Uniform distribution of sequences connected with the weighted sum-of-digits function

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Abstract

In this paper we consider sequences which are connected with the so-called weighted q-ary sum-of-digits function and give an *if and only if* condition under which such sequences are uniformly distributed modulo one. The sequences considered here contain the q-ary van der Corput sequence as well as the $(n\boldsymbol{\alpha})$-sequences as special cases.

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1 Introduction

A sequence $(\boldsymbol{x}_n)_{n\ge 0}$ in the d-dimensional unit-cube is said to be *uniformly distributed modulo one* if for all intervals $[\boldsymbol{a},\boldsymbol{b})\subseteq[0,1)^d$ we have

$$\lim_{N\to\infty}\frac{\#\{n\,:\,0\le n<N, \boldsymbol{x}_n\in[\boldsymbol{a},\boldsymbol{b})\}}{N}=\lambda_d([\boldsymbol{a},\boldsymbol{b})),$$

where λ_d denotes the d-dimensional Lebesgue measure. An excellent introduction into this topic can be found in the book of Kuipers and Niederreiter [7] or in the book of Drmota and Tichy [4].

In this paper we consider the uniform distribution properties of special sequences which are connected with the weighted sum-of-digits function and which are generalizations of many well known sequences.

Let $\gamma=(\gamma_0,\gamma_1,\ldots)$ be a sequence in $\mathbb{R}$ and let $q\in\mathbb{N}$, $q\ge 2$. For $n\in\mathbb{N}_0$ with base q representation $n=n_0+n_1q+n_2q^2+\cdots$ we define the *weighted q-ary sum-of-digits function* by

$$s_\gamma(n):=\gamma_0n_0+\gamma_1n_1+\gamma_2n_2+\cdots.$$

We remark that the weighted q-ary sum-of-digits function is a q-additive function, but it is not strongly q-additive (unless the weight-sequence γ is constant); see [5, 6] or [4] for the notion of (strongly) q-additive functions.

For $d\in\mathbb{N}$ let $\boldsymbol{\gamma}=(\boldsymbol{\gamma}_0,\boldsymbol{\gamma}_1,\ldots)$ be a sequences in $\mathbb{R}^d$ with $\boldsymbol{\gamma}_j=(\gamma_j^{(1)},\ldots,\gamma_j^{(d)})$, i.e., $\gamma_j^{(k)}$ denotes the k-th component of the j-th element of the sequence $\boldsymbol{\gamma}$. For $k\in\{1,\ldots,d\}$ let $\gamma^{(k)}=(\gamma_0^{(k)},\gamma_1^{(k)},\ldots)$ be the k-th coordinate sequence in $\mathbb{R}$. For $n\in\mathbb{N}_0$ define

$$s_{\boldsymbol{\gamma}}(n):=(s_{\gamma^{(1)}}(n),\ldots,s_{\gamma^{(d)}}(n)).$$

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Now we consider the d-dimensional sequence

$$(\{s_{\boldsymbol{\gamma}}(n)\})_{n\geq 0}, \tag{1}$$

where $\{\boldsymbol{x}\}$ denotes the fractional part of the vector $\boldsymbol{x}$ (applied component-wise), and ask under which conditions on the weight-sequence $\boldsymbol{\gamma}$ the sequence (1) is uniformly distributed modulo one?

Observe that the definition of the sequence in (1) covers many well known and extensively studied sequences as, for example:

1. If $d = 1$ and $\gamma_j = q^{-j-1}$ (here we simply write γ_j instead of $\gamma_j^{(1)}$) for all $j \in \mathbb{N}_0$, then the sequence $(\{s_\gamma(n)\})_{n\geq 0}$ is the q-ary van der Corput sequence which is of course well known to be uniformly distributed modulo one. See, for example, [7, 11].

