An exact formula for the L_2 discrepancy of the shifted Hammersley point set

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Abstract

In this note we prove an exact formula for the L_2 discrepancy of the shifted two-dimensional Hammersley point set in base 2. Our formula shows that this quantity only depends on the number of zero digits in the dyadic expansion of the shift and the cardinality of the point set. Our result is the solution to an open problem stated by O. Strauch. Further we obtain the best shifts from our formula and we show that the L_2 discrepancy of the shifted two-dimensional Hammersley point set in base 2 satisfies a central limit theorem.

Keywords: L_2 discrepancy, Hammersley point set, digital shift.
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1 Introduction

For a point set $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ in the two-dimensional unit cube $[0,1)^2$ the L_2 discrepancy is defined as

$$L_2(\mathcal{P}) = \int_0^1 \int_0^1 \left| \frac{A([0,\alpha)\times[0,\beta),\mathcal{P})}{N} - \alpha\beta \right| \mathrm{d}\alpha\,\mathrm{d}\beta,$$

where $A(E,\mathcal{P}) = |\mathcal{P}\cap E|$ for a subset $E \subseteq [0,1)^2$. This is a measure for the irregularity of distribution of the point set $\mathcal{P}$ in $[0,1)^2$, see [1, 3, 8, 10, 14].

In this note we consider the L_2 discrepancy of the shifted Hammersley point set which has been considered in a number of papers [2, 4, 5, 6, 7, 12, 15] and which is defined as follows.

For $m \in \mathbb{N}$ let x, y be two dyadic m-bit numbers with dyadic expansions

$$x = \frac{x_1}{2} + \cdots + \frac{x_m}{2^m} \quad \text{and} \quad y = \frac{y_1}{2} + \cdots + \frac{y_m}{2^m}.$$

Define

$$x \oplus y = z = \frac{z_1}{2} + \cdots + \frac{z_m}{2^m},$$

where $z_i = x_i + y_i \pmod 2$, $1 \le i \le m$.

Further let $\phi_2 : \mathbb{N}_0 \to [0,1)$ be the van der Corput radical inverse function defined by

$$\phi_2(n) = \frac{n_0}{2} + \frac{n_1}{2^2} + \cdots$$

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if $n \in \mathbb{N}_0$ has dyadic expansion $n = n_0 + n_1 2 + \cdots$ (note that this expansion is in fact finite).

For a dyadic m-bit number σ the σ-shifted two-dimensional Hammersley point set $\mathcal{H}_m(\sigma)$ in base 2 is given by the points

$$\left(\phi_2(n), \frac{n}{2^m} \oplus \sigma\right), \qquad n = 0, \ldots, 2^m - 1.$$

Note that the point sets $\mathcal{H}_m(\sigma)$ are examples of a so-called $(0, m, 2)$-net in base 2 (for an introduction on the theory of (t, m, s)-nets we refer to [11]).

In [7, Theorem 2 and Theorem 3] the authors gave tight lower and upper bounds on the L_2 discrepancy of $\mathcal{H}_m(\sigma)$ depending on the number of zeros in the dyadic expansion of the shift σ. This lead to the conjecture that the L_2 discrepancy of a σ-shifted Hammersley point set only depends on the number of zero digits in the dyadic expansion of the shift and not on the distribution of these (opposed to the star discrepancy as proved in [6]). Furthermore, finding an exact formula for $L_2(\mathcal{H}_m(\sigma))$ remained an open problem in [7] as recently pointed out by O. Strauch [13, Problem 1.1.8] in the unsolved problem section of the journal Uniform Distribution Theory.

In this note we close the gap between the lower and upper bounds in [7, Theorem 2 and Theorem 3] and give an exact formula for $L_2(\mathcal{H}_m(\sigma))$ depending only on the number of zero digits in the dyadic expansion of the shift and m. This gives the answer to the question of O. Strauch and proves the conjecture mentioned above. From our result we easily deduce the best shifts and we show that the L_2 discrepancy of the shifted two-dimensional Hammersley point set in base 2 satisfies a central limit theorem.

