the sense of Hlawka [6] and Korobov [7] (see also [14, 18]) which can be considered as special cases of digital (t, m, s)-nets in base b (see [4, 12, 14]). Here we only consider polynomial lattices over a prime base. For a more general construction we refer to [4, 13, 14].

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1.1 Polynomial Lattice Point Sets and Digital Shifts

For the construction of a polynomial lattice, choose a prime b and let $\mathbb{Z}_b$ be the finite field consisting of b elements. Furthermore let $\mathbb{Z}_b[x]$ be the field of polynomials over $\mathbb{Z}_b$, and let $\mathbb{Z}_b((x^{-1}))$ be the field of formal Laurent series over $\mathbb{Z}_b$, with elements of the form $\sum_{l=z}^{\infty} t_l x^{-l}$, where z is an arbitrary integer and the t_l are arbitrary elements in $\mathbb{Z}_b$. Note that the field of Laurent series contains the field of rational functions as a subfield. Given an integer $m \geq 1$, define a function $\nu_m : \mathbb{Z}_b((x^{-1})) \to [0,1)$ by

$$\nu_m\left(\sum_{l=z}^{\infty} t_l x^{-l}\right) := \sum_{l=\max(1,z)}^{m} t_l b^{-l}.$$

Furthermore, set

$$G_{b,m} := \{a \in \mathbb{Z}_b[x] \,:\, \deg(a) < m\} \quad \text{and} \quad G_{b,m}^* := G_{b,m} \setminus \{0\}.$$

Given a prime b, an integer $m \geq 1$, and a dimension $s \geq 1$, we choose an $f \in \mathbb{Z}_b[x]$ with $\deg(f) = m$ and s polynomials $g_1, \ldots, g_s \in \mathbb{Z}_b[x]$ and define

$$\boldsymbol{x}_h := \left(\nu_m\left(\frac{h(x)g_1(x)}{f(x)}\right), \ldots, \nu_m\left(\frac{h(x)g_s(x)}{f(x)}\right)\right), \quad h \in G_{b,m}.$$

The point set consisting of the $N = b^m$ points $\boldsymbol{x}_h$, $h \in G_{b,m}$, is denoted by $\mathscr{P}_{N,s}(\boldsymbol{g}, f)$, where $\boldsymbol{g} := (g_1, \ldots, g_s)$. Due to the many analogies of such a point set to good lattice points (see, e.g, [14, 18] and [17]), a QMC rule using $\mathscr{P}_{N,s}(\boldsymbol{g}, f)$ is called *polynomial lattice rule*, and $\mathscr{P}_{N,s}(\boldsymbol{g}, f)$ is called *polynomial lattice*. The polynomial f in the construction of $\mathscr{P}_{N,s}(\boldsymbol{g}, f)$ is referred to as the *modulus*, and the vector $\boldsymbol{g}$ is referred to as the *generating vector* of the polynomial lattice. Note that, due to the construction principle, we can reduce ourselves to considering only generating vectors $\boldsymbol{g} \in G_{b,m}^s$.

In this paper, we will be particularly interested in studying the properties of (randomly) *digitally shifted* point sets. Digital shifts yield an opportunity to randomize point sets and at the same time to preserve their basic structural properties. We will be concerned with different varieties of digital shifts which are introduced in the following.

(a) We first introduce the notion of a *general digital shift*. To be more precise, we give the formal definition for the one dimensional case. For higher dimensions each coordinate is randomized independently and therefore one just needs to apply the one dimensional randomization method to each coordinate independently.

Assume we are given a point set $\mathscr{P}_{b^m,1} = \{x_0, \ldots, x_{b^m-1}\}$ where x_n, $0 \leq n < b^m$, has b-adic digit expansion of the form

$$x_n = \frac{x_{n,1}}{b} + \frac{x_{n,2}}{b^2} + \cdots .$$

We then choose a number $\sigma = \sum_{i=1}^{\infty} \varsigma_i b^{-i}$, where the ς_i are independently and randomly chosen according to a uniform distribution on $\in \{0, 1, \ldots, b-1\}$ for $i \geq 1$. We then define

$$z_{n,i} \equiv x_{n,i} + \varsigma_i \pmod{b} \qquad \text{for } i \geq 1$$

with $z_{n,i} \in \{0, 1, \ldots, b-1\}$ and set

$$z_n = \frac{z_{n,1}}{b} + \frac{z_{n,2}}{b^2} + \cdots .$$

We then say that $\mathscr{P}_{b^m,1} \oplus \sigma = \{z_0, \ldots, z_{b^m-1}\}$ is the *(generally) digitally shifted version* of $\mathscr{P}_{b^m,1}$. Analogously, for an s-dimensional point set $\mathscr{P}_{b^m,s}$ and a (random) vector $\boldsymbol{\sigma} = (\sigma^{(1)}, \ldots, \sigma^{(s)})$, we denote the point set obtained by shifting the j-th coordinate of $\mathscr{P}_{b^m,s}$ by σ_j, $1 \leq j \leq s$, by $\mathscr{P}_{b^m,s} \oplus \boldsymbol{\sigma}$.

(b) We also will make use of a *digital shift of depth m*. Again, we give the formal definition for the one dimensional case, and for higher dimensions each coordinate is randomized independently.

Let the point set $\mathscr{P}_{b^m,1} = \{x_0, \ldots, x_{b^m-1}\}$, where x_n, $0 \le n < b^m$, has b-adic digit expansion of the form

$$x_n = \frac{x_{n,1}}{b} + \frac{x_{n,2}}{b^2} + \cdots + \frac{x_{n,m}}{b^m}.$$

Choose $\sigma_m = \varsigma_1 b^{-1} + \cdots + \varsigma_m b^{-m}$ with $\varsigma_i \in \{0, 1, \ldots, b-1\}$ uniformly i.i.d., define

$$z_{n,i} \equiv x_{n,i} + \varsigma_i \pmod{b} \qquad \text{for } 1 \le i \le m$$

with $z_{n,i} \in \{0, 1, \ldots, b-1\}$ and set

$$z_n = \frac{z_{n,1}}{b} + \cdots + \frac{z_{n,m}}{b^m}.$$

Now, for $0 \le n < b^m$, choose $\delta_n \in [0, b^{-m})$ uniformly i.i.d.. Then the digitally shifted point set $\{z'_0, \ldots, z'_{b^m-1}\}$ is defined by

$$z'_n = z_n + \delta_n.$$

This means that we apply the same digital shift σ_m to the first m digits, whereas the following digits are shifted independently for each x_n. This is why we refer to this kind of digital shift as a *digital shift of depth m* (see [4, 11]). In analogy to the general shift we denote a point set $\mathscr{P}_{b^m,s}$ that is digitally shifted by a shift of depth m, $\boldsymbol{\sigma}_m = (\sigma_m^{(1)}, \ldots, \sigma_m^{(s)})$, by $\mathscr{P}_{b^m,s} \oplus \boldsymbol{\sigma}_m$.

(c) Further we introduce the simplified version of a *digital shift of depth m*. With the notation from (b), the randomized point set $\{z'_0, \ldots, z'_{b^m-1}\}$ is defined by

$$z'_n = z_n + \frac{1}{2b^m}.$$

This means we apply the same digital shift σ_m of length m to the first m digits and then we add the quantity $1/(2b^m)$ to each point. Such a digital shift is called a *simplified digital shift (of depth m)*. We denote a point set $\mathscr{P}_{b^m,s}$ that is digitally shifted by a simplified digital shift of depth m by $\mathscr{P}_{b^m,s} \oplus \boldsymbol{\sigma}_m^{\text{simp}}$.