2. If $\boldsymbol{\gamma}_j = q^j \boldsymbol{\alpha}$ for all $j \in \mathbb{N}_0$ with $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$, then we obtain the sequence $(\{n\boldsymbol{\alpha}\})_{n\geq 1}$ which is well known to be uniformly distributed modulo one if and only if $1, \alpha_1, \ldots, \alpha_d$ are linearly independent over $\mathbb{Q}$. See, for example, [4, 7, 12].

3. If $\boldsymbol{\gamma}_j = \boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$ for all $j \in \mathbb{N}_0$, then we obtain the sequence $(\{s(n)\boldsymbol{\alpha}\})_{n\geq 1}$, where $s(\cdot)$ denotes the classical, i.e. unweighted q-ary sum-of-digits function. In the case $d = 1$ it was shown by Mendès France [10] and later by Coquet [1] that the sequence $(\{s(n)\alpha\})_{n\geq 1}$ is uniformly distributed modulo one if and only if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. See also [2, 3] and the references therein. We remark that this result even holds if the q-ary sum-of-digits function is replaced by an arbitrary strongly q-additive function; see [4].

4. If $d = 1$ and $\gamma_j = r_j \alpha$ (again we simply write γ_j instead of $\gamma_j^{(1)}$) with $r_j \in \mathbb{Z}$ for all $j \in \mathbb{N}_0$ where $\alpha \in \mathbb{R}$, then the following was proved (in fact in a more general setting) by Larcher [8]: the sequence $(\{s_\gamma(n)\})_{n\geq 0}$ is uniformly distributed modulo one if and only if
$$\sum_{k=0}^{\infty} \|hr_k\alpha\|^2 = \infty \quad \forall h \in \mathbb{N},$$
where for $x \in \mathbb{R}$, $\|x\| = \min_{k\in\mathbb{Z}} |x - k|$.

It is the aim of this paper to characterize the weight-sequences $\boldsymbol{\gamma} : \mathbb{N}_0 \to \mathbb{R}^d$ for which the sequence (1) is uniformly distributed modulo one. As corollary we obtain that the sequence (1) is uniformly distributed modulo one for almost all weight-sequences $\boldsymbol{\gamma} : \mathbb{N}_0 \to [0, 1)^d$. We close the paper with an interesting open question.

Throughout the paper let the base $q \in \mathbb{N}$, $q \geq 2$, and the dimension $d \in \mathbb{N}$ be fixed. By $\langle \cdot, \cdot \rangle$ we denote the usual inner product in $\mathbb{R}^d$. As above $\| \cdot \|$ denotes the *distance-to-the-nearest-integer function.*

2 Statement and proof of the results

The following theorem gives a full characterization of the sequences $\boldsymbol{\gamma} : \mathbb{N}_0 \to \mathbb{R}^d$ for which the sequence (1) is uniformly distributed modulo one. The proof is based on easy estimates for exponential sums and Weyl's criterion (see, for example, [4, 7]).

Theorem 1 *The sequence* $(\{s_{\boldsymbol{\gamma}}(n)\})_{n\geq 0}$ *is uniformly distributed modulo one if and only if for every* $\boldsymbol{h} \in \mathbb{Z}^d \setminus \{\boldsymbol{0}\}$ *one of the following properties hold:*

Either

$$\sum_{\substack{k=0 \\ \langle \boldsymbol{h}, \boldsymbol{\gamma}_k \rangle q \notin \mathbb{Z}}}^{\infty} \|\langle \boldsymbol{h}, \boldsymbol{\gamma}_k \rangle\|^2 = \infty$$

or there exists a $k \in \mathbb{N}_0$ *such that* $\langle \boldsymbol{h}, \boldsymbol{\gamma}_k \rangle \notin \mathbb{Z}$ *and* $\langle \boldsymbol{h}, \boldsymbol{\gamma}_k \rangle q \in \mathbb{Z}$.

Of course the condition from our theorem covers all special cases from the list of examples in Section 1. Before we give the proof of the theorem let us consider two of them.