2 Statement and Proof of the Results

We have

Theorem 1 *For $m \in \mathbb{N}$ let σ be a dyadic m-bit number and let l denote the number of zero digits in the dyadic expansion of σ. Then*

$$(2^m L_2(\mathcal{H}_m(\sigma)))^2 = \frac{m^2}{64} - \frac{19m}{192} - \frac{lm}{16} + \frac{l^2}{16} + \frac{l}{4} + \frac{3}{8} + \frac{m}{16 \cdot 2^m} - \frac{l}{8 \cdot 2^m} + \frac{1}{4 \cdot 2^m} - \frac{1}{72 \cdot 4^m}.$$

Remark 1 This formula generalizes the well known formula for the L_2 discrepancy of the unshifted two-dimensional Hammersley point set in base 2 which can be obtained by setting $l = m$, see [4, 12, 15]. Once again we point out that it only depends on m and the number of zero digits in the dyadic expansion of the shift σ.

The following easy consequence of Theorem 1 improves [7, Theorem 4].

Corollary 1 *For $m \geq 5$ we obtain the best L_2 discrepancy if we choose a shift with exactly $l = \left\lceil \frac{m-5}{2} + \frac{1}{2^m} \right\rceil$ zero digits in its dyadic expansion. Therefore it follows that*

$$\min_{\sigma\ m-bit} L_2(\mathcal{H}_m(\sigma)) = \begin{cases} \frac{1}{2^m}\left(\frac{5m}{192} + \frac{1}{8} + \frac{1}{2^{m+1}} - \frac{1}{72 \cdot 2^{2m}}\right)^{1/2} & \text{if } m \text{ is even,} \\ \frac{1}{2^m}\left(\frac{5m}{192} + \frac{9}{64} + \frac{7}{2^{m+4}} - \frac{1}{72 \cdot 2^{2m}}\right)^{1/2} & \text{if } m \text{ is odd.} \end{cases}$$

We can also show that the L_2 discrepancy of the shifted two-dimensional Hammersley point set satisfies a central limit theorem.

Corollary 2 *Let Z_m be the set of all dyadic m-bit numbers. For any real $x \geq \sqrt{\frac{5}{192}}$ we have*

$$\lim_{m\to\infty} \frac{1}{2^m} \# \left\{ \sigma \in Z_m : L_2(\mathcal{H}_m(\sigma)) \leq x \frac{\sqrt{m}}{2^m} \right\} = 2\Phi\left(\frac{192x^2-5}{3}\right) - 1,$$

where

$$\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} \mathrm{e}^{-\frac{t^2}{2}} \,\mathrm{d}t$$

denotes the normal distribution function.

For the proof of Theorem 1 we need the subsequent lemma for which we introduce a quantity that appears therein.

For integers $0 \leq u < m$ and dyadic m-bit numbers α, β and σ with dyadic expansions

$$\alpha = \frac{\alpha_1}{2} + \cdots + \frac{\alpha_m}{2^m}, \quad \beta = \frac{\beta_1}{2} + \cdots + \frac{\beta_m}{2^m} \quad \text{and} \quad \sigma = \frac{\sigma_1}{2} + \cdots + \frac{\sigma_m}{2^m}$$

we define

$$j(u) = \begin{cases} 0 & \text{if } u = 0, \\ 0 & \text{if } \alpha_{m+1-j} = \beta_j \oplus \sigma_j \\ & \text{for } j = 1, \ldots, u, \\ \max\{j \leq u : \alpha_{m+1-j} \neq \beta_j \oplus \sigma_j\} & \text{else,} \end{cases}$$

where here $\oplus$ denotes addition modulo 2. Further we set $\alpha_{m+1} := 0$. This function appears in the exact formula for the discrepancy function of $\mathcal{H}_m(\sigma)$. Since we do not need this formula here we refer to [7, Lemma 1] and [9, Theorem 1] for further details.