Geometrically, the simplified digital shift of depth m means that the randomized points are no longer on the left boundary of intervals $[ab^{-m}, (a+1)b^{-m})$ but they are moved to the midpoints of such intervals. Note that for the simplified digital shift, we only have b^m possibilities, which means only m digits need to be selected in performing a simplified digital shift. In comparison, the digital shift of depth m requires infinitely many digits.

It can be shown that a digital shift preserves (almost surely) some inherent structure of polynomial lattice rules (the (t, m, s)-net structure). As this property is not essential for the following we omit a further discussion in this direction and just refer to [4].

1.2 Weighted Sobolev Spaces

In this paper, we are going to consider the problem of numerically approximating the integral $I_s(F)$ of functions F that are contained in certain Hilbert spaces with a reproducing kernel. These spaces are weighted function spaces, i.e., the influence of the different variables is modelled by assigning suitable weights to the coordinates, as first done by Sloan and Woźniakowski in

[21]. For detailed information on integration problems in weighted function spaces and related topics we refer to the monographs [4, 15, 16]. In this paper we are going to consider two variants of weighted Sobolev spaces.

Before we give their definitions we introduce some notation which we require for the following: assume that $\boldsymbol{\gamma} = (\gamma_j)_{j=1}^{\infty}$ is a non-increasing sequence of positive weights, where $1 \geq \gamma_1 \geq \gamma_2 \geq \cdots$. For $s \in \mathbb{N}$ let $[s] := \{1, \ldots, s\}$. For $\mathfrak{u} \subseteq [s]$, $\boldsymbol{x}_{\mathfrak{u}}$ denotes the projection of $\boldsymbol{x} \in [0,1]^s$ onto $[0,1]^{|\mathfrak{u}|}$ consisting of the components whose indices are contained in $\mathfrak{u}$. Furthermore we write $(\boldsymbol{x}_{\mathfrak{u}}, \mathbf{1}) \in [0,1]^s$ for the point where those components of $\boldsymbol{x}$ whose indices are not in $\mathfrak{u}$ are replaced by 1.

The unanchored Sobolev space. On the one hand, we will be concerned with a weighted version of a so-called *unanchored Sobolev space* $\mathscr{H}_{\mathrm{sob},s,\boldsymbol{\gamma}}$ of functions defined over the s-dimensional unit cube $[0,1]^s$, for which the first mixed partial derivatives are square integrable. The reproducing kernel of $\mathscr{H}_{\mathrm{sob},s,\boldsymbol{\gamma}}$ is given by (see, e.g., [3, 4, 15, 21] for further information)

$$K(\boldsymbol{x}, \boldsymbol{y}) \quad := \quad \prod_{j=1}^{s} \left(1 + \gamma_j \left(\frac{1}{2} B_2\left(\{x_j - y_j\}\right) + \left(x_j - \tfrac{1}{2}\right)\left(y_j - \tfrac{1}{2}\right)\right)\right)$$

for $\boldsymbol{x} = (x_1, \ldots, x_s), \boldsymbol{y} = (y_1, \ldots, y_s) \in [0,1]^s$. Here, $\{z\}$ denotes the fractional part of a real number z, and B_2 is the second Bernoulli polynomial defined by $B_2(x) = x^2 - x + 1/6$. The inner product in $\mathscr{H}_{\mathrm{sob},s,\boldsymbol{\gamma}}$ is given by

$$\begin{aligned} \langle F, G \rangle_{\mathrm{sob},s,\boldsymbol{\gamma}} \quad := \quad & \sum_{\mathfrak{u} \subseteq [s]} \prod_{j \in \mathfrak{u}} \gamma_j^{-1} \int_{[0,1]^{|\mathfrak{u}|}} \left(\int_{[0,1]^{s-|\mathfrak{u}|}} \frac{\partial^{|\mathfrak{u}|} F}{\partial \boldsymbol{x}_{\mathfrak{u}}}(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}_{S \setminus \mathfrak{u}} \right) \times \\ & \times \left(\int_{[0,1]^{s-|\mathfrak{u}|}} \frac{\partial^{|\mathfrak{u}|} G}{\partial \boldsymbol{x}_{\mathfrak{u}}}(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}_{S \setminus \mathfrak{u}} \right) \mathrm{d}\boldsymbol{x}_{\mathfrak{u}}. \end{aligned}$$

The anchored Sobolev space. On the other hand, we are also going to consider the so-called *anchored Sobolev space* $\mathscr{H}'_{\mathrm{sob},s,\mathbf{1},\boldsymbol{\gamma}}$. The reproducing kernel of $\mathscr{H}'_{\mathrm{sob},s,\mathbf{1},\boldsymbol{\gamma}}$ is given by

$$K'(\boldsymbol{x}, \boldsymbol{y}) := \prod_{j=1}^{s} \left(1 + \gamma_j \min(1 - x_j, 1 - y_j)\right),$$

which has been studied, e.g., in [1, 2, 10, 19, 20]. The inner product in $\mathscr{H}'_{\mathrm{sob},s,\mathbf{1},\boldsymbol{\gamma}}$ is given by

$$\langle F, G \rangle_{\mathrm{sob},s,\mathbf{1},\boldsymbol{\gamma}} \quad := \quad \sum_{\substack{\mathfrak{u} \subseteq [s] \\ \mathfrak{u} \neq \emptyset}} \prod_{j \in \mathfrak{u}} \gamma_j^{-1} \int_{[0,1]^{|\mathfrak{u}|}} \frac{\partial^{|\mathfrak{u}|} F}{\partial \boldsymbol{x}_{\mathfrak{u}}}(\boldsymbol{x}_{\mathfrak{u}}, \mathbf{1}) \frac{\partial^{|\mathfrak{u}|} G}{\partial \boldsymbol{x}_{\mathfrak{u}}}(\boldsymbol{x}_{\mathfrak{u}}, \mathbf{1}) \, \mathrm{d}\boldsymbol{x}_{\mathfrak{u}}.$$

For a reproducing kernel Hilbert space $\mathscr{H} \in \{\mathscr{H}_{\mathrm{sob},s,\boldsymbol{\gamma}}, \mathscr{H}'_{\mathrm{sob},s,\mathbf{1},\boldsymbol{\gamma}}\}$ we are going to study the worst case error of integration using a point set $\mathscr{P}_{N,s}$ of N points in $[0,1)^s$,

$$e(\mathscr{P}_{N,s}, \mathscr{H}) := \sup_{\substack{F \in \mathscr{H} \\ \|F\| \leq 1}} \left| I_s(F) - Q_{N,s}(F) \right|,$$

where $\|\cdot\|$ denotes the norm in $\mathscr{H}$ induced by the inner product. To stress the dependence on the reproducing kernel we will also write $e(\mathscr{P}_{N,s}, L)$ instead of $e(\mathscr{P}_{N,s}, \mathscr{H})$ where $L \in \{K, K'\}$ denotes the corresponding reproducing kernel.

As mentioned above, digital shifts of a point set offer a convenient way of randomizing given quasi-Monte Carlo point sets. In particular, this has proven useful in deriving average type results on the integration error of such point sets in weighted Sobolev spaces. Indeed, one frequently studies the *mean square worst case integration error* of polynomial lattices $\mathscr{P}_{N,s}$, defined by

$$\widehat{e}^2(\mathscr{P}_{N,s},\mathscr{H}) := \mathbb{E}_{\boldsymbol{\sigma}}[e^2(\mathscr{P}_{N,s}\oplus\boldsymbol{\sigma},\mathscr{H})] = \int_{[0,1]^s} e^2(\mathscr{P}_{N,s}\oplus\boldsymbol{\sigma},\mathscr{H})\,\mathrm{d}\boldsymbol{\sigma},$$

i.e., one considers the expectation of the worst case integration error with respect to a randomly chosen general digital shift.