Example 1 Consider the q-ary van der Corput sequence, i.e., $d = 1$ and $\gamma_j = q^{-j-1}$ for all $j \in \mathbb{N}_0$. For $h \in \mathbb{Z} \setminus \{0\}$ let $k \in \mathbb{N}_0$ be maximal such that $q^k | h$. Then $hq^{-k-1} \notin \mathbb{Z}$ and $hq^{-k} \in \mathbb{Z}$. Hence from Theorem 1 we obtain the well known fact that the q-ary van der Corput sequence is uniformly distributed modulo one.

Example 2 Let $\boldsymbol{\gamma}_j = \boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$ for all $j \in \mathbb{N}_0$. Then for any $\boldsymbol{h} \in \mathbb{Z}^d \setminus \{\boldsymbol{0}\}$ we have

$$\sum_{\substack{k=0 \\ \langle \boldsymbol{h}, \boldsymbol{\gamma}_k \rangle q \notin \mathbb{Z}}}^{\infty} \|\langle \boldsymbol{h}, \boldsymbol{\gamma}_k \rangle\|^2 = \sum_{k=0}^{\infty} \|\langle \boldsymbol{h}, \boldsymbol{\alpha} \rangle\|^2 = \infty$$

if and only if $\langle \boldsymbol{h}, \boldsymbol{\alpha} \rangle \notin \mathbb{Z}$. But the last condition holds if and only if $1, \alpha_1, \ldots, \alpha_d$ are linearly independent over $\mathbb{Q}$.

For the proof of Theorem 1 we need the following easy lemmas. For the sake of completeness we give short verifications of these results.

Lemma 1 *Let* $x_0, \ldots, x_{q-1} \in \left(-\frac{1}{2}, \frac{1}{2}\right]$ *and define* $x := \max_{0 \leq j < q} |x_j|$. *Then we have*

$$\left| \sum_{j=0}^{q-1} \mathrm{e}^{2\pi \mathtt{i} x_j} \right| \geq q(1 - 4\pi^2 x^2).$$

Proof. We have

$$\left| \sum_{j=0}^{q-1} \mathrm{e}^{2\pi \mathtt{i} x_j} \right| \geq \left| \mathrm{Re} \left(\sum_{j=0}^{q-1} \mathrm{e}^{2\pi \mathtt{i} x_j} \right) \right| = \left| \sum_{j=0}^{q-1} \cos(2\pi x_j) \right| \geq q \cos(2\pi x) \geq q(1 - 4\pi^2 x^2).$$

□

Lemma 2 *For any* $x \in \mathbb{R}$ *we have*

$$\left| \sum_{n=0}^{q-1} \mathrm{e}^{2\pi \mathtt{i} x n} \right| \leq q - 4\|x\|^2.$$

Proof. We have

$$\begin{aligned}\left|\sum_{n=0}^{q-1} \mathrm{e}^{2\pi \mathtt{i} x n}\right| &\leq \left|1+\mathrm{e}^{2\pi \mathtt{i} x}\right| + q - 2 = 2\cos(\pi\|x\|) + q - 2 \\ &\leq 2\left(1 - \frac{\pi^2\|x\|^2}{\pi}\right) + q - 2 \leq q - 4\|x\|^2.\end{aligned}$$

□

Proof of Theorem 1. Let $\boldsymbol{h} \in \mathbb{Z}^d \setminus \{\boldsymbol{0}\}$. By Lemma 2 we have

$$\left|\sum_{n=0}^{q-1} \mathrm{e}^{2\pi \mathtt{i} \langle \boldsymbol{h}, \boldsymbol{\gamma}_k\rangle n}\right| \leq q - 4\|\langle \boldsymbol{h}, \boldsymbol{\gamma}_k\rangle\|^2.$$