Lemma 1 *With the definitions from above we have*

$$\sum_{2^m\alpha=1}^{2^m-1} 2^m \alpha(\alpha_{m-u} \oplus \alpha_{m+1-j(u)}) = 2^{2m-2} - 2^{m-2} + 2^{m+u-2} - 2^{m-1} \sum_{j=1}^{u} (\gamma_j \oplus 1) 2^{j-1},$$

where $\oplus$ denotes addition modulo 2 and where $\gamma_j := \beta_j \oplus \sigma_j$. (Here and in the following $\sum_{2^m\alpha=1}^{2^m-1}$ means summation over all $\alpha > 0$ m-bit (in dyadic expansion).)

Proof. We have

$$\begin{aligned} \Sigma &:= \sum_{2^m\alpha=1}^{2^m-1} 2^m \alpha(\alpha_{m-u} \oplus \alpha_{m+1-j(u)}) \\ &= \sum_{\alpha_{m+1-u},\ldots,\alpha_m=0}^{1} \left(\sum_{\alpha_1,\ldots,\alpha_{m-u}=0}^{1} 2^m \alpha(\alpha_{m-u} \oplus \alpha_{m+1-j(u)}) \right). \end{aligned}$$

Since $j(u)$ only depends on $\alpha_{m+1-u},\ldots,\alpha_{m+1-j(u)}$ and not on $\alpha_1,\ldots,\alpha_{m-u}$ we have

$$\begin{aligned}
&\sum_{\alpha_1,\ldots,\alpha_{m-u}=0}^{1} 2^m\alpha(\alpha_{m-u}\oplus\alpha_{m+1-j(u)})\\
&= 2^m\sum_{\alpha_1,\ldots,\alpha_{m-u}=0}^{1}\left(\frac{\alpha_1}{2}+\cdots+\frac{\alpha_m}{2^m}\right)(\alpha_{m-u}\oplus\alpha_{m+1-j(u)})\\
&= 2^m\sum_{\alpha_1,\ldots,\alpha_{m-u}=0}^{1}\left(\frac{\alpha_1}{2}+\cdots+\frac{\alpha_{m-u}}{2^{m-u}}\right)(\alpha_{m-u}\oplus\alpha_{m+1-j(u)})\\
&\quad +2^m\left(\frac{\alpha_{m-u+1}}{2^{m-u+1}}+\cdots+\frac{\alpha_m}{2^m}\right)\sum_{\alpha_1,\ldots,\alpha_{m-u}=0}^{1}(\alpha_{m-u}\oplus\alpha_{m+1-j(u)})\\
&= 2^m\sum_{\alpha_1,\ldots,\alpha_{m-u}=0}^{1}\left(\frac{\alpha_1}{2}+\cdots+\frac{\alpha_{m-u}}{2^{m-u}}\right)(\alpha_{m-u}\oplus\alpha_{m+1-j(u)})\\
&\quad +2^m\left(\frac{\alpha_{m-u+1}}{2^{m-u+1}}+\cdots+\frac{\alpha_m}{2^m}\right)2^{m-u-1}.
\end{aligned}$$

We study

$$S(\alpha_{m+1-j(u)}) := \sum_{\alpha_1,\ldots,\alpha_{m-u}=0}^{1}\left(\frac{\alpha_1}{2}+\cdots+\frac{\alpha_{m-u}}{2^{m-u}}\right)(\alpha_{m-u}\oplus\alpha_{m+1-j(u)}).$$