Regarding the mean square worst case integration error in the unanchored Sobolev space $\mathscr{H}_{\mathrm{sob},s,\boldsymbol{\gamma}}$, it was shown in [4, Theorem 12.14] that for any irreducible polynomial $f\in\mathbb{Z}_b[x]$ with $\deg(f)=m$ one can construct, component by component, a generating vector $\boldsymbol{g}\in(G_{b,m}^*)^s$ such that

$$\widehat{e}^2(\mathscr{P}_{N,s}(\boldsymbol{g},f),\mathscr{H}_{\mathrm{sob},s,\boldsymbol{\gamma}}) \le c_{s,b,\boldsymbol{\gamma},\lambda} b^{-m/\lambda}$$

for any $\lambda\in(1/2,1]$, with an explicitly known positive constant $c_{s,b,\boldsymbol{\gamma},\lambda}$. Under certain conditions on the weights $\boldsymbol{\gamma}$ one can show the property that $c_{s,b,\boldsymbol{\gamma},\lambda}$ (and hence also the worst-case error) depends only polynomially, or even does not depend at all, on the dimension s, i.e. we can obtain (strong) polynomial tractability, which is the technical notion for such a behavior. A result of the same tenor for the anchored Sobolev space $\mathscr{H}'_{\mathrm{sob},s,\mathbf{1},\boldsymbol{\gamma}}$ can be found in [2]. Furthermore, a generalization to the case where f is not necessarily irreducible is also possible by using results outlined in [8] (see [2, 8] for further details).

Even though the results in [2] and [4] are excellent, the error bounds presented in these papers are only valid for the mean square worst case integration error in the respective Sobolev spaces, i.e., one has no information about how the digital shift involved needs to be chosen. It remained an open question in these papers how a digital shift satisfying such bounds can be effectively found.

In this paper, we are going to partly answer this question and give component by component constructions not only of the underlying polynomial lattice rules, but also of the digital shifts such that we can achieve a small worst case integration error. Indeed, we are going to show a way of choosing, step by step, one component of the polynomial lattice and a digital shift for the same component at a time. By choosing this procedure, we remove the complete randomness of the digital shift involved and replace it by a constructive algorithm. This way of dealing with the problem is inspired by results in [19], where this idea was outlined for the case of shifted ordinary integration lattices. However, there is a certain price we have to pay for making the digital shifts more explicit, namely, our bounds on the worst case integration error are weaker than the probabilistic error bounds mentioned above. This trade-off between an explicit construction and the strength of the error bounds is also in line with the findings in [19].

The rest of the paper is structured as follows. In Section 2, we are going to present our results for the weighted Sobolev spaces $\mathscr{H}_{\mathrm{sob},s,\boldsymbol{\gamma}}$ and $\mathscr{H}'_{\mathrm{sob},s,\mathbf{1},\boldsymbol{\gamma}}$, including a detailed error analysis and the above mentioned construction algorithm. Finally, we are going to discuss tractability results in Section 3 and conclude in Section 4.

2 Component by Component Construction of Polynomial Lattice Points for Unanchored and Anchored Sobolev Spaces

In this section, we outline our results for the Sobolev spaces $\mathscr{H}\in\{\mathscr{H}_{\mathrm{sob},s,\boldsymbol{\gamma}},\mathscr{H}'_{\mathrm{sob},s,\mathbf{1},\boldsymbol{\gamma}}\}$ with reproducing kernel $L\in\{K,K'\}$ as defined in Subsection 1.2. It is well known that the squared worst case integration error in a reproducing kernel Hilbert space can be expressed in terms of the kernel function. In the particular case of the kernel K, using [4, Proposition 2.11] it is easily derived that for a point set $\mathscr{P}_{N,s} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ in $[0,1)^s$, where $\boldsymbol{x}_n = (x_{n,1}, \ldots, x_{n,s})$ for $0 \le n < N$, we have

$$e^2(\mathscr{P}_{N,s}, K) = -1 + \frac{1}{N^2} \sum_{n,h=0}^{N-1} \prod_{i=1}^{s} \left(1 + \gamma_i \left(\frac{B_2(|x_{n,i} - x_{h,i}|)}{2} + \left(x_{n,i} - \tfrac{1}{2}\right)\left(x_{h,i} - \tfrac{1}{2}\right)\right)\right). \quad (1)$$

In the same way one obtains for the kernel K'

$$\begin{aligned} e^2(\mathscr{P}_{N,s}, K') &= \prod_{i=1}^{s} \left(1 + \frac{\gamma_i}{3}\right) - \frac{2}{N} \sum_{n=0}^{N-1} \prod_{i=1}^{s} \left(1 + \frac{\gamma_i}{2}(1 - x_{n,i}^2)\right) \\ &\quad + \frac{1}{N^2} \sum_{n,h=0}^{N-1} \prod_{i=1}^{s} (1 + \gamma_i \min(1 - x_{n,i}, 1 - x_{h,i})). \end{aligned}$$

We now state a very useful lemma that is a first technical step towards making digital shifts yielding low worst case integration error constructible. The following result states that a simplified digital shift of depth m in the last component of a point set does better than the mean ordinary digital shift of depth m. We first introduce some notation. Assume we have a point set $\mathscr{P}_{N,s} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ in $[0,1)^s$, where $\boldsymbol{x}_n = (x_{n,1}, \ldots, x_{n,s})$ for $0 \le n < N$, and a point set $\mathscr{P}_{N,1} = \{x_{0,s+1}, \ldots, x_{N-1,s+1}\}$ in $[0,1)$. Then we denote by $\mathscr{P}_{N,s+1}(\mathscr{P}_{N,s}, \mathscr{P}_{N,1})$ the point set in $[0,1)^{s+1}$ consisting of the points $(x_{n,1}, \ldots, x_{n,s}, x_{n,s+1})$ for $0 \le n < N$.

Lemma 1 *Let $\mathscr{P}_{b^m,s}$ be a point set of b^m points in $[0,1)^s$, and let $\mathscr{P}_{b^m,1}$ be a point set of b^m points in $[0,1)$. Furthermore, let $\sigma_m \in \{ab^{-m} : 0 \le a < b^m\}$. Let $L \in \{K, K'\}$. Then it is true that*

$$e^2(\mathscr{P}_{b^m,s+1}(\mathscr{P}_{b^m,s}, \mathscr{P}_{b^m,1} \oplus \sigma_m^{\mathrm{simp}}), L) \le (b^m)^{b^m} \int_{\left[0,\frac{1}{b^m}\right]^{b^m}} e^2(\mathscr{P}_{b^m,s+1}(\mathscr{P}_{b^m,s}, \mathscr{P}_{b^m,1} \oplus \sigma_m), L) \,\mathrm{d}\boldsymbol{\delta},$$

where $\boldsymbol{\delta} = (\delta_0, \ldots, \delta_{b^m-1})$, and where

- *$\mathscr{P}_{b^m,1} \oplus \sigma_m^{\mathrm{simp}}$ denotes the point set obtained by applying the simplified digital shift of depth m, based on σ_m, to $\mathscr{P}_{b^m,1}$,*
- *$\mathscr{P}_{b^m,1} \oplus \sigma_m$ denotes the point set obtained by applying the ordinary digital shift of depth m, based on σ_m and $\boldsymbol{\delta} \in [0, b^{-m})^{b^m}$, to $\mathscr{P}_{b^m,1}$.*

Proof. We show the result only for $L = K$. The result for $L = K'$ follows in the same way.