But if $\langle \boldsymbol{h}, \boldsymbol{\gamma}_k\rangle \notin \mathbb{Z}$ and $\langle \boldsymbol{h}, \boldsymbol{\gamma}_k\rangle q \in \mathbb{Z}$ we also have

$$\left|\sum_{n=0}^{q-1} \mathrm{e}^{2\pi \mathtt{i} \langle \boldsymbol{h}, \boldsymbol{\gamma}_k\rangle n}\right| = 0.$$

For $j \in \mathbb{N}_0$ we have

$$\left|\frac{1}{q^j}\sum_{n=0}^{q^j-1} \mathrm{e}^{2\pi \mathtt{i} \langle \boldsymbol{h}, s_{\boldsymbol{\gamma}}(n)\rangle}\right| = \frac{1}{q^j}\prod_{k=0}^{j-1}\left|\sum_{n=0}^{q-1} \mathrm{e}^{2\pi \mathtt{i} \langle \boldsymbol{h}, \boldsymbol{\gamma}_k\rangle n}\right| \leq \prod_{\substack{k=0\\ \langle \boldsymbol{h}, \boldsymbol{\gamma}_k\rangle q \notin \mathbb{Z}}}^{j-1} \frac{q - 4\|\langle \boldsymbol{h}, \boldsymbol{\gamma}_k\rangle\|^2}{q} \prod_{\substack{k=0\\ \langle \boldsymbol{h}, \boldsymbol{\gamma}_k\rangle \notin \mathbb{Z} \wedge \langle \boldsymbol{h}, \boldsymbol{\gamma}_k\rangle q \in \mathbb{Z}}}^{j-1} 0.$$

Here and later on an empty product is considered to be one.

Let $N \in \mathbb{N}$ with base q representation $N = N_0 + N_1 q + \cdots + N_m q^m$ with $N_m \neq 0$. For $0 \leq j \leq m$ set $N(j) := N_j q^j + \cdots + N_m q^m$. Define $g(n) := \mathrm{e}^{2\pi \mathtt{i} \langle \boldsymbol{h}, s_{\boldsymbol{\gamma}}(n)\rangle}$. Then

$$\sum_{n=0}^{N-1} \mathrm{e}^{2\pi \mathtt{i} \langle \boldsymbol{h}, s_{\boldsymbol{\gamma}}(n)\rangle} = \sum_{n=0}^{N(m)-1} g(n) + \sum_{j=0}^{m-1}\sum_{n=N(j+1)}^{N(j)-1} g(n).$$

Now

$$\sum_{n=0}^{N(m)-1} g(n) = \sum_{l=0}^{N_m-1}\sum_{n=lq^m}^{(l+1)q^m-1} \mathrm{e}^{2\pi \mathtt{i} \langle \boldsymbol{h}, n_0\boldsymbol{\gamma}_0 + \cdots + n_m\boldsymbol{\gamma}_m\rangle} = \sum_{l=0}^{N_m-1} g(lq^m)\sum_{n=0}^{q^m-1} g(n),$$

and

$$\sum_{n=N(j+1)}^{N(j)-1} g(n) = g(N(j+1))\sum_{n=0}^{N_j q^j-1} g(n) = g(N(j+1))\sum_{l=0}^{N_j-1} g(lq^j)\sum_{n=0}^{q^j-1} g(n).$$

Therefore

$$\begin{aligned}\left|\sum_{n=0}^{N-1} \mathrm{e}^{2\pi \mathtt{i} \langle \boldsymbol{h}, s_{\boldsymbol{\gamma}}(n)\rangle}\right| &\leq \sum_{j=0}^{m}\left|\sum_{l=0}^{N_j-1} g(lq^j)\right| \cdot \left|\sum_{n=0}^{q^j-1} g(n)\right| \\ &\leq \sum_{j=0}^{m} N_j q^j \frac{1}{q^j}\left|\sum_{n=0}^{q^j-1} g(n)\right| \\ &\leq \sum_{j=0}^{r-1} N_j q^j + \sum_{j=r}^{m} N_j q^j \prod_{\substack{k=0\\ \langle \boldsymbol{h}, \boldsymbol{\gamma}_k\rangle q \notin \mathbb{Z}}}^{j-1} \left(\frac{q - 4\|\langle \boldsymbol{h}, \boldsymbol{\gamma}_k\rangle\|^2}{q}\right) \prod_{\substack{k=0\\ \langle \boldsymbol{h}, \boldsymbol{\gamma}_k\rangle \notin \mathbb{Z} \wedge \langle \boldsymbol{h}, \boldsymbol{\gamma}_k\rangle q \in \mathbb{Z}}}^{j-1} 0.\end{aligned}$$