(i) If $\alpha_{m+1-j(u)}=0$, then

$$\begin{aligned}
S(0) &= \sum_{\alpha_1,\ldots,\alpha_{m-u}=0}^{1}\left(\frac{\alpha_1}{2}+\cdots+\frac{\alpha_{m-u}}{2^{m-u}}\right)\alpha_{m-u}\\
&= \sum_{\alpha_1,\ldots,\alpha_{m-u-1}=0}^{1}\left(\frac{\alpha_1}{2}+\cdots+\frac{\alpha_{m-u-1}}{2^{m-u-1}}+\frac{1}{2^{m-u}}\right)\\
&= \frac{1}{2}+\sum_{a=0}^{2^{m-u-1}-1}\frac{a}{2^{m-u-1}} = 2^{m-u-2}.
\end{aligned}$$

(ii) If $\alpha_{m+1-j(u)}=1$, then we obtain, in a similar way,

$$S(1) = 2^{m-u-2}-\frac{1}{2}.$$

Together we have

$$S(\alpha_{m+1-j(u)}) = 2^{m-u-2}-\frac{\alpha_{m+1-j(u)}}{2}.$$

Hence

$$\begin{aligned}
&\sum_{\alpha_1,\ldots,\alpha_{m-u}=0}^{1} 2^m\alpha(\alpha_{m-u}\oplus\alpha_{m+1-j(u)})\\
&= 2^m\left(2^{m-u-2}-\frac{\alpha_{m+1-j(u)}}{2}\right)+2^{2m-u-1}\left(\frac{\alpha_{m-u+1}}{2^{m-u+1}}+\cdots+\frac{\alpha_m}{2^m}\right)
\end{aligned}$$

and therefore

$$\begin{aligned}\Sigma &= \sum_{\alpha_{m+1-u},\ldots,\alpha_m=0}^{1}\left(2^m\left(2^{m-u-2}-\frac{\alpha_{m+1-j(u)}}{2}\right)+2^{2m-u-1}\left(\frac{\alpha_{m-u+1}}{2^{m-u+1}}+\cdots+\frac{\alpha_m}{2^m}\right)\right)\\ &= 2^{2m-u-2}2^u+2^{2m-u-1}\sum_{a=0}^{2^u-1}\frac{a}{2^m}-2^{m-1}\sum_{\alpha_{m+1-u},\ldots,\alpha_m=0}^{1}\alpha_{m+1-j(u)}\\ &= 2^{2m-2}+2^m\frac{2^u-1}{4}-2^{m-1}\sum_{\alpha_{m+1-u},\ldots,\alpha_m=0}^{1}\alpha_{m+1-j(u)}.\end{aligned}$$

We evaluate the last sum in the above expression.

$$\begin{aligned}\sum_{\alpha_{m+1-u},\ldots,\alpha_m=0}^{1}\alpha_{m+1-j(u)} &= \sum_{j=0}^{u}\sum_{\substack{\alpha_{m+1-u},\ldots,\alpha_m=0\\ j(u)=j}}^{1}\alpha_{m+1-j}\\ &= \sum_{\substack{\alpha_{m+1-u},\ldots,\alpha_m=0\\ j(u)=0}}^{1}\alpha_{m+1}+\sum_{j=1}^{u}\sum_{\substack{\alpha_{m+1-u},\ldots,\alpha_m=0\\ j(u)=j}}^{1}\alpha_{m+1-j}\\ &= 0+\sum_{j=1}^{u}\sum_{\alpha_{m+2-j},\ldots,\alpha_m=0}^{1}\sum_{\substack{\alpha_{m+1-u}=\gamma_u\\ \vdots\\ \alpha_{m-j}=\gamma_{j+1}\\ \alpha_{m+1-j}=\gamma_j\oplus 1}}\alpha_{m+1-j}\\ &= \sum_{j=1}^{u}(\gamma_j\oplus 1)2^{j-1}.\end{aligned}$$

We obtain

$$\Sigma = 2^{2m-2}+2^m\frac{2^u-1}{4}-2^{m-1}\sum_{j=1}^{u}(\gamma_j\oplus 1)2^{j-1}$$

and the result follows. □

Now we give the proof of Theorem 1. Since large parts of the proof are identical with the proof of Theorem 2 in [7], we restrict ourselves to demonstrating only the new parts of our argument in greater detail. For further details, we refer the interested reader to [7].