For the rest of the paper we use the abbreviation

$$K_{\gamma_i}(x_{n,i}, x_{h,i}) := 1 + \gamma_i \left(\frac{B_2(|x_{n,i} - x_{h,i}|)}{2} + \left(x_{n,i} - \tfrac{1}{2}\right)\left(x_{h,i} - \tfrac{1}{2}\right)\right).$$

We have

$$\begin{aligned} &(b^m)^{b^m} \int_{\left[0,\frac{1}{b^m}\right]^{b^m}} e^2(\mathscr{P}_{b^m,s+1}(\mathscr{P}_{b^m,s}, \mathscr{P}_{b^m,1} \oplus \sigma_m), K) \,\mathrm{d}\boldsymbol{\delta} \\ &= -1 + \frac{1}{N^2} \sum_{n=0}^{b^m-1} K(\boldsymbol{x}_n, \boldsymbol{x}_n) b^m \int_0^{\frac{1}{b^m}} K_{\gamma_{s+1}}(z_{n,s+1} + \delta_n, z_{n,s+1} + \delta_n) \,\mathrm{d}\delta_n \\ &\quad + \frac{1}{N^2} \sum_{\substack{n,h \\ n \ne h}}^{b^m-1} K(\boldsymbol{x}_n, \boldsymbol{x}_h)(b^m)^2 \int_0^{\frac{1}{b^m}} \int_0^{\frac{1}{b^m}} K_{\gamma_{s+1}}(z_{n,s+1} + \delta_n, z_{h,s+1} + \delta_h) \,\mathrm{d}\delta_n \,\mathrm{d}\delta_h. \end{aligned}$$

Now

$$
\begin{aligned}
& b^m \int_0^{\frac{1}{b^m}} K_{\gamma_{s+1}}(z_{n,s+1}+\delta_n, z_{n,s+1}+\delta_n)\,\mathrm{d}\delta_n \\
&= 1+\gamma_i b^m \int_0^{\frac{1}{b^m}} \left(z_{n,s+1}+\delta_n-\tfrac{1}{2}\right)^2 \mathrm{d}\delta_n \\
&= 1+\gamma_i \left(\left(z_{n,s+1}-\tfrac{1}{2}\right)^2 + \left(z_{n,s+1}-\tfrac{1}{2}\right)\frac{1}{b^m}+\frac{1}{3b^{2m}} \right) \\
&\geq 1+\gamma_i \left(\left(z_{n,s+1}-\tfrac{1}{2}\right)^2 + \left(z_{n,s+1}-\tfrac{1}{2}\right)\frac{1}{b^m}+\frac{1}{2^2b^{2m}} \right) \\
&= K_{\gamma_{s+1}}\left(z_{n,s+1}+\frac{1}{2b^m}, z_{n,s+1}+\frac{1}{2b^m}\right).
\end{aligned}
$$

In the same way it is easy to check that

$$
\begin{aligned}
& (b^m)^2 \int_0^{\frac{1}{b^m}} \int_0^{\frac{1}{b^m}} K_{\gamma_{s+1}}(z_{n,s+1}+\delta_n, z_{h,s+1}+\delta_h)\,\mathrm{d}\delta_n\,\mathrm{d}\delta_h \\
&\geq K_{\gamma_{s+1}}\left(z_{n,s+1}+\frac{1}{2b^m}, z_{h,s+1}+\frac{1}{2b^m}\right)
\end{aligned}
$$

and hence the result follows. □

For our construction algorithm, we need some technical tools. First, of all, we define Walsh functions, a class of functions that frequently occurs in the analysis of polynomial lattice point sets (see, e.g., [4, Appendix A] for further information). We recall that in this paper we assume that the base b is an arbitrarily chosen, but fixed prime.

Definition 1 *For a non-negative integer k with base b representation $k=\kappa_0+\kappa_1 b+\kappa_2 b^2+\cdots+\kappa_r b^r$, with $\kappa_i\in\{0,1,\ldots,b-1\}$, $1\leq i\leq r$, we define the k-th Walsh function to the base b, ${}_b\mathrm{wal}_k:[0,1)\to\mathbb{C}$ by*

$$
{}_b\mathrm{wal}_k(x)=\exp(2\pi\mathtt{i}(\xi_1\kappa_0+\xi_2\kappa_1+\cdots+\xi_{r+1}\kappa_r)/b),
$$

for $x\in[0,1)$ with base b representation $x=\xi_1 b^{-1}+\xi_2 b^{-2}+\cdots$ (unique in the sense that infinitely many of the ξ_i must be different from $b-1$).

Since we assume that the base b is fixed, we shall omit the base b in ${}_b\mathrm{wal}$ and write wal for short.

We also need an auxiliary function, which will occur in our error analysis: for a positive integer k with base b representation $k=\kappa_0+\kappa_1 b+\cdots+\kappa_r b^r$ where $\kappa_i\in\{0,1,\ldots,b-1\}$ for all $0\leq i\leq r$ and $\kappa_r\neq 0$ we define

$$
\tau(k):=\frac{1}{b^{2r+2}}\left(\frac{1}{3}-\frac{1}{\sin^2(\kappa_r\pi/b)}\right).
$$

Furthermore, we set $\tau(0):=1/3$. Then for any $m\in\mathbb{N}$, we have

$$
\sum_{k=1}^{b^m-1}\tau(k)=\sum_{r=0}^{m-1}\frac{b^r}{b^{2r+2}}\left(\frac{1}{3}-\sum_{\kappa=1}^{b-1}\frac{1}{\sin^2(\kappa\pi/b)}\right)=-\frac{1}{b}\frac{b^2-2}{3(b-1)}\left(1-\frac{1}{b^m}\right), \tag{2}
$$

where we used the fact that $\sum_{\kappa=1}^{b-1}\sin^{-2}(\kappa\pi/b)=(b^2-1)/3$ as shown in [4, Appendix A].

Furthermore, we define

$$c(K) := \frac{b+1}{9} \quad \text{and} \quad c(K') := \frac{b+1}{3}.$$

We are now ready to show the following theorem which is the foundation of our construction algorithm.

Theorem 1 *Let $L \in \{K, K'\}$. Let b be a prime and let $m, s \in \mathbb{N}$ be given. Furthermore, let $f \in \mathbb{Z}_b[x]$ be irreducible with $\deg(f) = m$ and assume that $\mathscr{P}_{N,s}$ is a point set in $[0,1)^s$ with $N = b^m$ points such that*

$$e^2(\mathscr{P}_{N,s}, L) \leq \frac{1}{N} \prod_{j=1}^{s} (1 + \gamma_j c(L)).$$

Then there exists a $g_{s+1} \in G_{b,m} \setminus \{0\}$ and a $\sigma_m \in \{ab^{-m} : 0 \leq a < b^m\}$ such that

$$e^2(\mathscr{P}_{N,s+1}(\mathscr{P}_{N,s}, \mathscr{P}_{N,1}(g_{s+1}, f) \oplus \sigma_m^{\text{simp}}), L) \leq \frac{1}{N} \prod_{j=1}^{s+1} (1 + \gamma_j c(L)).$$

Proof. We show the result only for $L = K$. The result for $L = K'$ follows in the same way.

We first study the expression

$$\mathbb{E}_{\sigma_m}(e^2) := \mathbb{E}_{\sigma_m}\left[e^2(\mathscr{P}_{N,s+1}(\mathscr{P}_{N,s}, \mathscr{P}_{N,1}(g_{s+1}, f) \oplus \sigma_m), K)\right],$$

where $\mathbb{E}_{\sigma_m}$ means the expectation value with respect to the digital shift σ_m of depth m. We denote the points

- of $\mathscr{P}_{N,s}$ by $\boldsymbol{x}_n = (x_{n,1}, \ldots, x_{n,s})$, $0 \leq n \leq b^m - 1$,
- of $\mathscr{P}_{N,1}(g_{s+1}, f)$ by $x_{n,s+1}$, $0 \leq n \leq b^m - 1$,
- and of $\mathscr{P}_{N,1}(g_{s+1}, f) \oplus \sigma_m$ by $z_{n,s+1}$, $0 \leq n \leq b^m - 1$.