for any $r \in \mathbb{N}_0$.

We consider two cases

1. There exists a $k \in \mathbb{N}_0$ such that $\langle \boldsymbol{h}, \boldsymbol{\gamma}_k \rangle \notin \mathbb{Z}$ and $\langle \boldsymbol{h}, \boldsymbol{\gamma}_k \rangle q \in \mathbb{Z}$. Let k_0 be minimal with this property (of course k_0 is independent of N). Then we have

$$\left| \sum_{n=0}^{N-1} \mathrm{e}^{2\pi \mathtt{i} \langle \boldsymbol{h}, s_{\boldsymbol{\gamma}}(n) \rangle} \right| \leq \sum_{j=0}^{k_0} N_j q^j \prod_{\substack{k=0 \\ \langle \boldsymbol{h}, \boldsymbol{\gamma}_k \rangle q \notin \mathbb{Z}}}^{j-1} \left(\frac{q - 4 \| \langle \boldsymbol{h}, \boldsymbol{\gamma}_k \rangle \|^2}{q} \right) \leq (q-1) \sum_{j=0}^{k_0} q^j = q^{k_0+1} - 1.$$

2. For all $k \in \mathbb{N}_0$ we have $\langle \boldsymbol{h}, \boldsymbol{\gamma}_k \rangle \in \mathbb{Z}$ or $\langle \boldsymbol{h}, \boldsymbol{\gamma}_k \rangle q \notin \mathbb{Z}$. Then we have

$$\left| \sum_{n=0}^{N-1} \mathrm{e}^{2\pi \mathtt{i} \langle \boldsymbol{h}, s_{\boldsymbol{\gamma}}(n) \rangle} \right| \leq q^r + N \prod_{\substack{k=0 \\ \langle \boldsymbol{h}, \boldsymbol{\gamma}_k \rangle q \notin \mathbb{Z}}}^{r-1} \left(\frac{q - 4 \| \langle \boldsymbol{h}, \boldsymbol{\gamma}_k \rangle \|^2}{q} \right). \tag{2}$$

Define

$$\begin{aligned} x_r &:= q^r \Bigg/ \left(\prod_{\substack{k=0 \\ \langle \boldsymbol{h}, \boldsymbol{\gamma}_k \rangle q \notin \mathbb{Z}}}^{r-1} \left(\frac{q - 4 \| \langle \boldsymbol{h}, \boldsymbol{\gamma}_k \rangle \|^2}{q} \right) \right) \\ &= \left(\prod_{\substack{k=0 \\ \langle \boldsymbol{h}, \boldsymbol{\gamma}_k \rangle q \in \mathbb{Z}}}^{r-1} q \right) \prod_{\substack{k=0 \\ \langle \boldsymbol{h}, \boldsymbol{\gamma}_k \rangle q \notin \mathbb{Z}}}^{r-1} \left(\frac{q^2}{q - 4 \| \langle \boldsymbol{h}, \boldsymbol{\gamma}_k \rangle \|^2} \right) \geq q^r. \end{aligned}$$