Proof. For short we define $\Delta(\alpha,\beta)=A([0,\alpha)\times[0,\beta),\mathcal{P})-N\alpha\beta$, where $N=|\mathcal{P}|$. Then we have

$$\begin{aligned}\left(2^m L_2(\mathcal{H}_m(\sigma))\right)^2 &= \int_0^1\int_0^1\left(\Delta(\alpha,\beta)\right)^2\,\mathrm{d}\alpha\,\mathrm{d}\beta\\ &= \int_0^{1-2^{-m}}\int_0^{1-2^{-m}}\left(\Delta(\alpha,\beta)\right)^2\,\mathrm{d}\alpha\,\mathrm{d}\beta+\int_0^{1-2^{-m}}\int_{1-2^{-m}}^1\left(\Delta(\alpha,\beta)\right)^2\,\mathrm{d}\alpha\,\mathrm{d}\beta\\ &\quad+\int_{1-2^{-m}}^1\int_0^{1-2^{-m}}\left(\Delta(\alpha,\beta)\right)^2\,\mathrm{d}\alpha\,\mathrm{d}\beta+\int_{1-2^{-m}}^1\int_{1-2^{-m}}^1\left(\Delta(\alpha,\beta)\right)^2\,\mathrm{d}\alpha\,\mathrm{d}\beta\\ &=: I_1+I_2+I_3+I_4.\end{aligned}$$

In [7, Proof of Theorem 2] it was shown that

$$I_2 = I_3 = \frac{25}{36 \cdot 2^m} - \frac{5}{9 \cdot 4^m} - \frac{25}{36 \cdot 4^m} + \frac{2}{3 \cdot 8^m} - \frac{1}{9 \cdot 16^m}$$

and

$$I_4 = \frac{7}{6 \cdot 4^m} + \frac{1}{9 \cdot 16^m} - \frac{2}{3 \cdot 8^m}.$$

In [7] the authors gave lower and upper bounds on I_1 which then lead to lower and upper bounds on $L_2(\mathcal{H}_m(\sigma))$. Here we give the precise value of I_1. From [7, Proof of Theorem 2] we find that

$$\begin{aligned} I_1 &= \frac{1}{2^{2m}} \sum_{a=1}^{2^m-1} \sum_{b=1}^{2^m-1} \Delta\left(\frac{a}{2^m}, \frac{b}{2^m}\right)^2 + \\ &\quad + 2^{2m} \sum_{a=1}^{2^m-1} \sum_{b=1}^{2^m-1} \int_{\frac{a-1}{2^m}}^{\frac{a}{2^m}} \int_{\frac{b-1}{2^m}}^{\frac{b}{2^m}} \left(\frac{ab}{2^{2m}} - \alpha\beta\right)^2 \mathrm{d}\alpha\, \mathrm{d}\beta + \\ &\quad + 2^{m+1} \sum_{a=1}^{2^m-1} \sum_{b=1}^{2^m-1} \int_{\frac{a-1}{2^m}}^{\frac{a}{2^m}} \int_{\frac{b-1}{2^m}}^{\frac{b}{2^m}} \Delta\left(\frac{a}{2^m}, \frac{b}{2^m}\right) \left(\frac{ab}{2^{2m}} - \alpha\beta\right) \mathrm{d}\alpha\, \mathrm{d}\beta \\ &=: \Sigma_1 + \Sigma_2 + \Sigma_3. \end{aligned}$$

By [7, Lemma 6],

$$\Sigma_1 = \frac{1}{576}(9m^2 + 15m - 36lm + 36l^2 + 16 - 4^{2-m}).$$

Analyzing Σ_2 is a matter of straightforward computation and yields

$$\Sigma_2 = -\frac{1}{72 \cdot 16^m}(2^m - 1)^2(32 \cdot 2^m - 25 \cdot 4^m - 8).$$