Using Equation (1) and the definition of the second Bernoulli polynomial, we have

$$\begin{aligned}
\mathbb{E}_{\sigma_m}(e^2) &= -1 + \frac{1}{N^2} \sum_{n,h=0}^{N-1} \left(\prod_{j=1}^{s} K_{\gamma_j}(x_{n,j}, x_{h,j}) \right) \times \\
&\quad \times \Bigg(1 + \gamma_{s+1} \Bigg(\frac{\mathbb{E}_{\sigma_m}\left[(z_{n,s+1} - z_{h,s+1})^2\right] - \mathbb{E}_{\sigma_m}\left[|z_{n,s+1} - z_{h,s+1}|\right]}{2} + \frac{1}{12} + \\
&\quad + \mathbb{E}_{\sigma_m}\left[\left(z_{n,s+1} - \tfrac{1}{2}\right)\left(z_{h,s+1} - \tfrac{1}{2}\right)\right] \Bigg) \Bigg) \\
&= -1 + \frac{1}{N^2} \sum_{n,h=0}^{N-1} \left(\prod_{j=1}^{s} K_{\gamma_j}(x_{n,j}, x_{h,j}) \right) \times \\
&\quad \times \Bigg(1 + \gamma_{s+1} \Bigg(\frac{\mathbb{E}_{\sigma_m}\left[z_{n,s+1}^2\right] + \mathbb{E}_{\sigma_m}\left[z_{h,s+1}^2\right]}{2} + \\
&\quad + \frac{-2\mathbb{E}_{\sigma_m}\left[z_{n,s+1} z_{h,s+1}\right] - \mathbb{E}_{\sigma_m}\left[|z_{n,s+1} - z_{h,s+1}|\right]}{2} + \\
&\quad + \frac{1}{3} + \mathbb{E}_{\sigma_m}\left[z_{n,s+1} z_{h,s+1}\right] - \frac{1}{2}\left(\mathbb{E}_{\sigma_m}\left[z_{n,s+1}\right] + \mathbb{E}_{\sigma_m}\left[z_{h,s+1}\right]\right) \Bigg) \Bigg).
\end{aligned}$$

We now use [4, Lemma 16.38 (1) and (2)] according to which, for any $n \in \{0, 1, \dots, N-1\}$ we have $\mathbb{E}_{\sigma_m}[z_{n,s+1}] = \frac{1}{2}$ and $\mathbb{E}_{\sigma_m}\left[z_{n,s+1}^2\right] = \frac{1}{3}$. Consequently,

$$\mathbb{E}_{\sigma_m}(e^2) = -1 + \frac{1}{N^2} \sum_{n,h=0}^{N-1} \left(\prod_{j=1}^{s} K_{\gamma_j}(x_{n,j}, x_{h,j}) \right) \left(1 + \gamma_{s+1} \left(\frac{1}{6} - \frac{1}{2} \mathbb{E}_{\sigma_m}[|z_{n,s+1} - z_{h,s+1}|] \right) \right).$$

Furthermore, we employ [4, Lemma 16.38 (3)], which yields

$$\begin{aligned} \mathbb{E}_{\sigma_m}(e^2) &= -1 + \frac{1}{N^2} \sum_{n,h=0}^{N-1} \left(\prod_{j=1}^{s} K_{\gamma_j}(x_{n,j}, x_{h,j}) \right) \times \\ &\quad \times \left(1 + \gamma_{s+1} \left(\frac{1}{6} - \frac{1}{2} \sum_{k=0}^{N-1} \tau(k) \mathrm{wal}_k(x_{n,s+1} \ominus x_{h,s+1}) \right) \right) \\ &= e^2(\mathscr{P}_{N,s}, K) + \\ &\quad + \frac{1}{N^2} \sum_{n,h=0}^{N-1} \left(\prod_{j=1}^{s} K_{\gamma_j}(x_{n,j}, x_{h,j}) \right) \gamma_{s+1} \left(\frac{1}{6} - \frac{1}{2} \sum_{k=0}^{N-1} \tau(k) \mathrm{wal}_k(x_{n,s+1} \ominus x_{h,s+1}) \right), \end{aligned}$$

where $\ominus$ denotes digit-wise subtraction modulo b.

Separating out the case $k = 0$, and observing that $\mathrm{wal}_0(x) = 1$ for any x, yields

$$\begin{aligned} \mathbb{E}_{\sigma_m}(e^2) &= e^2(\mathscr{P}_{N,s}, K) - \\ &\quad - \frac{\gamma_{s+1}}{2} \frac{1}{N^2} \sum_{n,h=0}^{N-1} \left(\prod_{j=1}^{s} K_{\gamma_j}(x_{n,j}, x_{h,j}) \right) \sum_{k=1}^{N-1} \tau(k) \mathrm{wal}_k(x_{n,s+1} \ominus x_{h,s+1}) \\ &= e^2(\mathscr{P}_{N,s}, K) - \frac{\gamma_{s+1}}{2} \frac{1}{N^2} \sum_{n=0}^{N-1} \left(\prod_{j=1}^{s} K_{\gamma_j}(x_{n,j}, x_{n,j}) \right) \sum_{k=1}^{N-1} \tau(k) \mathrm{wal}_k(0) \\ &\quad - \frac{\gamma_{s+1}}{2} \frac{1}{N^2} \sum_{\substack{n,h=0 \\ n \neq h}}^{N-1} \left(\prod_{j=1}^{s} K_{\gamma_j}(x_{n,j}, x_{h,j}) \right) \sum_{k=1}^{N-1} \tau(k) \mathrm{wal}_k(x_{n,s+1} \ominus x_{h,s+1}). \end{aligned}$$

Now we use the fact that $\mathrm{wal}_k(0) = 1$ and (2), we obtain

$$\begin{aligned} & e^2(\mathscr{P}_{N,s}, K) - \frac{\gamma_{s+1}}{2} \frac{1}{N^2} \sum_{n=0}^{N-1} \left(\prod_{j=1}^{s} K_{\gamma_j}(x_{n,j}, x_{n,j}) \right) \sum_{k=1}^{N-1} \tau(k) \mathrm{wal}_k(0) \leq \\ &\leq e^2(\mathscr{P}_{N,s}, K) + \frac{\gamma_{s+1}}{2} \frac{1}{b} \frac{b^2 - 2}{3(b-1)} \left(1 - \frac{1}{b^m} \right) \frac{1}{N^2} \sum_{n=0}^{N-1} \prod_{j=1}^{s} K_{\gamma_j}(x_{n,j}, x_{n,j}) \\ &\leq e^2(\mathscr{P}_{N,s}, K) + \gamma_{s+1} \frac{b+1}{9} \frac{1}{N^2} \sum_{n=0}^{N-1} \prod_{j=1}^{s} \left(1 + \gamma_j \left(\frac{1}{12} + \left(x_{n,j} - \frac{1}{2} \right)^2 \right) \right) \\ &\leq e^2(\mathscr{P}_{N,s}, K) + \gamma_{s+1} \frac{b+1}{9} \frac{1}{N^2} \sum_{n=0}^{N-1} \prod_{j=1}^{s} \left(1 + \frac{\gamma_j}{3} \right) \\ &\leq \frac{1}{N} \prod_{j=1}^{s} \left(1 + \gamma_j \frac{b+1}{9} \right) + \gamma_{s+1} \frac{b+1}{9} \frac{1}{N} \prod_{j=1}^{s} \left(1 + \gamma_j \frac{b+1}{9} \right) \end{aligned}$$

$$= \frac{1}{N}\prod_{j=1}^{s+1}\left(1+\gamma_j\frac{b+1}{9}\right).$$

We now analyze, for $n,h\in\{0,1,\ldots,N-1\}$, $n\neq h$, the expression

$$M_{G_{b,m}} := \frac{1}{b^m-1}\sum_{g_{s+1}\in G_{b,m}\setminus\{0\}}\sum_{k=1}^{N-1}\tau(k)\mathrm{wal}_k(x_{n,s+1}\ominus x_{h,s+1}).$$