Therefore $x_r \to \infty$ as $r \to \infty$. Choose r such that $x_r \leq N < x_{r+1}$. Then we have

$$q^r \leq N \prod_{\substack{k=0 \\ \langle \boldsymbol{h}, \boldsymbol{\gamma}_k \rangle q \notin \mathbb{Z}}}^{r-1} \left(\frac{q - 4 \| \langle \boldsymbol{h}, \boldsymbol{\gamma}_k \rangle \|^2}{q} \right). \tag{3}$$

On the other hand we have

$$\prod_{\substack{k=0 \\ \langle \boldsymbol{h}, \boldsymbol{\gamma}_k \rangle q \notin \mathbb{Z}}}^{r} \left(\frac{q - 4 \| \langle \boldsymbol{h}, \boldsymbol{\gamma}_k \rangle \|^2}{q} \right) \geq \prod_{k=0}^{r} \left(\frac{q - 4 \| \langle \boldsymbol{h}, \boldsymbol{\gamma}_k \rangle \|^2}{q} \right) \geq \prod_{k=0}^{r} \frac{1}{q} = \frac{1}{q^{r+1}}$$

and hence

$$N < q^{r+1} \Bigg/ \left(\prod_{\substack{k=0 \\ \langle \boldsymbol{h}, \boldsymbol{\gamma}_k \rangle q \notin \mathbb{Z}}}^{r} \left(\frac{q - 4 \| \langle \boldsymbol{h}, \boldsymbol{\gamma}_k \rangle \|^2}{q} \right) \right) \leq q^{2(r+1)}.$$

Thus we have $\log_q \sqrt{N} < r + 1$ resp. $\lfloor \log_q \sqrt{N} \rfloor \leq r$ and hence

$$\prod_{\substack{k=0 \\ \langle \boldsymbol{h}, \boldsymbol{\gamma}_k \rangle q \notin \mathbb{Z}}}^{r-1} \left(\frac{q - 4 \| \langle \boldsymbol{h}, \boldsymbol{\gamma}_k \rangle \|^2}{q} \right) \leq \prod_{\substack{k=0 \\ \langle \boldsymbol{h}, \boldsymbol{\gamma}_k \rangle q \notin \mathbb{Z}}}^{\lfloor \log_q \sqrt{N} \rfloor - 1} \left(\frac{q - 4 \| \langle \boldsymbol{h}, \boldsymbol{\gamma}_k \rangle \|^2}{q} \right). \tag{4}$$

From (2), (3) and (4) we find

$$\begin{aligned}\left|\sum_{n=0}^{N-1} \mathrm{e}^{2\pi \mathtt{i}\langle \boldsymbol{h}, s_{\boldsymbol{\gamma}}(n)\rangle}\right| &\leq 2N \prod_{\substack{k=0\\ \langle \boldsymbol{h},\boldsymbol{\gamma}_k\rangle q\notin\mathbb{Z}}}^{\lfloor \log_q \sqrt{N}\rfloor -1} \left(\frac{q-4\|\langle \boldsymbol{h},\boldsymbol{\gamma}_k\rangle\|^2}{q}\right)\\ &\leq 2N\mathrm{e}^{\sum_{\substack{k=0\\ \langle \boldsymbol{h},\boldsymbol{\gamma}_k\rangle q\notin\mathbb{Z}}}^{\lfloor \log_q \sqrt{N}\rfloor -1} \log\left(\frac{q-4\|\langle \boldsymbol{h},\boldsymbol{\gamma}_k\rangle\|^2}{q}\right)}\\ &\leq 2N\mathrm{e}^{-\frac{4}{q}\sum_{\substack{k=0\\ \langle \boldsymbol{h},\boldsymbol{\gamma}_k\rangle q\notin\mathbb{Z}}}^{\lfloor \log_q \sqrt{N}\rfloor -1} \|\langle \boldsymbol{h},\boldsymbol{\gamma}_k\rangle\|^2}\end{aligned}$$

In both of the above cases we obtain $\frac{1}{N}\sum_{n=0}^{N-1} \mathrm{e}^{2\pi \mathtt{i}\langle \boldsymbol{h}, s_{\boldsymbol{\gamma}}(n)\rangle} \to 0$ as $N\to\infty$. Hence the result follows by Weyl's criterion.