So it remains to deal with Σ_3. Here, we find that

$$\begin{aligned} \frac{1}{2^m}\Sigma_3 &= 2 \sum_{a=1}^{2^m-1} \sum_{b=1}^{2^m-1} \Delta\left(\frac{a}{2^m}, \frac{b}{2^m}\right) \frac{ab}{2^{2m}} \frac{1}{2^{2m}} \\ &\quad - 2 \sum_{a=1}^{2^m-1} \sum_{b=1}^{2^m-1} \Delta\left(\frac{a}{2^m}, \frac{b}{2^m}\right) \frac{1}{4} \frac{1}{2^{4m}} (4ab - 2(a+b) + 1) \\ &= \frac{1}{2^{4m}} \sum_{a=1}^{2^m-1} \sum_{b=1}^{2^m-1} (a+b) \Delta\left(\frac{a}{2^m}, \frac{b}{2^m}\right) \\ &\quad - \frac{1}{2^{4m+1}} \sum_{a=1}^{2^m-1} \sum_{b=1}^{2^m-1} \Delta\left(\frac{a}{2^m}, \frac{b}{2^m}\right) \\ &=: \Sigma_4 - \Sigma_5. \end{aligned}$$

We start with Σ_4.

$$\begin{aligned} \Sigma_4 &= \frac{1}{2^{4m}} \sum_{a=1}^{2^m-1} \sum_{b=1}^{2^m-1} a \Delta\left(\frac{a}{2^m}, \frac{b}{2^m}\right) + \frac{1}{2^{4m}} \sum_{a=1}^{2^m-1} \sum_{b=1}^{2^m-1} b \Delta\left(\frac{a}{2^m}, \frac{b}{2^m}\right) \\ &=: \frac{1}{2^{4m}} \left(\Sigma_{4,1} + \Sigma_{4,2}\right). \end{aligned}$$

Using [7, Lemma 1], we find that

$$\begin{aligned}\Sigma_{4,1} &= \sum_{\substack{u=0\\ \sigma_{u+1}=0}}^{m-1}\sum_{2^m\beta=1}^{2^m-1}\|2^u\beta\|\sum_{2^m\alpha=1}^{2^m-1}2^m\alpha(\alpha_{m-u}\oplus\alpha_{m+1-j(u)})\\ &\quad-\sum_{\substack{u=0\\ \sigma_{u+1}=1}}^{m-1}\sum_{2^m\beta=1}^{2^m-1}\|2^u\beta\|\sum_{2^m\alpha=1}^{2^m-1}2^m\alpha(\alpha_{m-u}\oplus\alpha_{m+1-j(u)}),\end{aligned}$$

where $\|x\|$ denotes the distance to the nearest integer of a real number x.

By Lemma 1, we obtain

$$\begin{aligned}\Sigma_{4,1} &= \sum_{\substack{u=0\\ \sigma_{u+1}=0}}^{m-1}\sum_{2^m\beta=1}^{2^m-1}\|2^u\beta\|\left(2^{2m-2}+2^{m-2}+2^{m+u-2}-2^{m-1}\sum_{j=1}^{u}(\gamma_j\oplus 1)2^{j-1}\right)\\ &\quad-\sum_{\substack{u=0\\ \sigma_{u+1}=1}}^{m-1}\sum_{2^m\beta=1}^{2^m-1}\|2^u\beta\|\left(2^{2m-2}+2^{m-2}+2^{m+u-2}-2^{m-1}\sum_{j=1}^{u}(\gamma_j\oplus 1)2^{j-1}\right),\end{aligned}$$