Using the definition of the points of a polynomial lattice, we obtain

$$\begin{aligned} M_{G_{b,m}} &= \frac{1}{b^m-1}\sum_{g_{s+1}\in G_{b,m}\setminus\{0\}}\sum_{k=1}^{N-1}\tau(k)\mathrm{wal}_k\left(\nu_m\left(\frac{n(x)g_{s+1}(x)}{f(x)}\right)\ominus\nu_m\left(\frac{n(x)g_{s+1}(x)}{f(x)}\right)\right) \\ &= \frac{1}{b^m-1}\sum_{g_{s+1}\in G_{b,m}\setminus\{0\}}\sum_{k=1}^{N-1}\tau(k)\mathrm{wal}_k\left(\nu_m\left(\frac{(n\ominus h)(x)g_{s+1}(x)}{f(x)}\right)\right). \end{aligned}$$

Now, since $n\neq h$, and since g_{s+1} runs through all of $G_{b,m}\setminus\{0\}$, we can write

$$M_{G_{b,m}} = \frac{1}{b^m-1}\sum_{k=1}^{N-1}\tau(k)\sum_{g_{s+1}\in G_{b,m}\setminus\{0\}}\mathrm{wal}_k\left(\nu_m\left(\frac{g_{s+1}(x)}{f(x)}\right)\right).$$

Furthermore, again, since g_{s+1} runs through all of $G_{b,m}\setminus\{0\}$, and since f is irreducible, we can rewrite $M_{G_{b,m}}$ as

$$\begin{aligned} M_{G_{b,m}} &= \frac{1}{b^m-1}\sum_{k=1}^{N-1}\tau(k)\sum_{g=1}^{b^m-1}\mathrm{wal}_k\left(\frac{g}{b^m}\right) \\ &= \frac{1}{b^m-1}\sum_{k=1}^{N-1}\tau(k)\left(\sum_{g=0}^{b^m-1}\mathrm{wal}_k\left(\frac{g}{b^m}\right)-\mathrm{wal}_k(0)\right) \\ &= -\frac{1}{b^m-1}\sum_{k=1}^{N-1}\tau(k) \\ &= \frac{1}{b^m-1}\frac{1}{b}\frac{b^2-2}{3(b-1)}\left(1-\frac{1}{b^m}\right), \end{aligned}$$

where we used the fact that $\sum_{g=0}^{b^m-1}\mathrm{wal}_k(g/b^m)=0$, and (2). We can therefore conclude that $M_{G_{b,m}}\geq 0$. Furthermore, it is easily seen that

$$K_{\gamma_j}(x_{n,j},x_{h,j}) = 1+\gamma_j\left(\frac{B_2(|x_{n,j}-x_{h,j}|)}{2}+\left(x_{n,j}-\tfrac{1}{2}\right)\left(x_{h,j}-\tfrac{1}{2}\right)\right)\geq 0.$$

This implies that

$$\frac{1}{b^m-1}\sum_{g_{s+1}\in G_{b,m}\setminus\{0\}}-\frac{\gamma_{s+1}}{2}\frac{1}{N^2}\sum_{\substack{n,h=0\\ n\neq h}}^{N-1}\left(\prod_{j=1}^{s}K_{\gamma_j}(x_{n,j},x_{h,j})\right)\sum_{k=1}^{N-1}\tau(k)\mathrm{wal}_k(x_{n,s+1}\ominus x_{h,s+1})\leq 0.$$

Putting all of these results together, we see that there exists a $g_{s+1}\in G_{b,m}\setminus\{0\}$ such that

$$\mathbb{E}_{\sigma_m}\left[e^2(\mathscr{P}_{N,s+1}(\mathscr{P}_{N,s},\mathscr{P}_{N,1}(g_{s+1},f)\oplus\sigma_m),K)\right]\leq\frac{1}{N}\prod_{j=1}^{s+1}\left(1+\gamma_j\frac{b+1}{9}\right).$$

Thus, there exists $g_{s+1} \in G_{b,m} \setminus \{0\}$ and a special σ_m such that the digital shift of depth m based on σ_m satisfies

$$e^2(\mathscr{P}_{N,s+1}(\mathscr{P}_{N,s}, \mathscr{P}_{N,1}(g_{s+1}, f) \oplus \sigma_m), K) \le \frac{1}{N} \prod_{j=1}^{s+1} \left(1 + \gamma_j \frac{b+1}{9}\right).$$

And, finally, invoking Lemma 1, we see that it is sufficient to consider the simplified digital shift of depth m based on σ_m in the above expression. □

Based on Theorem 1, we can now formulate our construction algorithms for polynomial lattice points with low worst-case integration error in the spaces $\mathscr{H}_{\mathrm{sob},s,\boldsymbol{\gamma}}$ and $\mathscr{H}'_{\mathrm{sob},s,\mathbf{1},\boldsymbol{\gamma}}$, respectively.

Algorithm 1 **($L = K$)** *Let $m, s \in \mathbb{N}$, and $f \in \mathbb{Z}_b[x]$ (b a prime) be irreducible with $\deg(f) = m$ be given, and set $N := b^m$.*

(1) Set $g_1 = 1$.

(2) Find $\sigma_m^{(1)} \in \{ab^{-m} : 0 \le a < b^m\}$ to minimize

$$e^2\left(\mathscr{P}_{N,1}(1, f) \oplus \left(\sigma_m^{(1)}\right)^{\mathrm{simp}}, K\right) = -1 + \frac{1}{N^2} \sum_{n,h=0}^{N-1} K_{\gamma_1}(z_{n,1}, z_{h,1}),$$

where $z_{n,1}$ denotes the n-th point of $\mathscr{P}_{N,1}(1, f) \oplus \left(\sigma_m^{(1)}\right)^{\mathrm{simp}}$.

(3) For $d = 1, 2, \ldots, s-1$, suppose we already found $g_1, \ldots, g_d$ and $\sigma_m^{(1)}, \ldots, \sigma_m^{(d)}$. Proceed as follows.