Assume now that there is a $\boldsymbol{h}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}$ such that

$$\sum_{\substack{k=0\\ \langle \boldsymbol{h},\boldsymbol{\gamma}_k\rangle q\notin\mathbb{Z}}}^{\infty} \|\langle \boldsymbol{h},\boldsymbol{\gamma}_k\rangle\|^2 < \infty$$

and for all $k\in\mathbb{N}_0$ we have $\langle \boldsymbol{h},\boldsymbol{\gamma}_k\rangle\in\mathbb{Z}$ or $\langle \boldsymbol{h},\boldsymbol{\gamma}_k\rangle q\notin\mathbb{Z}$.

Then we have

$$\sum_{k=0}^{\infty}\|\langle \boldsymbol{h},\boldsymbol{\gamma}_k\rangle\|^2 = \sum_{\substack{k=0\\ \langle \boldsymbol{h},\boldsymbol{\gamma}_k\rangle q\notin\mathbb{Z}}}^{\infty} \|\langle \boldsymbol{h},\boldsymbol{\gamma}_k\rangle\|^2 + \sum_{\substack{k=0\\ \langle \boldsymbol{h},\boldsymbol{\gamma}_k\rangle q\in\mathbb{Z}}}^{\infty} \|\langle \boldsymbol{h},\boldsymbol{\gamma}_k\rangle\|^2 < \infty.$$

For $j\in\mathbb{N}_0$ we have

$$\left|\frac{1}{q^j}\sum_{n=0}^{q^j-1} \mathrm{e}^{2\pi \mathtt{i}\langle \boldsymbol{h}, s_{\boldsymbol{\gamma}}(n)\rangle}\right| = \frac{1}{q^j}\prod_{k=0}^{j-1}\left|\sum_{n=0}^{q-1} \mathrm{e}^{2\pi \mathtt{i}\langle \boldsymbol{h},\boldsymbol{\gamma}_k\rangle n}\right|.$$

Here we have

$$\sum_{n=0}^{q-1} \mathrm{e}^{2\pi \mathtt{i}\langle \boldsymbol{h},\boldsymbol{\gamma}_k\rangle n} \neq 0$$

for all $k\in\mathbb{N}_0$. This is clear for the case $\langle \boldsymbol{h},\boldsymbol{\gamma}_k\rangle\in\mathbb{Z}$. If $\langle \boldsymbol{h},\boldsymbol{\gamma}_k\rangle\notin\mathbb{Z}$, then we have $\langle \boldsymbol{h},\boldsymbol{\gamma}_k\rangle q\notin\mathbb{Z}$ and the inequality holds as well.

With Lemma 1 and since $\|nx\|\leq n\|x\|$ for all $n\in\mathbb{N}_0$ we obtain

$$\left|\sum_{n=0}^{q-1} \mathrm{e}^{2\pi \mathtt{i}\langle \boldsymbol{h},\boldsymbol{\gamma}_k\rangle n}\right| \geq q\left(1-4\pi^2\max_{0\leq n<q}\|\langle \boldsymbol{h},\boldsymbol{\gamma}_k\rangle n\|^2\right) > q\left(1-4\pi^2 q\|\langle \boldsymbol{h},\boldsymbol{\gamma}_k\rangle\|^2\right).$$

Let $0<c<1$ and let $l\in\mathbb{N}$ be large enough such that

$$1-4\pi^2 q\sum_{k>l}\|\langle \boldsymbol{h},\boldsymbol{\gamma}_k\rangle\|^2 > c > 0.$$