which, in turn, by [7, Lemma 4] equals

$$\begin{aligned}&\frac{2^m}{4}\left((l-(m-l))2^{2m-2}+((m-l)-l)2^{m-2}+\sum_{\substack{u=0\\ \sigma_{u+1}=0}}^{m-1}2^{m+u-2}-\sum_{\substack{u=0\\ \sigma_{u+1}=1}}^{m-1}2^{m+u-2}\right)\\ &+2^{m-1}\sum_{\substack{u=0\\ \sigma_{u+1}=1}}^{m-1}\sum_{2^m\beta=1}^{2^m-1}\|2^u\beta\|\sum_{j=1}^{u}(\gamma_j\oplus 1)2^{j-1}-2^{m-1}\sum_{\substack{u=0\\ \sigma_{u+1}=0}}^{m-1}\sum_{2^m\beta=1}^{2^m-1}\|2^u\beta\|\sum_{j=1}^{u}(\gamma_j\oplus 1)2^{j-1}.\end{aligned}$$

For given $u\in\{0,\dots,m-1\}$ let us study the expression

$$\begin{aligned}\sum_{2^m\beta=1}^{2^m-1}\|2^u\beta\|\sum_{j=1}^{u}(\gamma_j\oplus 1)2^{j-1} &= \sum_{\beta_1,\dots,\beta_m=0}^{1}\|2^u\beta\|\sum_{j=1}^{u}(\beta_j\oplus\sigma_j\oplus 1)2^{j-1}\\ &= \sum_{j=1}^{u}2^{j-1}\sum_{\beta_1,\dots,\beta_m=0}^{1}(\beta_j\oplus\sigma_j\oplus 1)\,\|2^u\beta\|\\ &= \sum_{j=1}^{u}2^{j-1}\sum_{\substack{\beta_1,\dots,\beta_{j-1},\beta_{j+1},\dots,\beta_u=0\\ \beta_j=\sigma_j}}^{1}\ \sum_{\beta_{u+1},\dots,\beta_m=0}^{1}\|2^u\beta\|.\end{aligned}$$

Since the value of $\|2^u\beta\|$ does not depend on $\beta_1,\dots,\beta_u$, the latter expression equals

$$\begin{aligned}&\sum_{j=1}^{u}2^{j-1}2^{u-1}\sum_{\beta_{u+1},\dots,\beta_m=0}^{1}\|2^u\beta\| =\\ &= \sum_{j=1}^{u}2^{j-1}2^{u-1}\left(\sum_{\beta_{u+2},\dots,\beta_m=0}^{1}\left(\sum_{i=2}^{m-u}\frac{\beta_{u+i}}{2^i}+\frac{1}{2}-\sum_{i=2}^{m-u}\frac{\beta_{u+i}}{2^i}\right)\right)\\ &= 2^{m-3}(2^u-1).\end{aligned}$$

Hence,

$$\begin{aligned}
\Sigma_{4,1} &= \frac{2^m}{4}((l-(m-l))2^{2m-2}+((m-l)-l)2^{m-2}) \\
&\quad +\frac{2^m}{4}\sum_{\substack{u=0\\ \sigma_{u+1}=0}}^{m-1} 2^{m+u-2} - \frac{2^m}{4}\sum_{\substack{u=0\\ \sigma_{u+1}=1}}^{m-1} 2^{m+u-2} \\
&\quad +2^{m-1}\sum_{\substack{u=0\\ \sigma_{u+1}=1}}^{m-1} 2^{m-3}(2^u-1) - 2^{m-1}\sum_{\substack{u=0\\ \sigma_{u+1}=0}}^{m-1} 2^{m-3}(2^u-1) \\
&= \frac{2^m}{4}((l-(m-l))2^{2m-2}+((m-l)-l)2^{m-2}) \\
&\quad +\frac{2^m}{4}\sum_{\substack{u=0\\ \sigma_{u+1}=0}}^{m-1} 2^{m+u-2} - \frac{2^m}{4}\sum_{\substack{u=0\\ \sigma_{u+1}=1}}^{m-1} 2^{m+u-2} + \frac{2^m}{4}\sum_{\substack{u=0\\ \sigma_{u+1}=1}}^{m-1} 2^{m+u-2} - \frac{2^m}{4}\sum_{\substack{u=0\\ \sigma_{u+1}=0}}^{m-1} 2^{m+u-2} \\
&\quad +2^{2m-4}(l-(m-l)) \\
&= (2l-m)2^{3m-4}.
\end{aligned}$$