(3a) Find $g_{d+1} \in G_{b,m} \setminus \{0\}$ to minimize

$$-\frac{\gamma_{d+1}}{2} \frac{1}{N^2} \sum_{n,h=0}^{N-1} \left(\prod_{j=1}^{d} K_{\gamma_j}(x_{n,j}, x_{h,j})\right) \sum_{k=1}^{N-1} \tau(k) \mathrm{wal}_k(x_{n,d+1} \ominus x_{h,d+1}),$$

where $x_{n,d+1}$ denotes the $(d+1)$-th component (obtained by the means of g_{d+1}) of the n-th point of

$$\mathscr{P}_{N,d+1}\left(\mathscr{P}_{N,d}((g_1, \ldots, g_d), f) \oplus \left(\left(\sigma_m^{(1)}, \ldots, \sigma_m^{(d)}\right)\right)^{\mathrm{simp}}, \mathscr{P}_{N,1}(g_{d+1}, f)\right).$$

(3b) Find $\sigma_m^{(d+1)} \in \{ab^{-m} : 0 \le a < b^m\}$ to minimize

$$\begin{aligned} & e^2\left(\mathscr{P}_{N,d+1}((g_1, \ldots, g_{d+1}), f) \oplus \left(\left(\sigma_m^{(1)}, \ldots, \sigma_m^{(d+1)}\right)\right)^{\mathrm{simp}}, K\right) = \\ &= \quad -1 + \frac{1}{N^2} \sum_{n,h=0}^{N-1} \left(\prod_{j=1}^{d} K_{\gamma_j}(x_{n,j}, x_{h,j})\right) \times \\ & \quad \times \left(1 + \gamma_{d+1}\left(\frac{(z_{n,d+1} - z_{h,d+1})^2 - |z_{n,d+1} - z_{h,d+1}|}{2} + \right.\right. \\ & \quad \left.\left. + \frac{1}{12} + \left(z_{n,d+1} - \tfrac{1}{2}\right)\left(z_{h,d+1} - \tfrac{1}{2}\right)\right)\right), \end{aligned}$$

where $z_{n,d+1}$ denotes the $(d+1)$-th component of the n-th point of

$$\mathscr{P}_{N,d+1}((g_1, \ldots, g_{d+1}), f) \oplus \left(\left(\sigma_m^{(1)}, \ldots, \sigma_m^{(d+1)}\right)\right)^{\mathrm{simp}}.$$

Algorithm 2 **($L = K'$)** *Let $m, s \in \mathbb{N}$, and $f \in \mathbb{Z}_b[x]$ (b a prime) be irreducible with $\deg(f) = m$ be given, and set $N := b^m$.*

(1) Set $g_1 = 1$.

(2) Find $\sigma_m^{(1)} \in \{ab^{-m} : 0 \le a < b^m\}$ to minimize

$$e^2\left(\mathscr{P}_{N,1}(1,f) \oplus \left(\sigma_m^{(1)}\right)^{\mathrm{simp}}, K'\right) = 1 + \frac{\gamma_1}{3} - \frac{2}{N}\sum_{n=0}^{N-1}\left(1 + \frac{\gamma_1}{2}\left(1 - z_{n,1}^2\right)\right) + \\ + \frac{1}{N^2}\sum_{n,h=0}^{N-1}\left(1 + \gamma_1 \min(1 - z_{n,1}, 1 - z_{h,1})\right),$$

where $z_{n,1}$ denotes the n-th point of $\mathscr{P}_{N,1} \oplus \left(\sigma_m^{(1)}\right)^{\mathrm{simp}}$.

(3) For $d = 1, 2, \ldots, s-1$, suppose we already found $g_1, \ldots, g_d$ and $\sigma_m^{(1)}, \ldots, \sigma_m^{(d)}$. Proceed as follows.

(3a) Find $g_{d+1} \in G_{b,m} \setminus \{0\}$ to minimize

$$-\frac{\gamma_{d+1}}{2}\frac{1}{N^2}\sum_{n,h=0}^{N-1}\left(\prod_{j=1}^{d}(1 + \gamma_j \min(1 - x_{n,j}, 1 - x_{h,j}))\right)\sum_{k=1}^{N-1}\tau(k)\mathrm{wal}_k(x_{n,d+1} \ominus x_{h,d+1}),$$

where $x_{n,d+1}$ denotes the $(d+1)$-th component (obtained by the means of g_{d+1}) of the n-th point of

$$\mathscr{P}_{N,d+1}\left(\mathscr{P}_{N,d}((g_1, \ldots, g_d), f) \oplus \left(\left(\sigma_m^{(1)}, \ldots, \sigma_m^{(d)}\right)\right)^{\mathrm{simp}}, \mathscr{P}_{N,1}(g_{d+1}, f)\right).$$

(3b) Find $\sigma_m^{(d+1)} \in \{ab^{-m} : 0 \le a < b^m\}$ to minimize

$$e^2\left(\mathscr{P}_{N,d+1}((g_1, \ldots, g_{d+1}), f) \oplus \left(\left(\sigma_m^{(1)}, \ldots, \sigma_m^{(d+1)}\right)\right)^{\mathrm{simp}}, K\right) = \\ = \prod_{j=1}^{d+1}\left(1 + \frac{\gamma_j}{3}\right) - \frac{2}{N}\sum_{n=0}^{N-1}\left(\prod_{j=1}^{d}\left(1 + \frac{\gamma_j}{2}\left(1 - x_{n,j}^2\right)\right)\right)\left(1 + \frac{\gamma_{d+1}}{2}\left(1 - z_{n,d+1}^2\right)\right) + \\ + \frac{1}{N^2}\sum_{n,h=0}^{N-1}\left(\prod_{j=1}^{d}(1 + \gamma_j \min(1 - x_{n,j}, 1 - x_{h,j}))\right) \times \\ \times (1 + \gamma_{d+1}\min(1 - z_{n,d+1}, 1 - z_{h,d+1})),$$

where $z_{n,d+1}$ denotes the $(d+1)$-th component of the n-th point of

$$\mathscr{P}_{N,d+1}((g_1, \ldots, g_{d+1}), f) \oplus \left(\left(\sigma_m^{(1)}, \ldots, \sigma_m^{(d+1)}\right)\right)^{\mathrm{simp}}.$$

Remark 1 *As in [3, Appendix B] it is easily shown that*

$$\sum_{k=1}^{b^m-1}\tau(k)\mathrm{wal}_k(y) = \begin{cases} -\frac{1}{3}\left(1 - \frac{1}{b^m}\right) & \text{if } y = 0, \\ -\frac{1}{3} + 2\frac{|y_{i_0}|(b - |y_{i_0}|)}{b^{i_0+1}} & \text{if } y_{i_0} \neq 0 \text{ and } y_i = 0\ \forall 1 \le i < i_0. \end{cases}$$

Therefore it follows that the cost of constructing an s-dimensional point set with Algorithm 1 or Algorithm 2 is of order $O(N^3 s^2)$ if $N = b^m$. This is in accordance with the findings in [20].

We can now show the following theorem.

Theorem 2 *Let $L \in \{K, K'\}$. Let $m, s \in \mathbb{N}$, and $f \in \mathbb{Z}_b[x]$ (b a prime) be irreducible with $\deg(f) = m$ be given, and set $N := b^m$. Then Algorithm 1 and Algorithm 2, respectively, construct a polynomial lattice rule $\mathscr{P}_{N,s}((g_1, \ldots, g_s), f)$ and a vector $\left(\sigma_m^{(1)}, \ldots, \sigma_m^{(s)}\right) \in \{ab^{-m} : 0 \le a < b^m\}^s$ such that*

$$e^2\left(\mathscr{P}_{N,d}((g_1, \ldots, g_d), f) \oplus \left(\left(\sigma_m^{(1)}, \ldots, \sigma_m^{(d)}\right)\right)^{\mathrm{simp}}, L\right) \le \frac{1}{N}\prod_{j=1}^{d}(1 + \gamma_j c(L))$$

for every $d \in \{1, \ldots, s\}$.

Proof. We show the result only for $L = K$. The result for $L = K'$ follows in the same way.