For $j > l$ we have

$$\begin{aligned}\left|\frac{1}{q^j}\sum_{n=0}^{q^j-1} \mathrm{e}^{2\pi\mathtt{i}\langle \boldsymbol{h}, s_{\boldsymbol{\gamma}}(n)\rangle}\right| &\geq \prod_{k=0}^{l}\frac{1}{q}\left|\sum_{n=0}^{q-1}\mathrm{e}^{2\pi\mathtt{i}\langle \boldsymbol{h},\boldsymbol{\gamma}_k\rangle n}\right|\prod_{k=l+1}^{j}\left(1-4\pi^2 q\|\langle \boldsymbol{h},\boldsymbol{\gamma}_k\rangle\|^2\right)\\ &\geq c'\left(1-4\pi^2 q\sum_{k>l}\|\langle \boldsymbol{h},\boldsymbol{\gamma}_k\rangle\|^2\right) > c'\cdot c > 0.\end{aligned}$$

and by Weyl's criterion $(\{s_{\boldsymbol{\gamma}}(n)\})_{n\geq 0}$ is not uniformly distributed modulo one. $\square$

Corollary 1 *The sequence $(\{s_{\boldsymbol{\gamma}}(n)\})_{n\geq 0}$ is uniformly distributed modulo one for almost all sequences $\boldsymbol{\gamma} : \mathbb{N}_0 \to [0,1)^d$.*

Proof. We consider the sequence of random variables $X_1, X_2, \ldots$ uniformly i.i.d. in $[0,1)^d$. For $\boldsymbol{h} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$, we have $\mathbb{E}(\|\langle \boldsymbol{h}, X_i\rangle\|^2) = 1/12$ and hence it follows from Kolmogorov's strong law of large numbers that for $n \to \infty$ we have

$$\frac{\|\langle \boldsymbol{h}, X_1\rangle\|^2 + \cdots + \|\langle \boldsymbol{h}, X_n\rangle\|^2}{n} \to \frac{1}{12} \qquad \text{a.e..}$$

Therefore

$$\sum_{k=0}^{\infty}\|\langle \boldsymbol{h},\boldsymbol{\gamma}_k\rangle\|^2 = \infty$$

for almost all sequences $\boldsymbol{\gamma} : \mathbb{N}_0 \to [0,1)^d$ and hence

$$\sum_{k=0}^{\infty}\|\langle \boldsymbol{h},\boldsymbol{\gamma}_k\rangle\|^2 = \infty \qquad \forall \boldsymbol{h} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$$

for almost all sequences $\boldsymbol{\gamma} : \mathbb{N}_0 \to [0,1)^d$. The result follows from Theorem 1. $\square$

Finally we state an

Open question: Let $q_1, \ldots, q_d \geq 2$ be pairwisely coprime integers. Under which conditions on the weight-sequences $\gamma^{(k)} = (\gamma_0^{(k)}, \gamma_1^{(k)}, \ldots)$ in $\mathbb{R}$, $k \in \{1, \ldots, d\}$, is the sequence

$$(\{(s_{q_1,\gamma^{(1)}}(n), \ldots, s_{q_1,\gamma^{(1)}}(n))\})_{n\geq 0} \tag{5}$$

uniformly distributed modulo one? (Here we wrote $s_{q,\gamma}(\cdot)$ for the weighted q-ary sum-of-digits function to stress the dependence on the base q.)

For example if $\gamma_i^{(k)} = q_k^{-i-1}$ for all $k \in \{1, \ldots, d\}$ and all $i \in \mathbb{N}_0$, then we obtain the d-dimensional Halton sequences which is well known to be uniformly distributed modulo one. If $\gamma_i^{(k)} = \alpha_k \in \mathbb{R}$ for all $k \in \{1, \ldots, d\}$ and all $i \in \mathbb{N}_0$, then it was shown by Drmota and Larcher [3] that the sequence (5) is uniformly distributed modulo one if and only if $\alpha_1, \ldots, \alpha_d \in \mathbb{R} \setminus \mathbb{Q}$. But also the classical $(n\boldsymbol{\alpha})$-sequence is contained in this concept.

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