It follows that

$$\frac{1}{2^{4m}}\Sigma_{4,1} = \frac{1}{2^m}\left(\frac{l}{8}-\frac{m}{16}\right).$$

Furthermore, it has been shown in [7] that we also have

$$\frac{1}{2^{4m}}\Sigma_{4,2} = \frac{1}{2^m}\left(\frac{l}{8}-\frac{m}{16}\right),$$

such that we arrive at

$$\Sigma_4 = \frac{1}{2^{m-1}}\left(\frac{l}{8}-\frac{m}{16}\right).$$

On the other hand we already had in [7]

$$\Sigma_5 = \frac{1}{2^{2m}}\left(\frac{l}{8}-\frac{m}{16}\right),$$

thus,

$$\frac{1}{2^m}\Sigma_3 = \frac{2^{m+1}-1}{4^m}\left(\frac{l}{8}-\frac{m}{16}\right)$$

Putting all results together, we obtain

$$\begin{aligned}
I_1+I_2+I_3+I_4 &= \frac{m^2}{64}-\frac{19m}{192}-\frac{lm}{16}+\frac{l^2}{16}+\frac{l}{4}+\frac{3}{8} \\
&\quad +\frac{m}{16\cdot 2^m}-\frac{l}{8\cdot 2^m}+\frac{1}{4\cdot 2^m}-\frac{1}{72\cdot 4^m}
\end{aligned}$$

as claimed. □

Finally we give the proof of Corollary 2.

Proof. We denote the right hand side of the formula in Theorem 1 by $d(m,l)$. Then we have

$$\frac{1}{2^m}\#\left\{\sigma\in Z_m : L_2(\mathcal{H}_m(\sigma))\le x\frac{\sqrt{m}}{2^m}\right\}=\frac{1}{2^m}\sum_{\substack{l=0\\ \sqrt{d(m,l)}\le x\cdot\sqrt{m}}}^{m}\binom{m}{l}$$

We have $\sqrt{d(m,l)}\le x\cdot\sqrt{m}$ iff $a_m(x)\le l\le b_m(x)$, where

$$a_m(x)=\frac{m}{2}-2+\frac{1}{2^m}-\frac{\sqrt{44-288\cdot 2^m-72\cdot 2^{2m}+3\cdot 2^{2m}m(192x^2-5)}}{6\cdot 2^m}$$

and

$$b_m(x)=\frac{m}{2}-2+\frac{1}{2^m}+\frac{\sqrt{44-288\cdot 2^m-72\cdot 2^{2m}+3\cdot 2^{2m}m(192x^2-5)}}{6\cdot 2^m}.$$

Therefore (at least for m large enough)

$$\frac{1}{2^m}\#\left\{\sigma\in Z_m : L_2(\mathcal{H}_m(\sigma))\le x\frac{\sqrt{m}}{2^m}\right\}=\frac{1}{2^m}\sum_{a_m(x)\le l\le b_m(x)}\binom{m}{l}.$$

Since for $x\ge\sqrt{\frac{5}{192}}$ we have

$$\lim_{x\to\infty}\frac{a_m(x)-\frac{m}{2}}{\sqrt{\frac{m}{4}}}=-\sqrt{\frac{192x^2-5}{3}}$$

and

$$\lim_{x\to\infty}\frac{b_m(x)-\frac{m}{2}}{\sqrt{\frac{m}{4}}}=+\sqrt{\frac{192x^2-5}{3}}$$

the result follows from the central limit theorem. □

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