We show the result by induction on d. For $d = 1$, we obtain, similar to the proof of Theorem 1,

$$\begin{aligned}
&\mathbb{E}_{\sigma_m}\left[e^2\left(\mathscr{P}_{N,1}(1, f) \oplus \sigma_m^{(1)}, K\right)\right] = \\
&= -1 + \frac{1}{N^2}\sum_{n,h=0}^{N-1}\left(1 + \gamma_1\left(\frac{1}{6} - \frac{1}{2}\sum_{k=0}^{N-1}\tau(k)\mathrm{wal}_k(x_{n,1} \ominus x_{h,1})\right)\right) \\
&= -\frac{1}{N^2}\sum_{n,h=0}^{N-1}\frac{\gamma_1}{2}\sum_{k=1}^{N-1}\tau(k)\mathrm{wal}_k(x_{n,1} \ominus x_{h,1}) \\
&= \frac{1}{b}\frac{b^2-2}{3(b-1)}\left(1 - \frac{1}{b^m}\right)\frac{\gamma_1}{2N} - \frac{1}{N^2}\sum_{\substack{n,h=0\\ n\neq h}}^{N-1}\frac{\gamma_1}{2}\sum_{k=1}^{N-1}\tau(k)\mathrm{wal}_k(x_{n,1} \ominus x_{h,1}) \\
&= \frac{1}{b}\frac{b^2-2}{3(b-1)}\left(1 - \frac{1}{b^m}\right)\frac{\gamma_1}{2N} - \frac{\gamma_1}{2N^2}\sum_{\substack{n,h=0\\ n\neq h}}^{N-1}\sum_{k=1}^{N-1}\tau(k)\mathrm{wal}_k(x_{n,1} \ominus x_{h,1}) \\
&= \frac{1}{b}\frac{b^2-2}{3(b-1)}\left(1 - \frac{1}{b^m}\right)\frac{\gamma_1}{2N} - \frac{\gamma_1}{2N^2}\sum_{k=1}^{N-1}\tau(k)\sum_{\substack{n,h=0\\ n\neq h}}^{N-1}\mathrm{wal}_k\left(\nu_m\left(\frac{n \ominus l}{f}\right)\right) \\
&= \frac{1}{b}\frac{b^2-2}{3(b-1)}\left(1 - \frac{1}{b^m}\right)\frac{\gamma_1}{2N} + \frac{\gamma_1}{2N}\sum_{k=1}^{N-1}\tau(k) \\
&= \frac{\gamma_1}{N}\frac{1}{b}\frac{b^2-2}{3(b-1)}\left(1 - \frac{1}{b^m}\right) \\
&\le \frac{\gamma_1}{N}\frac{b+1}{9},
\end{aligned}$$

and the result follows for $d = 1$.

Suppose we have already shown the result for some fixed d, then the induction step to $d+1$ follows immediately by the proof of Theorem 1. □

3 Tractability

We now briefly discuss a concept stemming from complexity theory, namely that of *(polynomial) tractability* and *strong tractability*. As our discussion follows standard arguments, we only state a few crucial points regarding our results. For further information on tractability, we refer the interested reader to the monographs [15, 16].

For the following, let $\mathscr{H} \in \{\mathscr{H}_{\mathrm{sob},s,\boldsymbol{\gamma}}, \mathscr{H}'_{\mathrm{sob},s,\mathbf{1},\boldsymbol{\gamma}}\}$ and let L be the corresponding kernel, i.e., $L \in \{K, K'\}$. We first define the *initial error* of multivariate integration (i.e., the error without sampling a function) in $\mathscr{H}$ by

$$e_{0,s}(L) := \sup_{\substack{F \in \mathscr{H} \\ \|F\| \leq 1}} |I_s(F)|\,,$$

where $\|\cdot\|$ denotes the norm in $\mathscr{H}$ induced by the inner product. According to [4, Proposition 2.11] we have $e_{0,s}^2(L) = \int_{[0,1]^{2s}} L(\boldsymbol{x}, \boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}$ and hence for both spaces considered, it is easily checked that the initial error equals one.

The task we consider is to reduce the initial error by a factor of $\varepsilon \in (0,1)$. We define

$$N_{\min}(\varepsilon, s, L) := \min\{N \in \mathbb{N} : \ \exists\ \mathscr{P}_{N,s} : \ e(\mathscr{P}_{N,s}, L) \leq \varepsilon\}$$

and say that the integration problem in $\mathscr{H}$ is *(polynomially) QMC-tractable* if there exist nonnegative integers c, p, q such that

$$N_{\min}(\varepsilon, s, L) \leq c s^q \varepsilon^{-p}$$

holds for all $s \in \mathbb{N}$ and all $\varepsilon \in (0,1)$.

We further say that the integration problem in $\mathscr{H}$ is *strongly* (polynomially) QMC-tractable, if the above inequality holds with $q = 0$.

We now have the following result.

Theorem 3 *Let $L \in \{K, K'\}$.*

(1) Suppose that

$$\Sigma_1 := \limsup_{s \to \infty} \frac{\sum_{j=1}^{s} \gamma_j}{\log s} < \infty.$$

Then, for any $s, m \in \mathbb{N}$, $N = b^m$, the point set $\mathscr{P}_{N,s}$ constructed by Algorithm 1 or 2, respectively, yields

$$e(\mathscr{P}_{N,s}, L) \leq c_\delta s^{(\Sigma_1 + \delta) c(L)/2} N^{-1/2},$$

for any $\delta > 0$ where $c_\delta > 0$. In particular, we obtain QMC-tractability for the integration problem in $\mathscr{H} \in \{\mathscr{H}_{\mathrm{sob},s,\boldsymbol{\gamma}}, \mathscr{H}'_{\mathrm{sob},s,\mathbf{1},\boldsymbol{\gamma}}\}$.

(2) Suppose that

$$\Sigma_2 := \sum_{j=1}^{\infty} \gamma_j < \infty.$$

Then, for any $s, m \in \mathbb{N}$, $N = b^m$, the point set $\mathscr{P}_{N,s}$ constructed by Algorithm 1 or 2, respectively, yields

$$e(\mathscr{P}_{N,s}, L) \leq \exp\left(\Sigma_2 \frac{c(L)}{2}\right) N^{-1/2}.$$

In particular, we obtain strong QMC-tractability for the integration problem in $\mathscr{H} \in \{\mathscr{H}_{\mathrm{sob},s,\boldsymbol{\gamma}}, \mathscr{H}'_{\mathrm{sob},s,\mathbf{1},\boldsymbol{\gamma}}\}$.

Proof. Let γ_j be a sequence of nonnegative weights. Then we have

$$\prod_{j=1}^{s} (1 + c(L)\gamma_j) = \exp\left(\sum_{j=1}^{s} \log(1 + c(L)\gamma_j)\right) \leq \exp\left(c(L) \sum_{j=1}^{s} \gamma_j\right) = s^{c(L) \sum_{j=1}^{s} \gamma_j / \log s},$$

so the result follows by standard arguments. $\square$

4 Conclusion and Outlook

In this paper, we have shown how to explicitly construct digitally shifted polynomial lattice point sets that yield a low worst-case integration error in the spaces $\mathscr{H}_{\mathrm{sob},s,\boldsymbol{\gamma}}, \mathscr{H}'_{\mathrm{sob},s,\mathbf{1},\boldsymbol{\gamma}}$. Even though the results in [2] and [3] (see also [4]) give better error bounds, those are not as explicit as the results presented here as they regard the *mean square integration error*. The algorithms presented here, explicitly construct the digital shift involved in a component by component fashion. The task of closing the gap in the integration error between the explicit construction here and the less explicit constructions in the above-mentioned papers seems to be demanding and is left open for future research.

We also remark that the anchor $\mathbf{1}$ in $\mathscr{H}'_{\mathrm{sob},s,\mathbf{1},\boldsymbol{\gamma}}$ could be chosen more generally (cf. [20]), though this was, for the sake of simplicity, not explicitly done here.

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Authors' address:
Institut für Finanzmathematik, Universität Linz, Altenbergerstr. 69, 4040 Linz, Austria
E-mail: {`peter.kritzer,friedrich.pillichshammer`}`@jku.at`